Approximate BER Expressions for FSO Communication Systems with Multiuser Diversity

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***Abstract*—This paper tackles the performance analysis of multiuser diversity (MD) free space optical (FSO) communication systems. In particular, we evaluate the performance of the n-th best user point-to-multipoint selection scheme and the performance of the multipoint-to-multipoint user grouping scheme. We derive approximate bit error rate (BER) expressions under lognormal and gamma-gamma scintillation. The approximation technique is based on the recurrence relations between the probability density functions (pdf's) of order statistics and on moment matching methods for deriving the parameters of the approximating pdf. Results highlight the accuracy of the derived expressions in predicting the BER performance.**

***Index Terms*—FSO, Opportunistic Scheduling, Order Statistics, Atmospheric Turbulence, Lognormal, Gamma-Gamma.**

I. Introduction

Atmospheric turbulence-induced scintillation constitutes a major impairment that severely degrades the performance of free space optical (FSO) communication systems. This motivated an extensive research that investigated the cooperative diversity techniques as cost-effective fading mitigation solutions. The existing literature on cooperative FSO systems is rich and covers amplify-and-forward (AF) modes, decode-and-forward (DF) modes, serial-relaying, selective parallel-relaying and all-active parallel-relaying in the context of point-to-point [1], [2], point-to-multipoint [3]–[7] or multipoint-to-multipoint communications [8], [9].

This work focuses on the point-to-multipoint FSO systems [3]–[7] that are also referred to as opportunistic scheduling or multiuser diversity (MD) schemes capable of achieving spatial diversity gains through multiuser selective transmissions. In MD scenarios, a central node communicates with N distant users where, in a given time slot, the n-th best user is selected given that the $N-n$ users with better channel conditions might have no information to communicate under given traffic conditions [6]. Evidently, this entails the best user selection MD scheme ($n=N$) as a special case. We also consider the multipoint-to-multipoint FSO user-grouping systems [8], [9] where the information bits of the N users are grouped and transmitted over a subgroup of the N available links entailing the N-th, $(N-1)$-th, $(N-2)$-th, $\cdots$ best channels.

The performance of the best user among $N=2$ users was analyzed in [4] where the bit error rate (BER) was expressed in terms of the extended generalized bivariate Meijer G-function. The asymptotic performance with large number of users has been carried out in [5] using the extreme value theory. Both [4] and [5] considered the gamma-gamma (generalized-K) composite channel model. Excluding the special cases $N=2$ and $N \gg 1$, no exact expressions of the BER have been reported in the literature given that the expressions of the order statistics probability density functions (pdf) get very involved rendering all subsequent calculations very challenging. Approximate BER expressions were reported under the form of power series in [6] for lognormal and gamma-gamma scintillation while an exact performance analysis was carried out in [7] using the Gauss-Laguerre quadrature rule under exponentiated Weibull turbulence channels.

Regarding multipoint-to-multipoint systems, an asymptotic analysis was carried out in [8] for any number of users with maximum-ratio-combining (MRC) and equal-gain-combining (EGC). On the other hand, [9] targeted the two-user case with selective-combining (SC) where a joint encoding scheme based on pulse position modulation (PPM) was proposed and analyzed.

In this paper, we target the BER performance analysis of the n-th best user-selection and N-user-grouping MD schemes under the lognormal and gamma-gamma fading channels that are widely used to model the FSO channels under weak and strong turbulence conditions, respectively. We provide approximate BER expressions that are very accurate over the entire signal-to-noise ratio (SNR) range. In this context, instead of adopting the series expansions of the conditional BER, order statistics pdf and the average BER integral [6], [7], appropriate distributions are proposed to approximate the exact pdf of the channel of the n-th best user. This results in more tractable expressions with a high level of accuracy. A theoretical analysis guides the search for the appropriate approximating pdf that is further optimized numerically. The presented calculation methodology is based on the recurrence relations that exist between the pdf's of the order statistics [10] that avoid approximating the order statistic pdf for all values of n and N. To the authors' best knowledge, these relations were never exploited before for evaluating the performance of communication systems.

II. System Model

Consider a N-user FSO system with intensity modulation and direct detection (IM/DD). The received electrical signal at the output of the n-th user's transceiver can be written as:

$$y_n = \eta h_n x_n + w_n \quad ; \quad n=1,\dots,N \tag{1}$$

where x_n and y_n stand for the transmitted and received signals, respectively. η is the optical-to-electrical conversion ratio and w_n is the noise at the n-th user's transceiver that is assumed to be additive white Gaussian noise (AWGN) with zero mean and variance $N_0/2$. In (1), h_n stands for the irradiance of the n-th channel. The channel irradiances $h_1, \dots, h_N$ are assumed to be independent and identically distributed (iid). In this context, for the sake of simplicity, all users are assumed to be at the same distance from the central node in the case of point-to-multipoint communications. For multipoint-to-multipoint systems, the distances separating any pair of transceivers are practically the same following from the network architecture [8] where the transmitting/receiving FSO units are placed on the same building.

The presented performance analysis is based on two statistical models that are widely used to model the atmospheric turbulence over FSO channels. Namely, we consider the log-normal and gamma-gamma models that are often adopted for the weak and strong turbulence conditions, respectively [6].

The lognormal pdf is given by:

$$f_{\text{LN}}(h;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma h}\exp\left(-\frac{(\ln(h)-\mu)^2}{2\sigma^2}\right) \; ; \; h \geq 0 \quad (2)$$

where the parameters μ and σ satisfy the relation $\mu = -\sigma^2$ so that the mean path intensity is unity.

The gamma-gamma pdf is given by ($h \geq 0$):

$$f_{\gamma\gamma}(h;\alpha,\beta) = \frac{2(\alpha\beta)^{(\alpha+\beta)/2}}{\Gamma(\alpha)\Gamma(\beta)} h^{(\alpha+\beta)/2-1} K_{\alpha-\beta}\left(2\sqrt{\alpha\beta h}\right) \quad (3)$$

where $\Gamma(.)$ is the Gamma function and $K_c(.)$ is the modified Bessel function of the second kind of order c.

The channel parameters σ and (α, β) in (2) and (3) are related to the Rytov variance $\sigma_R^2 = 1.23 C_n^2 k^{7/6} d^{11/6}$ through equations (6) and (3)-(4) in [6] and [8], respectively. d is the link distance, $k = \frac{2\pi}{\lambda}$ is the wave number and C_n^2 denotes the refractive index structure parameter.

Finally, we assume that the transmitted symbols are carved from a unipolar Q-ary Pulse Amplitude Modulation (PAM) constellation that is suitable for IM/DD [8]. The signal set is given by $\{q(2A) \; ; \; q = 0, \dots, Q-1\}$ where $A^2 = \frac{3}{2(Q-1)(2Q-1)}$ in order to normalize the energy of the unipolar PAM signal set.

III. Performance Analysis: Preliminaries

A. Point-to-Multipoint Systems

Consider a N-user point-to-multipoint FSO system. At a given time slot, and based on the underlying channel conditions and the availability of the different FSO links, the cental node selects to communicate with the user possessing the n-th best channel [3]–[7].

The average BER (with Q-PAM) of the n-th best user can be written as [8]:

$$P_e^{(n)}(Q) = \frac{2(Q-1)}{Q\log_2 Q} \times \int_0^{+\infty} Q\left(\sqrt{\frac{3\log_2 Q}{(Q-1)(2Q-1)}\frac{E_b}{N_0}}\eta h\right) f_{n,N}(h)\mathrm{d}h \quad (4)$$

where E_b stands for the bit energy and $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{t^2}{2}}\mathrm{d}t$ is the Q-function.

In (4), $f_{n,N}(h)$ is the pdf of the n-th best channel irradiance out of a total of N irradiances. Based on order statistics, this pdf is given by [11, 2.2.2]:

$$f_{n,N}(h) = \frac{N!}{(n-1)!(N-n)!} f(h)\left[F(h)\right]^{n-1}\left[1-F(h)\right]^{N-n} \quad (5)$$

where the pdf $f(h)$ corresponds to either one of the pdf's in (2)-(3) while $F(h)$ corresponds to the cumulative distribution function (cdf).

For lognormal scintillation:

$$F(h) = F_{\text{LN}}(h;\mu,\sigma) = 1 - Q\left(\frac{\ln x - \mu}{\sigma}\right) \quad (6)$$

while for gamma-gamma scintillation:

$$F(h) = F_{\gamma\gamma}(h;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} G_{1,3}^{2,1}\left[\alpha\beta h\middle|\,{}_{\alpha,\beta,0}^{\;\;1}\right] \quad (7)$$

where $G_{p,q}^{m,n}[.]$ is the Meijer G-function.

B. Multipoint-to-Multipoint Systems

Consider a N-user multipoint-to-multipoint FSO system. Such systems revolve around the grouping of the FSO transmitters and receivers in order to achieve enhanced diversity levels [8]. In a more detailed manner, instead of having each one of the N users transmitting $\log_2 Q$ bits of information along its dedicated FSO link in a noncooperative manner, the total $N\log_2 Q$ bits are jointly encoded and grouped to generate a number of PAM symbols that are transmitted along the best FSO links.

In a more formal way, designate by q_n the number of bits that will be transmitted along the n-th best channel for $n = 1, \dots, N$ with $\sum_{n=1}^{N} q_n = N\log_2 Q$. Based on the expression given in (4), the average BER of the cooperative system can be written as follows [8]:

$$P_e(\mathbf{q}) = \frac{1}{N\log_2 Q}\sum_{n=1}^{N} q_n P_e^{(n)}(2^{q_n}) \quad (8)$$

where $\mathbf{q}$ is the N-dimensional vector that is defined as $\mathbf{q} \triangleq [q_1, \dots, q_N]$. This vector defines the bit allocation along the N links or, in an equivalent manner, it defines the way in which the users are grouped.

For example, $\mathbf{q} = [\log_2 Q, \dots, \log_2 Q]$ corresponds to the non-cooperative scheme with no user grouping. On the other hand, $\mathbf{q} = [0, \dots, 0, N\log_2 Q]$ corresponds to the all-user grouping where all information bits of all users are transmitted over the N-th best channel. It was proven in [8] that this scheme results in the highest diversity gain at the expense of a degenerated coding gain since a PAM constellation with cardinality $2^{N \log_2 Q} = Q^N$ is needed. A compromise that achieves higher performance levels for small-to-average SNR values corresponds to allocating the information bits not only to the N-th best channel but also to the $(N-1)$-th, $(N-2)$-th, $\cdots$ best channels. In this way, the cardinalities of the used PAM signal sets are alleviated resulting in higher coding gains. Evidently, the elements of $\mathbf{q}$ must be ordered in an increasing order for an enhanced average BER.

Based on (4) and (8), the optimal bit allocation vector $\mathbf{q} = [q_1, \ldots, q_N]$ can be obtained by performing the following optimization:

$$\begin{aligned} \min \quad & \sum_{n=1}^{N} \frac{(2^{q_n}-1)}{2^{q_n}} \times \\ & \int_0^{+\infty} Q\left(\sqrt{\frac{3q_n}{(2^{q_n}-1)(2^{q_n+1}-1)} \frac{E_b}{N_0}} \eta h\right) f_{n,N}(h) \mathrm{d}h \\ \text{st.} \quad & \begin{cases} q_1, \ldots, q_N \in \mathbb{N}, \\ \sum_{n=1}^{N} q_n = N \log_2 Q, \\ q_1 \leq q_2 \leq \cdots \leq q_N. \end{cases} \end{aligned} \quad (9)$$

IV. Performance Analysis: Average BER

Evaluating the BER in (4), and consequently the BER in (9), for arbitrary values of N and n is extremely involved and no closed-form exact expressions seem to exist. This is justified by the complexity of the pdf expressions in (2)-(3) and the cdf expressions in (6)-(7). This motivates introducing approximate expressions for the order statistics pdf $f_{n,N}(h)$ in (5) that result in accurate approximations of the BER.

A. Recurrence Relations

Instead of tackling the approximation of the pdf $f_{n,N}(h)$ for all values of N and n, the proposed approach consists of making use of the recurrence relations that exist between the pdf's of order statistics [10] (for $n = 1, \ldots, N-1$):

$$f_{n+1,N}(h) = \frac{N}{n} f_{n,N-1}(h) - \frac{N-n}{n} f_{n,N}(h) \quad (10)$$

In particular, (10) allows for expressing $f_{n,N}(h)$ as a linear combination of the pdf's $f_{1,1}, f_{1,2}, \ldots, f_{1,N}$ of the minimum random variables (rv's) or as a linear combination of the pdf's $f_{1,1}, f_{2,2}, \ldots, f_{N,N}$ of the maximum rv's. Among these two approaches, we found that the latter one results in more accurate results since it is less sensitive to approximation errors. In fact, the pdf $f_{K,K}(h)$ of the maximum is spread over a wider range of h. On the other hand, the pdf $f_{1,K}(h)$ of the minimum is more compressed towards small values of h entailing a lower tolerance to the approximation imprecision.

Consequently, in the adopted calculation methodology, the pdf $f_{n,N}(h)$ will be written as:

$$f_{n,N}(h) = \sum_{K=n}^{N} w_{N,n,K} f_{K,K}(h) \quad ; \quad n = 1, \ldots, N \quad (11)$$

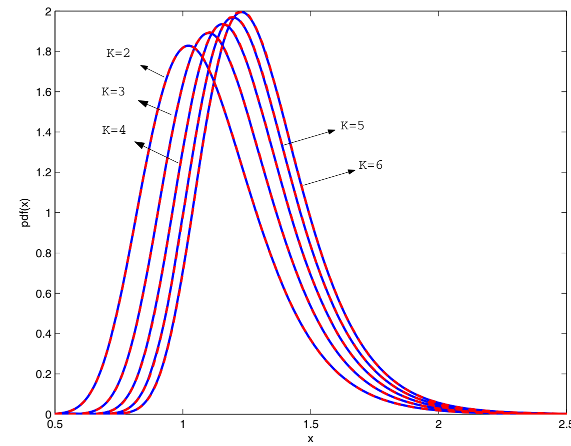

Fig. 1. Exact pdf's (in solid lines) and the approximate pdf's (in dashed lines) corresponding to eq. (13).

where $\{w_{N,n,K}\}_{K=n}^{N}$ correspond to the weights that depend on the specific values of N and n with $w_{N,N,N} = 1$. For example, for $N = 5$, these weights are reported below in the matrix $\mathbf{W}_N$ whose (n, K)-th element is equal to $w_{N,n,K}$:

$$\mathbf{W}_5 = \begin{bmatrix} 5 & -10 & 10 & -5 & 1 \\ 0 & 10 & -20 & 15 & -4 \\ 0 & 0 & 10 & -15 & 6 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

where, evidently, the elements of each row add up to one.

B. Approximating $f_{K,K}(h)$ with Lognormal Scintillation

$f_{K,K}(h)$ corresponds to the pdf of the rv $\mathfrak{h} \triangleq \max(h_1, \ldots, h_K)$. This rv can be approximated as follows:

$$\mathfrak{h} \approx (h_1^m + \cdots + h_K^m)^{\frac{1}{m}} \quad (13)$$

where m is a large enough positive number.

The approximation in (13) is inspired from the well-known smoothing techniques that are commonly applied for solving minimax optimization problems where equivalent differentiable functions are used to approximate the non-differentiable minimax function [12]. The intuition behind this approximation is as follows. Since the rv's $h_1, \ldots, h_K$ are all positive, then raising each one of them to a high power will accentuate the largest one as compared to the others. To the authors' best knowledge, this method that is widely used for solving optimization problems has never been used for solving probability problems.

The approximation in (13) is highlighted numerically in Fig. 1 that compares the exact pdf (corresponding to the rv to the left-hand side of the equation) with the approximate pdf (corresponding to the combination of rv's to the right-hand side of the equation) for $m = 100$. We set $C_n^2 = 1.7 \times 10^{-14}$ $\mathrm{m}^{-2/3}$ and the link distance to 3 km (all details on the simulation parameters are provided in Section V). Results highlight the close match between the exact and approximate

TABLE I
MEAN SQUARE ERROR IN APPROXIMATING $f_{K,K}(h)$

K	2	3	4	5	6
MSE Lognormal	3.769×10^{-7}	1.981×10^{-6}	4.472×10^{-6}	7.549×10^{-6}	9.861×10^{-6}
MSE Gamma-Gamma	3.152×10^{-7}	3.304×10^{-7}	3.332×10^{-7}	3.381×10^{-7}	3.405×10^{-7}

pdf's for different values of K, thus, supporting the validity of this approximation.

Now, since $\{h_k\}_{k=1}^K$ are iid lognormal rv's with parameters μ and σ, then $\{h_k^m\}_{k=1}^K$ will be iid lognormal rv's with parameters $m\mu$ and $m\sigma$ (where $m > 0$). Therefore, $\mathfrak{h}^m = \sum_{k=1}^K h_k^m$ corresponds to the sum of K lognormal rv's and, hence, can be accurately approximated by a lognormal distribution following from the rich literature on the lognormal-sum approximation problem [13], [14]. As a conclusion, $\mathfrak{h}$, in its turn, will be approximated by the lognormal distribution implying that the lognormal pdf constitutes a strong candidate approximation to $f_{K,K}(h)$.

While the lognormal pdf approximates $f_{K,K}(h)$ with an acceptable level of accuracy, our numerical analysis has shown that this accuracy can be dramatically enhanced if this approximation is further refined to a more general two-component lognormal-mixture pdf. Based on this approach, $f_{K,K}(h)$ can be approximated as follows:

$$f_{K,K}(h) \approx \sum_{i=1}^{2} a_{K,i} f_{\text{LN}}(h; \mu_{K,i}, \sigma_{K,i}) \tag{14}$$

with $a_{K,1} + a_{K,2} = 1$ and where $f_{\text{LN}}(.)$ is defined in (2).

Moment matching techniques [14] can be efficiently applied to calculate the five parameters $a_{K,1}, \mu_{K,1}, \mu_{K,2}, \sigma_{K,1}$ and $\sigma_{K,2}$. Matching the first five moments of $\mathfrak{h}$ to those of the lognormal-mixture results in the following set of equations:

$$\sum_{i=1}^{2} a_{K,i} e^{s\mu_{K,i} + \frac{1}{2}s^2\sigma_{K,i}^2} = \text{E}\left[\mathfrak{h}^s\right] \;\; ; \;\; s = 1, \ldots, 5 \tag{15}$$

where the moments of $\mathfrak{h}$ can be evaluated either numerically or by applying the explicit expressions derived in [15]. The system of transcendental equations in (15) can be readily solved using built-in packages that are available in the majority of the existing computing softwares.

As a conclusion, based on (11) and (14), the order statistics pdf $f_{n,N}(h)$ can be approximated by the mixture of $2(N-n+1)$ lognormal pdf's. Approximating $Q(x)$ by $\frac{1}{12}e^{-\frac{x^2}{2}} + \frac{1}{4}e^{-\frac{2x^2}{3}}$ and solving (4) results in:

$$\begin{aligned} P_e^{(n)}(Q) \approx & \frac{2(Q-1)}{Q\log_2 Q} \sum_{K=n}^{N} w_{N,n,K} \sum_{i=1}^{2} a_{K,i} \\ & \left[\frac{1}{12}\text{Fr}\left(\frac{3\log_2 Q}{2(Q-1)(2Q-1)}\frac{E_b}{N_0}\eta^2 A_{K,i}, 0; \sigma_{K,i}\right)\right. \\ & \left. + \frac{1}{4}\text{Fr}\left(\frac{2\log_2 Q}{(Q-1)(2Q-1)}\frac{E_b}{N_0}\eta^2 A_{K,i}, 0; \sigma_{K,i}\right)\right] \end{aligned} \tag{16}$$

where $A_{K,i} \triangleq e^{2\mu_{K,i} + 2\sigma_{K,i}^2}$ and $\text{Fr}(a, 0; b)$ is the lognormal density frustration function [16]:

$$\text{Fr}(a, 0; b) = \int_0^{+\infty} \frac{1}{\sqrt{2\pi}bx} \exp\left(-ax^2\right) \exp\left[-\frac{(\ln(x) + b^2)^2}{2b^2}\right] \mathrm{d}x \tag{17}$$

Finally, replacing (16) in (8) results in the approximate BER expression for multipoint-to-multipoint systems.

C. Approximating $f_{K,K}(h)$ with Gamma-Gamma Scintillation

From (5), $f_{K,K}(h) = Kf(h)F^{K-1}(h)$. The first two terms in the expansion of $f(h)$ near the origin are $a_0h^{\beta-1} - a_1h^{\beta}$ where $a_0 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\beta)}(\alpha\beta)^{\beta}$ and $a_1 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(\alpha\beta)^{\beta+1}}{(\alpha-\beta-1)}$ [2]. The quest for the approximating pdf is motivated by the approach that has been proposed in [2], [17] and starts by observing that $f(h) \approx a_0h^{\beta-1}\left(1 - \frac{a_1}{a_0}h\right) \approx a_0h^{\beta-1}e^{-\frac{a_1}{a_0}h}$ since $e^{-x} \approx 1 - x$ for $|x| \ll 1$. Handling $F(h)$ in the same way results in $F(h) \approx \frac{a_0}{\beta}h^{\beta} - \frac{a_1}{\beta+1}h^{\beta+1} \approx \frac{a_0}{\beta}h^{\beta}e^{-\frac{a_1}{a_0}\frac{\beta}{\beta+1}h}$. Consequently, $f_{K,K}(h)$ can be approximated as follows:

$$f_{K,K}(h) \approx K\frac{a_0^K}{\beta^{K-1}}h^{K\beta-1}e^{-\frac{a_1}{a_0}\left(1+\frac{(K-1)\beta}{\beta+1}\right)h} \tag{18}$$

showing that the approximating pdf is proportional to a gamma pdf with shape parameter $K\beta$ and scale parameter $\frac{a_0}{a_1}\left(1 + \frac{(K-1)\beta}{\beta+1}\right)^{-1}$.

The above analysis motivates examining the gamma distribution as a candidate approximating distribution to the exact pdf $f_{K,K}(h)$. A consequent numerical analysis showed that $f_{K,K}(h)$ can be approximated to a higher level of accuracy by a two-component gamma-mixture pdf:

$$f_{K,K}(h) \approx \sum_{i=1}^{2} a_{K,i} f_{\gamma}(h; \alpha_{K,i}, \beta_{K,i}) \tag{19}$$

with $a_{K,1} + a_{K,2} = 1$ where $f_{\gamma}(h; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}h^{\alpha-1}e^{-\beta h}$ stands for the gamma pdf.

The parameters $a_{K,1}, \alpha_{K,1}, \alpha_{K,2}, \beta_{K,1}$ and $\beta_{K,2}$ can be obtained by matching the moments of the exact and approximate pdf's:

$$\sum_{i=1}^{2} a_{K,i} \frac{\Gamma(\alpha_{K,i} + s)}{\Gamma(\alpha_{K,i})\beta_{K,i}^s} = \text{E}\left[\mathfrak{h}^s\right] \;\; ; \;\; s = 1, \ldots, 5 \tag{20}$$

where this system of transcendental equations can be solved numerically in a straightforward manner.

As a conclusion, based on (11) and (19), the order statistics pdf $f_{n,N}(h)$ can be approximated by the mixture of $2(N-n+1)$ gamma pdf's in the case of gamma-gamma turbulence. Approximating $Q(x)$ by $\frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}$ and solving (4) using [18,

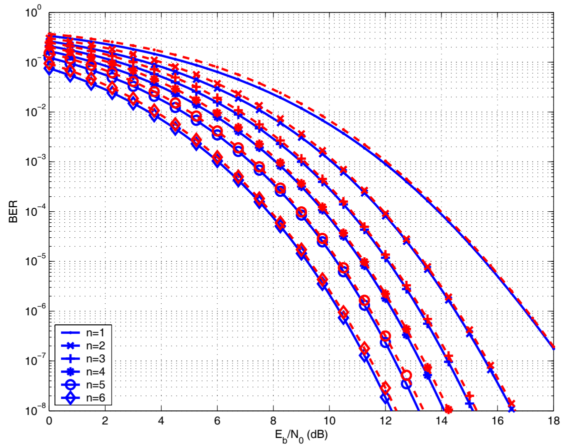

Fig. 2. Performance of the n-th best user among $N = 6$ users under lognormal scintillation. Solid and dashed lines correspond to the exact and approximate BER's, respectively.

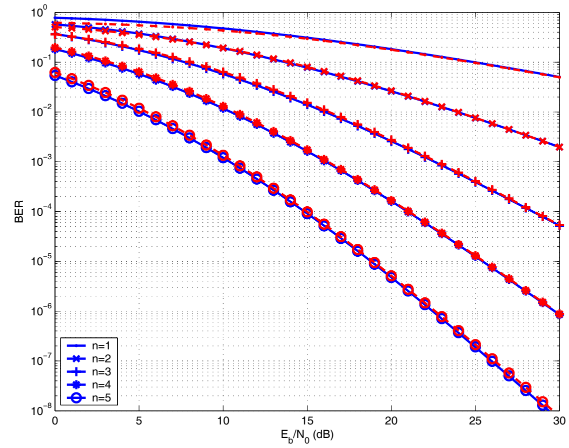

Fig. 3. Performance of the n-th best user among $N = 5$ users under gamma-gamma scintillation. Solid and dashed lines correspond to the exact and approximate BER's, respectively.

3.462] and [18, 9.240] results in:

$$P_e^{(n)}(Q) \approx \frac{2(Q-1)}{Q\log_2 Q}\sum_{K=n}^{N} w_{N,n,K} \sum_{i=1}^{2}\frac{a_{K,i}}{\alpha_{K,i}-1}\times\left(\sqrt{\frac{2E_b'}{N_0}}\frac{\eta}{\beta_{K,i}}\right)^{-\alpha_{K,i}}\times \left[\frac{1}{\Gamma\left(\frac{\alpha_{K,i}}{2}\right)}\Phi\left(\frac{\alpha_{K,i}-1}{2},\frac{1}{2};\frac{\beta_{K,i}^2 N_0}{2E_b'\eta^2}\right)- \frac{\frac{\beta_{K,i}}{\eta}\sqrt{\frac{2N_0}{E_b'}}}{\Gamma\left(\frac{\alpha_{K,i}-1}{2}\right)}\Phi\left(\frac{\alpha_{K,i}}{2},\frac{3}{2};\frac{\beta_{K,i}^2 N_0}{2E_b'\eta^2}\right)\right] \quad (21)$$

where $\Phi(.)$ stands for the confluent hypergeometric function and, for notational simplicity, we set $E_b' = \frac{3\log_2 Q}{(Q-1)(2Q-1)}E_b$.

V. Numerical Results

In what follows, the operating wavelength and the refractive index structure constant are set to $\lambda = 1550$ nm and $C_n^2 = 1.7 \times 10^{-14}$ $\text{m}^{-2/3}$, respectively. We also set $Q = 2$ implying that the point-to-multipoint systems use 2-PAM (or equivalently On-Off-Keying) while the multipoint-to-multipoint systems use a number of PAM constellations whose maximum cardinality is 2^N. Finally, all users are assumed to be at a distance of 3 km from the central node. In what follows, the exact BER expressions are obtained from (4) where the conditional BER is numerically integrated over the exact pdf.

The mean-square-errors (MSE) between the exact and approximate versions of $f_{K,K}(h)$ are reported in Table-I. The MSE values are very small reflecting a high accuracy in approximating $f_{K,K}(h)$ and, consequently, in approximating $f_{n,N}(h)$ for all values of N and n.

Fig. 2 shows the performance of point-to-multipoint systems with $N = 6$ users under lognormal scintillation. Results show the close match between the exact and proposed BER's over the entire SNR range. The accuracy of the derived BER expressions is also highlighted in Fig. 3 in the case of gamma-gamma scintillation with $N = 5$ users. This figure also highlights the improved diversity orders that can be achieved by the MD scheme. Similar findings were observed for other values of the link distances but are not reported here for the sake of brevity.

Fig. 4 shows the performance of multipoint-to-multipoint systems with $N = 5$ users under lognormal scintillation. We investigate different grouping strategies of the users as determined by the vector $\mathbf{g}$. Results show that the achievable performance levels are highly dependent on the adopted grouping strategy thus highlighting the importance of the grouping optimization that needs to be carried out according to (9). In this context, some groupings are even outperformed by non-cooperative systems for the considered SNR range. For the considered scenario, the grouping $\mathbf{g} = [0\ 0\ 1\ 2\ 2]$ results in the best BER performance. In this case, 2 bits are allocated to the best channel over which 4-PAM symbols are transmitted, 2 bits are allocated to the second-best channel over which 4-PAM symbols are transmitted and 1 bit is allocated to the third-best channel over which 2-PAM symbols are transmitted.

Fig. 5 shows the performance of multipoint-to-multipoint systems under gamma-gamma scintillation. The performance of the best grouping strategy (determined from (9)) is shown for 2, 3, 4 and 5 users. Results highlight the impact of increasing the number of users on the system performance. For example, at a BER of 10^{-4}, 5-user systems outperform 2-user systems by around 12 dB. In this context, the best bit-allocation (or user-grouping) strategy is highly dependent on the number of users N and on the specific SNR. For $N = 2$ and $N = 4$, the groupings $\mathbf{g} = [0\ 2]$ and $\mathbf{g} = [0\ 0\ 1\ 3]$, respectively, show the best performance over the entire SNR range. For $N = 3$, $\mathbf{g} = [0\ 1\ 2]$ shows the best performance for SNR values below 30 dB while $\mathbf{g} = [0\ 0\ 3]$ shows the best performance for SNR values exceeding 30 dB. Finally,

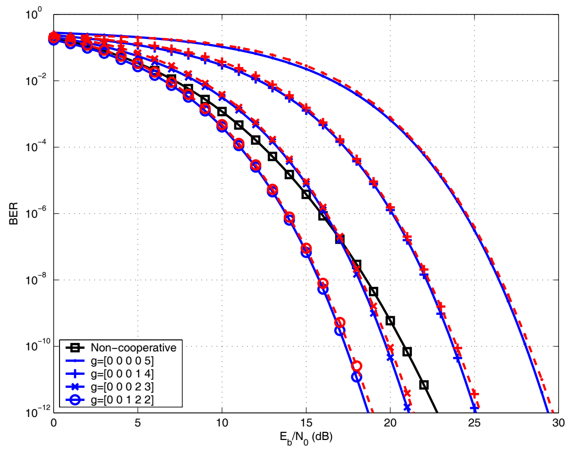

Fig. 4. Performance of multipoint-to-multipoint systems with $N = 5$ users under lognormal scintillation for different user groupings. Solid and dashed lines correspond to the exact and approximate BER's, respectively.

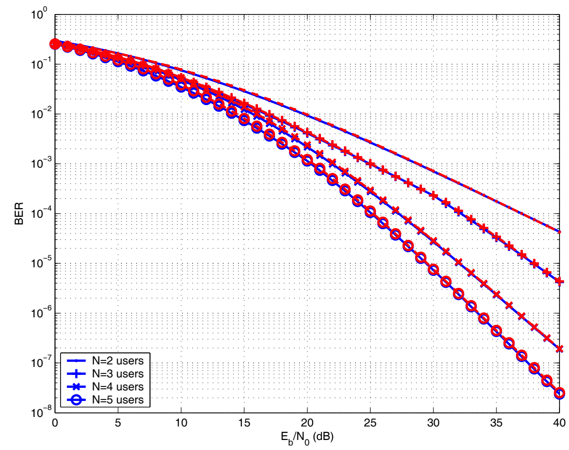

Fig. 5. Performance of multipoint-to-multipoint systems with the best grouping strategy under gamma-gamma scintillation. Solid and dashed lines correspond to the exact and approximate BER's, respectively.

for $N = 5$, $\mathbf{g} = [0\ 0\ 1\ 2\ 2]$ (resp. $\mathbf{g} = [0\ 0\ 0\ 2\ 3]$) achieves the optimal BER value for SNR values below (resp. above) 20 dB.

VI. Conclusion

We have proven that the pdf of the n-th best channel out of N channels can be accurately approximated by a weighted sum of lognormal pdf's and gamma pdf's in the cases of lognormal and gamma-gamma fading channels, respectively. At a next time, the proposed novel approximations were used to derive tractable and accurate BER expressions that contribute to shedding more light on the performance of MD FSO systems under weak and strong atmospheric turbulence conditions. The derived BERs are more compact compared to the conventional power series representations and they are accurate over a very wide SNR range.

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