

## **From Physical Model To Proof For Understanding Via DGS: Interplay Among Environments**

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### **Abstract**

The widespread use of Dynamic Geometry Software (DGS) is raising many interesting questions and discussions as to the necessity, usefulness and meaning of proof in school mathematics. With these questions in mind, a didactical sequence on the topic “Conics” was developed in a teacher education course tailored for pre-service secondary math methods course. The idea of the didactical sequence is to introduce “Conics” using a concrete manipulative approach (paper folding) then an explorative DGS-based construction activity embedding the need for a proof. For that purpose, the DGS software serves as an intermediary tool, used to bridge the gap between the physical model and the formal symbolic system of proof. The paper will present an analysis of participants’ geometric thinking strategies, featuring proof as an embedded process in geometric construction situations.

### **Introduction**

Mathematical proof has been a focus of reflection throughout the various stages that mathematics education has undergone. With each new wave of school mathematics the place, status, importance and even format of proof were subject to debates (Hanna, 2000; Hanna & Jahnke, 1996). For many years now, the widespread use of Dynamic Geometry Software (DGS) has been raising many new interesting questions. Laborde (2000) discussed major concerns identified in the literature, which she condensed in one disturbing question: “Is proof activity in danger with the use of dynamic geometry systems?” An extensive body of discussions and debates is found as to the necessity, usefulness, meaning, and types of proof in school mathematics (e.g. Arzarello, Micheletti, 1998; Christou, Mousoulides, Pittalis & Pitta-Pantazi, 2004; Hoyles & Healy, 1999; Leung & Lopez-Real, 2002), particularly when it is so easy to move elements of a geometric figure and observe many examples that support a certain conjecture. Pandiscio (2002) reports that pre-service teachers expressed their concern that with the use of DGS, students might believe that formal proofs are unnecessary; although they still believed that a formal proof is different from a proof by many examples. Others contend, on the other hand, that DGS can be used to help students see the need for deductive reasoning (Wares, 2004).

Since most of the frames of thought agree to the fact that proof is at the heart of mathematics (Knuth, 2002), the literature in math education is witnessing, since the 1990s (Hoyles & Healy, 1999, Jones, 2000; Laborde, 2000; Mariotti, 2000), numerous attempts to investigate and engineer teaching strategies and learning situations whereby DGS is used for enhancing proving abilities. “Quasi-empirical investigations” (Connor & Moss, 2007; de Villiers, 2004) are acquiring more and more importance, highlighting functions of proof that were traditionally undermined. Examples of such functions are explanation, understanding, insight, validation and discovery. These non-deductive methods of investigation, which rely on experimental, intuitive and inductive reasoning (de Villiers, 2004), are seen to provide more meaningful contexts for teaching-learning geometry with DGS than the classical approach of proof as a way of obtaining certainty. According to de Villiers (2007), the latter approach “stems largely from a narrow formalist view that the only function of proof is the verification of the correctness of mathematical statements”.

The present paper was motivated by the following assumptions: a) the development and widespread of DGS use are changing the way geometry is taught and learned, b) the tools available to geometry teachers and students in the classroom affect the nature of geometry in school math curricula as well as its teaching approaches, and c) learning does not happen automatically when students use DGS. Tasks should be reflectively designed to incite knowledge

construction (Laborde, 2001). This paper reports a study attempting to contribute to the discussions about proof in Dynamic Geometry Environments (DGEs) and to suggest a type of situations whereby proof emerges naturally to fulfill a need felt by the learner, instead of being required by an external authority (the teacher or the curriculum). The experimented problem-situation would raise a genuine need for proof as a way to foster understanding of unforeseen mathematical relationships. The idea of the situation is based on a belief that in a DGE, construction tasks are full-fledged problem solving situations in which instances of proof are needed while the construction process is taking place. A special feature of the suggested situation is that proving is not explicitly required per se, but the need for a proof emerges naturally while participants try to find a point with specific properties.

### **Context**

Not long ago, geometry in the Lebanese curriculum was for a long time taught in an abstract way. Most geometric objects were introduced with formal definitions followed by properties and theorems generated deductively. Very little attention was given to intuition, perception, or figure construction. Particularly the topic *Conics* was taught with an algebraic approach, with almost no geometrical connection at the beginning. The very first contact that students have with the topic is made through the following definition at the beginning of the chapter in the math textbook:

A fixed straight line (D) and a fixed point F being given in a plane, we call *conic of focus F and directrix (D)* the set of the points M of the plane, the ratio of whose distances from F and from (D) is equal to a given positive number e. The number e is called the *eccentricity* of the conic.

Beside the complexity of language and unfamiliar phrases used in this definition (e.g. the ratio of whose distances from F and from (D)), four new terms (conic, focus, directrix and eccentricity) are introduced for the first time. No visual representation of parabola, ellipse or hyperbola is shown until the fifth page of the chapter. After the definition, the ratio relationship is quickly transformed into a complex algebraic equation involving two variables x and y, coordinates of point M in a presumed coordinate system, and a parameter e named eccentricity.

Once the three types of conics are distinguished (by name and not by shape) depending on the different values of eccentricity, their algebraic equations get treated separately, as three independent entities, with no geometric connection between them. The geometrical definitions and properties of the three types of conics are only presented at a later stage, after the long algebraic work.

### **The study**

A didactical sequence on the topic “Conics” was developed in a teacher education course tailored for pre-service secondary math methods course. Many of the teachers participating in this course have learned geometry under the old curriculum, in an abstract and formal way. The sequence is designed to put them in a learning situation fostering metacognitive dialogues. It is composed of several problem-situations, connected in a way to gradually offer new tools to promote learning. In the limited scope of this paper, I will only present two phases of the sequence:

Phase 1: Producing conics through paper folding (See Fig.1)

Task1: Having, on wax paper, a straight line and a point not on the line, make many folds of the paper, by overlapping the point with random points of the line. Describe the result.

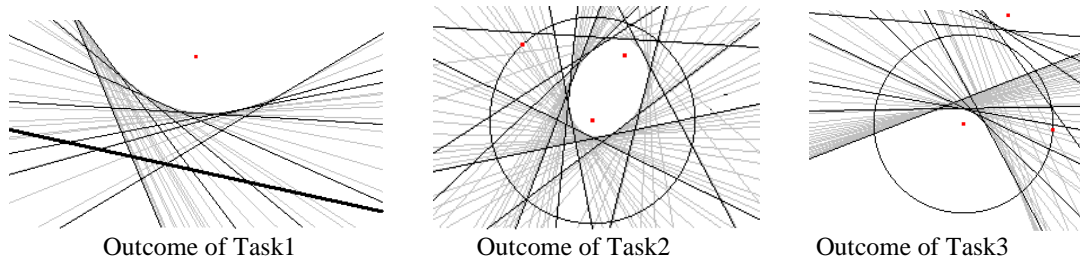
Task2: Having, on wax paper, a circle and a point inside the circle, make many folds of the paper, by overlapping the point with random points of the circle. Describe the result.

Task3: Having, on wax paper, a circle and a point outside the circle, make many folds of the paper, by overlapping the point with arbitrary points of the circle. Describe the result.

Phase 2: DGS model of the paper folding tasks

Construct using Cabri-Géomètre a model of the paper folding activity. In this DGS model, the three conics are just visually perceived (envelopes of the sets of straight lines) but are not actually

constructed (drawn). The task is to physically construct the conics by finding, constructing and tracing the point of tangency between the presumed conic and the moving straight line.



**Fig.1.** The outcomes of the three paper folding tasks

### Method

Data from participants' work were collected over six semesters (an average of six student teachers enrolled every semester). The aim was to investigate participants' thinking and proving strategies and to identify what they consider to be an acceptable, sufficient, plausible proof.

Following are the data collection techniques:

- Observation: Observation notes were taken during participants' work.
- Audio taping: Working in pairs or in groups of three, the participants were encouraged to discuss their reasoning and think aloud about their strategies. These dialogues were audio-recorded, transcribed and analyzed.
- Record of computer files: Participants' work on the computer was saved every few minutes in different consecutive Cabri files, which allowed a follow-up of the construction and proving attempts, through a follow-up of their figure constructions and manipulations.

### Some results

Within the scope of this paper I will present some of the global reflections and conclusions that the empirical data raised. The above problem situations evolved into an alternative use of proof and path to it, in a dynamic geometry environment. While DGS is most commonly used for exploring geometric figures, formulating conjectures, verifying conjectures or properties, the present problem situations propose a different context. DGS's function here is not to create or to confirm a conviction about a geometric property or relationship. Such conviction was already created by a different artifact, the paper folding model. It is this latter model which has led to the conviction that, by considering the family of perpendicular bisectors of a moving segment, a conic (or a silhouette of it) is created. This compelling observation raised the need for a proof that would explain "why" the three forms are generated.

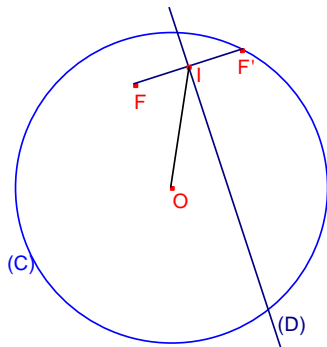
The status of the proof in this situation is rather explanatory, aiming at connecting the concrete model to the abstract properties of the geometrical figures and at understanding "why", by the same paper folding process on three figures, one can generate those three geometric objects that participants studied previously as geometrically separate and different. Without being asked, participants started looking for an explanation, trying to connect the concrete model to what they knew about the more formal properties of the geometric objects involved: the parabola as the set of points equidistant from a point and a straight line, the ellipse as the set of points whose sum of distances from two fixed points is constant, and the hyperbola as the set of points whose difference of distances from two fixed points is constant.

As the connection turned to be a too complex task, the second problem was proposed, whereby Cabri is used to bridge the gap between the physical model and the formal symbolic system of geometric proof. The function of DGS here is to explore and facilitate the explanation of an uprising, obviously valid fact, rather than to verify or validate a less compelling observed one. Thus the situation shifted from the more common use: *drag to find a pattern, state a conjecture and verify it*, to: *we know the result to be true from concrete experimental investigation. Let us try*

to explain why it is true in terms of other well-known geometric properties; in other words, how it is a logical consequence of these other properties.

The record of observations revealed the progress toward deductive proof through the processes of interplay between three models of the same problem: At a first stage, between the paper folding model and the DGS model, and at a later stage between DGS and paper-and-pencil model, in a proof-for-understanding situation. An interesting phenomenon was observed by which, after hasty and random manipulations of the DGS geometric figures, experimentation gradually became more rational. Participants started thinking about the possibility of a point to be the required one before moving it to check its trace. They tried to deductively check the validity of the constant sum property before venturing to move the point. Most groups resorted to sketching the figure or parts of it on paper while looking for a point with the required property. For most of the groups, the final solution of the problem was achieved when they worked on paper, after extensive explorations with the DGS figure.

In the case of the ellipse, for example, a group of participants started dragging and tracing the perpendicular bisector in an attempt to visualize the possible position of the point of tangency. For them, point I seemed to be the solution (see Fig.2).

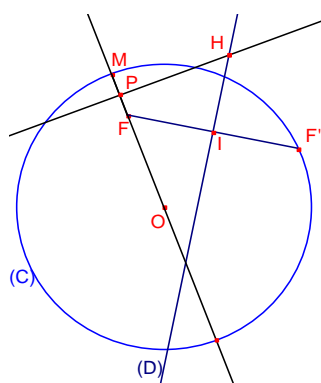


**Fig. 2.** Attempt to prove that  $IO + IF$  constant

Despite the fact that dragging and tracing the point did not yield an ellipse, participants still tried to deductively investigate whether it satisfies the constant sum property. “No, it doesn’t work... We want to find something constant... a constant sum... I think it should be the radius. Can we prove that  $IO + IF = R$ ?”

This reflects interplay between perceptive, empirical and deductive evidence, in the context of DGS. The empirical evidence was undeniable, point I does not generate the ellipse. But the perceptive impression, and the way the three points F, I and O are relatively positioned, incited the participants to look for a formal relationship that would make point I the searched point, despite empirical evidence.

This is an example that goes against common cases where learners don’t see the relevance of verbal deductive proofs because perceptive and empirical evidence is to them enough of a conviction. Participants were convinced that if they succeed to deductively prove that point I satisfies the constant sum property, they would have a more valid reason than empirical evidence to say that I is the point of tangency.



**Fig.3.** Many points were dragged

Then they tried to select several “plausibly” selected points and trace them to check if they produce the ellipse. Those selected points are mainly points of intersection between significant objects in the figure: I, midpoint of  $FF'$ , P, midpoint of FM, M, intersection of OF with the circle, H, intersection of the respective perpendicular bisectors of  $FF'$  and FM (see Fig.3), and other points created by the participants through joining points, extending segments, constructing other perpendicular bisectors, then considering intersections.

Throughout the above exploratory stage, interesting instances of attempting deductive proof started to appear, then to take more and more place in the process, as the figure became more complex and the exploration more tedious.

## Conclusion

The dynamic geometry tool provided, in the analyzed situation, a mid-way representation of the problem, between the physical model, which provided the convincing evidence, and the more abstract deductive thinking (proof). Insight and progress toward deductive proof (finally conducted on a figure sketched on paper and not on the DGS figure) were fostered by two roles of the DGS software:

- An active positive role, by which manipulation of the figure leads to better understanding the geometric relationships
- A passive negative role, by which the DGS figures act as an obstacle that learners should overcome by resorting to deductive proof. Indeed, creating new points and objects, moving basic elements, “messaging-up” with the figure, make it so complex, “fluid” and “evasive” that learners would need to sketch what they consider relevant parts of the figure on a solid support, on paper.

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