On the Steiner Walk Problem

by

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To my beloved grandparents

Vladimir and Tamara

Your Memory fills me with strength and drives me forward.

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Abstract

Given a Euclidean graph G = (V, E) with a special set of vertices, called the terminals, the Euclidean Steiner Walk problem asks for a walk through the vertices of the graph that starts and ends at a particular point, called the center, such that every terminal vertex is visited and the total cost of the walk (or distance) is minimized.

Despite its wealth of real applications, Euclidean Steiner Walk (ESW) seems to have been neglected thus far by the computing community. We show, easily, that the problem is NP-Complete and present a suite of heuristic approaches for computing efficient approximate solutions to the problem. A Graphical User Interface is also provided for users of our software and to allow developers to conduct further experimental work on the problem.

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Chapter 1

Introduction

This chapter introduces the Steiner Walk Problem, its applications and a number of related problems.

1.1 The Steiner Walk Problem

Consider the following problem: A postman needs to deliver post to specific mail boxes distributed over a known geographic area where post destinations are fixed and known. Each day, the postman has to deliver mail to a set of mail boxes. He/she would like to find an optimal tour starting from the post office, covering the needed set of addresses and then back to post office. The same problem applies to a delivery person with a potential set of clients from which only a subset needs to be served at a time.

Actually the above postman example can be transformed into a problem on a Euclidean weighted graph G = (V, E), where V is the set of vertices and E is the set of edges. In this graph, each mailbox in the given geographic area is represented by a vertex v. An edge e_{ij} is the road between vertex i and j with length w_{ij}. The set of mail boxes to be visited by the postman form a subset V' of V. The vertices in V' are called terminals.

Thus our postman problem can be treated as a Euclidean Steiner Walk problem. Figure 1 shows an example of a Euclidean Steiner Walk problem instance. The graph has three terminal nodes; any one of them can be the center. The problem is to find an optimal walk that contains all terminals and minimize the cost.

> **Definition 1.1** *Given a Euclidean weighted graph G = (V, E) with a set V' of marked vertices and a source node s. The Euclidean Steiner Walk problem asks to find an optimal tour that starts at s, visits all nodes in V' and ends at s. Vertices in V' are called terminals while those in V\V' are called Steiner vertices.*

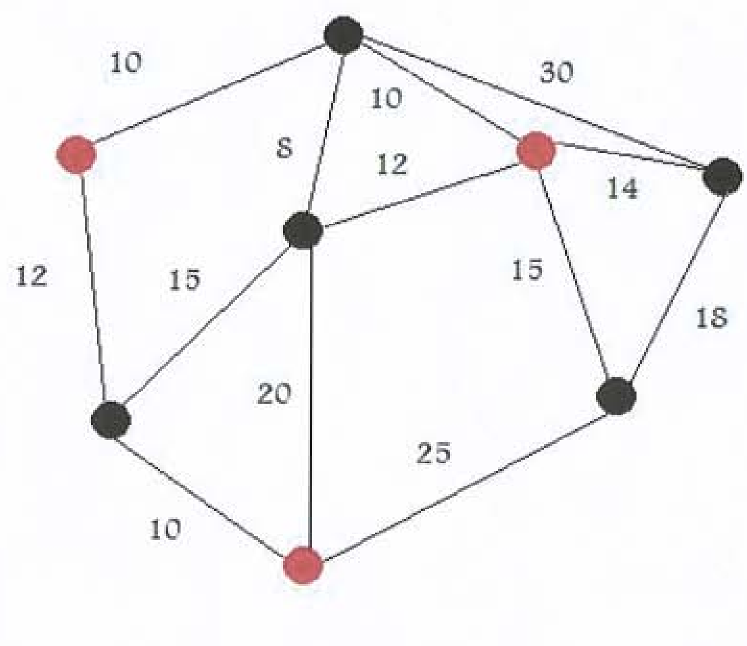

Figure 1.Graph with 3 terminal vertices

In the following section, we show that the Euclidean Steiner Walk problem is an NP-complete combinatorial optimization problem. The problem is named after the German mathematician Jacob Steiner, famous for his contribution to projective geometry. It arises in many contexts such as transportation, delivery and order picking. Despite its importance and applicability, this problem seems to have received little ot no attention from researchers so far.

1.2 NP Hardness

This section briefly presents preliminary definitions and concepts about complexity classes, and different types of algorithms. It also shows that the Euclidean Steiner walk problem is NP-hard.

1.2.1 Definitions and Notions

Problems can be broadly classified into two types; decision problems and optimization problems. A decision problem is a problem whose solution is either yes or no [Cormen, et al. 1990]. An example of decision problems is the graph coloring problem; given a graph G = (V, E) decided whether G can be colored using k colors. An optimization problem, on the other hand, is a problem where a quantity needs to be either maximized or minimized. An optimization graph coloring problem asks to color a given graph using the minimum number of colors. Every optimization problem has its decision version [Cormen, et al. 1990].

1.2.2 NP Class

P is the set of problems solvable on a deterministic sequential machine by a polynomial time algorithm, i.e. O (n^k) for some constant k where n is the input size to the problem. Even though many problems are in P, this is not the case for a large number of interesting problems. NP is the set of decision problems that have polynomial time verification algorithms, i.e. positive solutions can be verified in polynomial time. Every problem in NP can be solved in exponential time by exhaustive search. A problem is referred to be NP-complete if it is in NP and as hard as any problem in NP [Cormen, et al. 1990]. This means that every problem in NP can be reduced to a problem in NP complete class in polynomial time. Thus if any NP-complete problem can be solved in polynomial time, then every problem in NP has a polynomial time algorithm [Cormen, et al. 1990]. NP hard is the set of optimization problems whose decision version is in NP complete.

1.2.3 Proof for NP-Completeness

The travelling salesman problem (TSP) is one of the classical old and well known NP complete problems. In this section, we will use TSP to prove that Euclidean Steiner Walk problem is NP-complete. Given a Euclidean TSP instance G = (V, E), find a

Steiner Walk for that instance where all nodes are terminals; V'=V. Suppose that the Steiner walk for G has cost W_s and contains an ordered list of vertices. We can go over the list and delete any repeated vertex. The removal of repeated vertices is possible due to the Euclidean property of G, and given that G is a complete graph. Suppose that the solution has the following order of vertices:

$V_1, V_2, V_5, V_8, V_6, V_5, V_{10} \ldots V_n.$

Since the distance from V_6 to V_{10} is shorter than the distance from V_6 to V_5 and then from V_5 to V_{10}. We can safely remove V_5 in its second appearance, and shortcut the distance. Thus |cost of Optimal TSP| >= |cost of Optimal Steiner Walk|

Since by solving Euclidean Steiner Walk problem we can find the optimal travelling salesman tour, the Euclidean Steiner Walk problem is NP hard.

1.2.4 Overview of Heuristic Algorithms

Finding exact solutions for large instances of NP problems may not be always affordable. Heuristic algorithms try to find sub optimal solution in polynomial time [Kokash, 2006]. A heuristic algorithm is able to produce an acceptable solution to a problem in an acceptable time, but for which there is no formal proof of its correctness. Also, it may be correct but may not be proven to produce an optimal solution or to use reasonable resources [Wang, 2008]. There is a wide spectrum of different heuristic approaches and techniques, all try to find a near optimal solution while keeping the complexity of the algorithm within the polynomial range. Kokash (2006) and

Misevicius et al. (2004) summarize different heuristic approaches used to solve NP complete problems.

Metaheuristics algorithms combine heuristic techniques in the hope to get a better solution. Genetic algorithms (GA) are search based algorithms inspired from Darwin's theory of evolution and natural selection of species. A genetic algorithm starts with a set of solutions; each solution is represented by chromosomes and called population. Solutions from one population evolve to form a new population driven by a hope that the new solution will be better than the old one. Solutions are selected based on some fitness value. The more suitable they are the more chances they have to evolve. This is repeated until some condition is satisfied.

1.3 Applications

1.3.1 Transportation and Delivery

Many transportation problems seem to naturally map to Steiner Walk problem. The reason is pretty simple; in transportation problems we always look at optimizing a certain parameter like distance, time, costs, etc. In many transportation scenarios, we don't need to visit all nodes in the graph, but still look at optimal tour to cover the needed marked places to visit.

Consider a postman who goes to work every morning travelling from one post address to another to deliver parcels or letters in a certain assigned geographic area. It would be very convenient if the postman can get a list of address to deliver in the most optimal order. How about a delivery man who needs to deliver orders to a number of destinations distributed in his area? A musical band that is willing to go on a tour throughout the world? A salesman who travels from door to door marketing products? Would it be very efficient if one can get an ordered list of destination to minimize the total distance travelled or the total cost spent?

Consider a manufacturing company takes order to manufacture certain products. The company will need to buy supplies for an order. The order in which supplies are purchased is not important. Thus it would be convenient to have a list of stores to visit that minimize the time or distance travelled.

In the case of transportation company that wishes to find the best bus routes. A bus route has a number of stop stations where passengers wait. In would be highly suitable if the bus follows an optimal route minimizing the total distance travelled.

1.3.2 Manufacturing

Consider an assembly line machine whose purpose is to drill holes in a certain piece of material. The material could vary from a circuit board, to a frame of a vehicle or even a piece of wood to be used in building furniture. The machine contains a drill that is repositioned by special motors causing the drill to slide among the tracks enabling the drill to move to any point with the given material. It would be very efficient and time saving if the motors could reposition the drill among the given points in the optimal order.

In a similar scenario, consider a robot arm assigned to solder a number of connections on a printed circuit board. The shortest tour that visits the assigned solder points defines the most efficient path for the robot. A similar application arises in minimizing the amount of time taken by a graphics plotter to draw a given figure.

1.3.3 Networking

Consider the case of a network architect who wants to design the most efficient ring topology that will connect a number of special nodes in a network graph.

1.4 Related Problems

The Euclidean Steiner Walk Problem was not investigated in literature the way it is formulated in this thesis although one can find in literature a number of related problems, which either ask a different question or require a different solution. The following section presents three problems related to Steiner Walk Problem, mainly universal TSP, Steiner tree, and Hamiltonian problems.

1.4.1 Universal TSP

In the universal TSP problem, we are given a complete weighted undirected graph G = (V, E) where V is the set of vertices and E is the set of edges. The objective is to find a universal tour that covers all nodes in V such that for all subsets S in V, the induced sub-tours resulting from following the universal tour and skipping the vertices that are not included in the subset is optimal. In other words, the goal is to minimize the ratio of the length of the induced sub-tour over a subset of vertices S divided by the length of the optimal tour on S for all subsets S in V.

1.4.2 Steiner Tree

The Steiner tree problem asks for a minimum cost tree interconnecting a subset V' of vertices in a weighted undirected graph G = (V, E) where V is the set of vertices and E is the set of edges. The solution to the problem is called Minimal Steiner Tree and it may contain vertices in the subset V – V'.

1.4.3 Hamiltonian Problem

The Hamiltonian problem is one of the classic NP-complete problems tracing its origins back to 1850's. Hamiltonian path problem poses a question of finding a path that visits every node in a non weighted undirected graph G = (V, E) exactly once. The solution to the problem is either by finding such a path or claiming that there is no such path.

1.5 Overview and Contributions

In this thesis, we focus on heuristic approaches for the Euclidean Steiner Walk problem. The Steiner Walk problem is an NP-complete problem that has not been formulated in the way it is presented in this thesis. The problem has many real world applications especially in the field of transportation and delivery. The aim of this thesis is to define the Euclidean Steiner Walk problem, shed light on its applications and importance, present heuristic approaches for the problem and develop a user friendly tool that implements the different heuristic approaches for the Steiner walk problem.

The structure of this thesis is as follows:

Chapter 2 reviews previous works done on three related problems; mainly the Steiner tree problem and the universal travelling salesman problem. The reason behind reviewing related problems is that no previous work is found on our problem.

Chapter 3 presents four heuristic approaches to the Euclidean Steiner Walk problem.

Chapter 4 summarizes the experimental results conducted and analyses the algorithms' performance.

Finally chapter 5 draws conclusion and focuses on future work and further studies that can be done in this domain.

Chapter 3

Proposed Solutions

We present a few preliminary lemmas and remarks, and then we present our algorithmic approaches.

3.1 The Steiner Walk Problem Overview

Lemma 1: No optimal Steiner walk uses an edge more than twice.

Proof:

Consider a graph G = (V, E). Suppose that the optimal Steiner walk uses an edge more than twice.

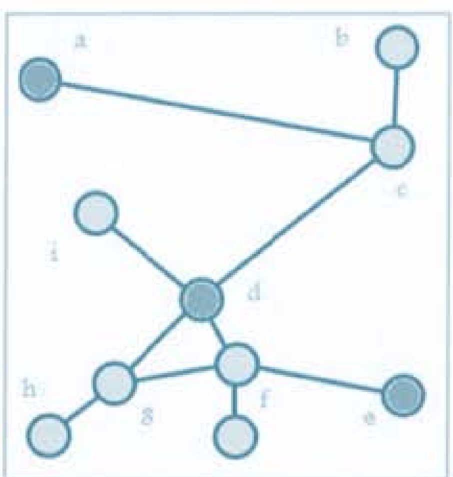

Figure 2. A graph with 3 terminal nodes

Let's say the optimal walk is a-c-d-f-e-f-e-f-d-c-a. The walk uses the edge (f, e) more than twice. We can get a smaller walk by deleting the repeated e-f pair from the above walk. If the edge (e,f) appears, but not successively, more than twice, then we distinguish two possibilities for the walk:

1. e-f-a ….x….b-e-f-c.. or
2. e-f-a ….x….b-f-e-c….y…d-e-f..

In the first case, we could replace the walk by the following: e-b …x…a-f-c. In the second, we replace the walk that covers the same vertices as follows: e-c…y…d-e-f-a…x…b-f… QED.

Thus an optimal walk cannot contain an edge more than twice (when e-f is followed, but not successively, by f-e).

Lemma 2: In any planar graph G= (V, E), number of edges <= 3|V| -6.

Proof: This is a well-known corollary of Euler's formula.

Corollary 1: In any planar graph G= (V, E), there exists a vertex v with degree <=5.

Proof: This is a well known corollary of Lemma 2.

Triangle inequality: for all sets of vertices A, B, C. the cost of travelling directly from A to B is lower than the cost of travelling from A to C and then from C to B.

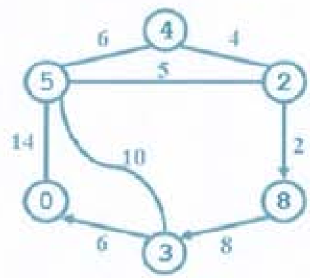

Figure 3. A graph with triangle inequality property (figure obtained from Chawla S.)

Lemma 3: Shortest paths satisfy the triangle inequality; i.e.

$\delta(s,v) \leq \delta(s,u) + w(u,v) \; for \; all \; (u,v) \in E$

Proof:

The shortest path from s to u followed by the edge (u, v) constitutes a path from s to v. The length of this path must be greater than or equal to the shortest path from s to v. Thus triangle inequality property holds true in all pairs shortest paths graph.

Lemma 4: The cost of the optimal Steiner walk is at most twice the cost of the Steiner tree.

Proof:

We can construct a Steiner walk by traversing a Steiner tree which will result in visiting each edge in the tree at most twice as shown in figure 3; thus the cost of optimal Steiner walk < 2|cost of optimal Steiner Tree|.

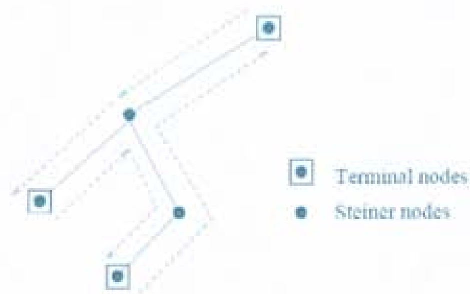

Figure 4. Constructing Steiner walk using Steiner tree (figure obtained from Chawla S.)

3.2 The Greedy Approach

The idea behind this algorithm is to try to find a Steiner walk that is a cycle, with no edge used more than once. The greedy approach constructs a shortest path matrix M for the terminals. The algorithm sorts the edges in M as pairs of vertices; each pair has a certain weight which is the weight of the corresponding edge. Then the algorithm starts with the pair having minimum weight, it then checks if adding the edge creates a cycle. The only cycle that is accepted is the one containing all terminals. In order not to get stuck in a local optimum, the algorithm iterates over all pairs, in each iteration, the algorithm select a different pair, paying no consideration to its cost as a starting pair. The algorithm will then select a Steiner walk with minimum total cost. The steps of the greedy approach are described below.

INPUT: Euclidean graph G= (V, E) and a set V' of marked vertices

OUTPUT: An ordered list V'' of vertices such that V'' contains V' and having a minimum cost.

The Greedy Approach:

STEP 1: Construct the shortest path matrix M for the terminals set. Sort the edges in M in increasing order.

STEP 2: REPEAT for all e in M

STEP 2.1: Start a new solution by adding e

STEP 2.2: Find an edge m with minimum cost

STEP 2.3: IF m does not create a cycle, THEN add m to solution

STEP 2.4: IF m creates a cycle AND there are still terminals not covered, THEN delete m.

STEP 3: IF walk found, THEN calculate cost and go to STEP 2.

STEP 4: Select a Steiner walk with minimum cost.

3.3 The MiniMAL Steiner Tree (ST) Walkthrough Approach

In this heuristic, we construct a Steiner tree T (not necessarily minimum), by iteratively adding Steiner nodes to T whenever there are terminal nodes that cannot be connected to T but through a Steiner node. Note that T will never have a Steiner leaf. A Steiner walk is then constructed by a walkthrough over the resulting ST. The walkthrough begins at a leaf (which must be a terminal node). It then traverses the ST by going to an unvisited node, when such a step is not possible anymore; the algorithm tries to find an unvisited terminal with the shortest distance to the current node. If no such terminal is found, the algorithm goes back until it finds a node that has unvisited neighbors. The steps of the algorithm are described below.

INPUT: Euclidean graph G= (V, E) and a set V' of marked vertices

OUTPUT: An ordered list V'' of vertices such that V'' contains V' and having a minimum cost.

ST_WALKTHROUGH

STEP 1: Find terminal spanning tree T.

STEP 2: Start with a leaf node s, add s to the walk.

STEP 3: Repeatedly go to an unvisited neighbor and add it to the walk.

STEP 3.1: If no unvisited neighbor is found:

STEP 3.1.1: Find an unvisited terminal t with shortest distance from the current node.

STEP 3.1.1.1: IF t is found, add t to the walk and go to STEP 3.

STEP 3.1.1.2: IF t is not found, go back to the previous node n, add n to walk and go to STEP 3.

The algorithm runs in O (n^2) time and constructs a walk whose cost is at most 2|cost of TST|.

The ST algorithm works best when traversing a Minimum Steiner Tree rather than our proposed ST. A Steiner tree algorithm gives the minimum tree spanning all terminal vertices, thus the cost of the Steiner walk resulting from the Steiner tree traversal will be better. The problem here is that Steiner tree problem is by itself an NP-hard problem. Solving the Steiner tree problem exactly requires exponential time, which is unaffordable in our case. On the other hand, solving the Steiner tree using approximation algorithms will result in double approximation, which is one level of approximation to find the Steiner tree and the second level of approximation to find the Steiner walk out of the resulted Steiner tree. Since Minimum Steiner tree has a 1.5 approximation algorithm, our ST-Walkthrouigh method yields a factor-3 approximation algorithm, mainly because a tree edge is not used more than twice.

3.4 The Traveling Salesman Approach

The travelling salesman (TSP) heuristic reduces the input graph to a TSP instance graph. Given a graph G = (V, E) with V' is the set of terminal nodes in G, we can easily obtain a complete graph G' = (V', E') where E' is the set of shortest paths between any pair of vertices in V'. Find a tour using Christofide's algorithms and then insert appropriate Steiner nodes to the walk to get a Steiner walk. Christofide's algorithm runs in O (n^3) during which an MST for the reduced graph G' is calculated, and then edges are added between MST nodes with odd degrees, as shown in figure 6. Then Euler walk is found by traversing each edge in the resulted MST.

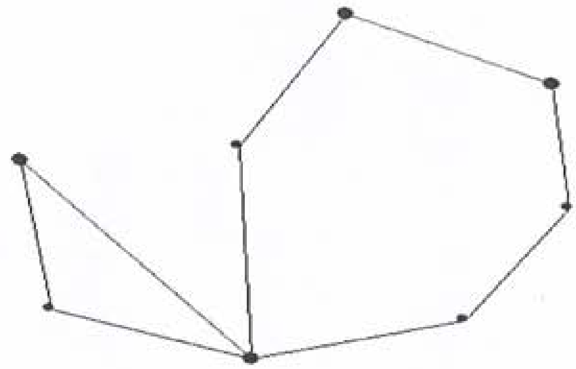

Figure 5. Adding minimum weight edges to MST

The steps of the travelling salesman approach are described below. The algorithms runs in $O(n^3)$ time.

INPUT: Euclidean graph G= (V, E) and a set V' of marked vertices

OUTPUT: An ordered list V'' of vertices such that V'' contains V' and having a minimum cost.

TSP_HEURISTIC

STEP 1: Construct a complete graph G'=(V',E') where V' is the set of all terminal vertices and E' is the set of edges and shortest paths between any pair of vertices in V'

STEP 2: Apply Christofide's Algorithm to G' to get walk W

STEP 3: For every consecutive nodes v_1 and v_2 in W, if v_1 and v_2 are not adjacent in G, insert the nodes from the shortest path from v_1 to v_2 to the walk.

3.5 The Cluster and Conquer (CC) heuristic

The Cluster and Conquer heuristic starts by clustering the input graph into a specified number of small clusters. The algorithm then finds minimum Steiner walks in each cluster and connects the optimal walks. Here are the main steps:

- Construct a complete weighted graph G' consisting of the terminal nodes as vertices, such that the weight of a edge uv is the length of shortest path between terminals u and v in G.
- Find a minimum spanning tree in G'.
- Cluster the graph G:
  - Repeatedly select the minimum edge e_{ij}:
    - If vertices i and j are not clustered, create a new cluster and add i and j to the cluster.
    - If either one of the vertices i, j is clustered in a cluster c:
      - If c is not full, add the other vertex to c.
      - Else create a new cluster, add the other vertex to that cluster and store vertex e_{ij} in a vector of marked edges.
      - Delete e_{ij}

. The cluster size is specified by a strict upper bound. The reason behind this is the need to keep the cluster size small for acceptable time calculation of the optimal walk in the cluster. The algorithm then connects the clusters to construct a Steiner Walk. The steps of the algorithm are described below:

INPUT: Euclidean graph G= (V, E) and a set V' of marked vertices

OUTPUT: An ordered list V'' of vertices such that V'' contains V' and having a minimum cost.

CLUSTER_CONQUER

STEP 1: Cluster the graph into clusters with specific cluster size k.

STEP 2: Find minimum walk in each cluster.

STEP 3: Connect minimum walks in all clusters to get the Steiner walk.

Chapter 4

Experimental Results

A graphical user interface (GUI) tool is developed implementing the different proposed approaches. The tool is written in visual C#. We used Gengraph tool to generate random planar graphs. Since Gengraph generates graphML (graph markup language) files, our tool allows for the conversion from graphML to text file in the format needed. The tool also allows for the conversion from TSPLIB files and text files since many geographic graphs and maps are represented in the TSPLIB format. Appendix A shows snapshots of the developed tool.

To the best of our knowledge, there is no such tool or algorithm that finds Steiner Walk in Euclidean graphs, so we are not able to compare our results to some previous work. This section presents experimental results and compares the performance of the different algorithmic approaches presented in Chapter 3.

To test the efficiency of our algorithms, we computed the optimal Steiner Walk for small graph instances and compared the results with our proposed heuristics. The performance of the heuristics was within the factor 2 of the optimal for graphs of order less than 50.

The experiments were performed on a Pentium IV 2.00 GHz machine with 3.00 GB of RAM. Table 1 shows a summary of the results. The detailed experimental results are presented in Appendix B.

Table 1. Summary of the experimental results showing the cost of Steiner walk using the different proposed approaches

Graph		terminals	No. of test cases	TST wins	TSP wins	CC wins	PM wins
n	e						
200	584	10	18	1	0	4	13
200	586	20	19	0	0	8	11
200	586	30	10	0	0	3	7
200	586	40	5	0	0	2	3
200	586	50	9	0	0	7	2
300	884	30	30	0	0	12	18
300	884	40	30	0	0	16	14
300	884	50	30	0	0	25	5
300	884	60	30	0	0	25	5
300	884	70	20	0	0	15	5
300	884	80	20	0	0	17	3
300	884	90	20	0	0	18	2
400	1000	40	7	0	0	3	4
400	1000	50	5	0	0	1	4
400	1000	60	8	0	0	2	6
400	1000	70	5	0	0	4	1
400	1000	80	5	0	0	4	1
400	1179	40	10	0	0	9	1
400	1179	50	10	0	0	9	1
400	1179	60	10	0	0	8	2
400	1179	70	10	0	0	8	2
400	1179	80	10	0	0	8	2
400	1179	90	10	0	0	10	0
400	1184	40	5	0	0	3	2
400	1184	50	5	0	0	5	0
400	1184	60	5	0	0	5	0
400	1184	70	5	0	0	5	0
400	1184	80	5	0	0	3	2
400	1184	90	5	0	0	5	0
400	1178	40	5	0	0	4	1
400	1178	50	5	0	0	3	2

400	**1178**	**60**	**5**	**0**	**0**	**5**	**0**
400	**1178**	**70**	**5**	**0**	**0**	**4**	**1**
400	**1178**	**80**	**5**	**0**	**0**	**5**	**0**
400	**1178**	**90**	**5**	**0**	**0**	**4**	**1**

In most of the test cases, cluster and conquer approach outperformed the other three approaches. We have noticed that as the graph increases in size, Cluster and Conquer approach yields better results than its peers. Even though, in the above table, it may seem that the first two approaches have worse performance. We could not generalize this conclusion, since the results show that when the best cost is given by PM approach, it does not necessarily imply that the next to best cost is given by CC approach. In many cases, TSP approach gives next to best cost. Terminal spanning tree approach does not give optimal Steiner walks on large graph instances. However it gives good results on small graphs that have nodes less than 100. The terminal spanning tree approach gives better Steiner walks than TSP in some cases especially when the graph size is small and the terminals set is small.

The first approach (TST) runs faster than the other three approaches while the Cluster and Conquer approach takes the longest running time. The time taken by each experiment is recorded in Appendix B.

Figures 7 to 9, shows the number of wins of each proposed heuristics as the terminal size changes. Each figure use graphs of fixed order. Figures 10 to 12 show the average run times of the different heuristics on different terminal size and different graph orders.

The time reported is system time and is recorded in milliseconds. The TST and the TSP heuristics run much faster than the other two approaches. The C&C approach is the slowest and its run time lies within a bounded range for graphs of same size. With graphs of order 200, the C&C algorithm runs in 4 to 5 seconds, while with graphs of order 300, the C&C algorithm runs in 25 to 30 seconds. The average run times of the greedy approach increase as the terminal size increases. With graphs of order 300, as an example, the average run times of the greedy approach varies from less than 2 seconds for 30 terminals to about 25 seconds for 90 terminals.

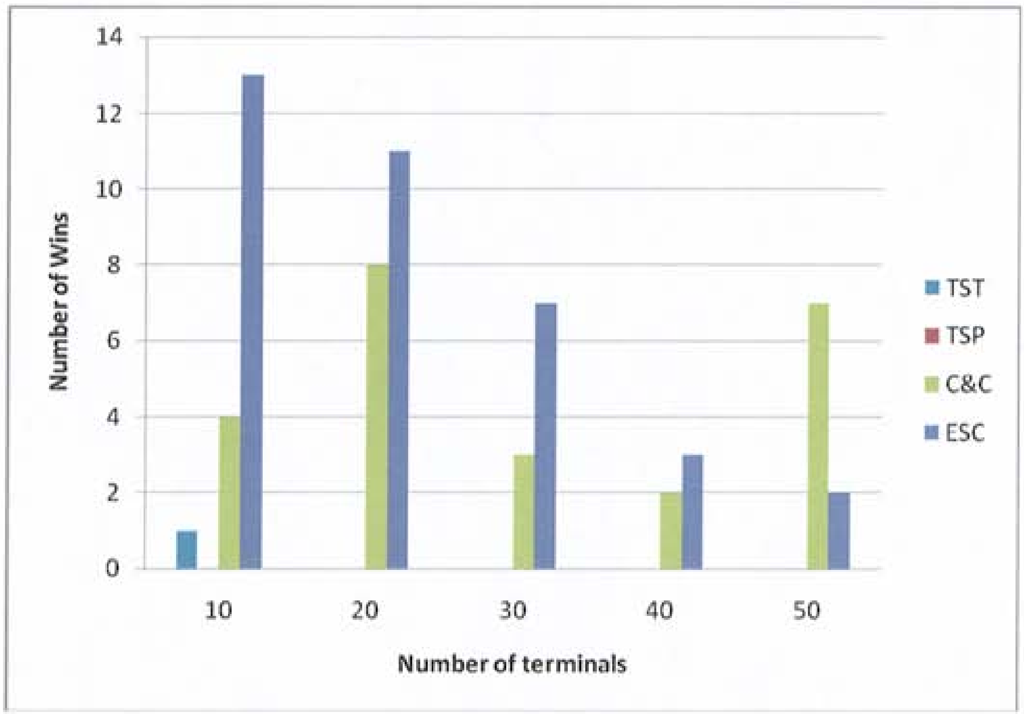

Figure 6. Number of Wins in graphs of order 200

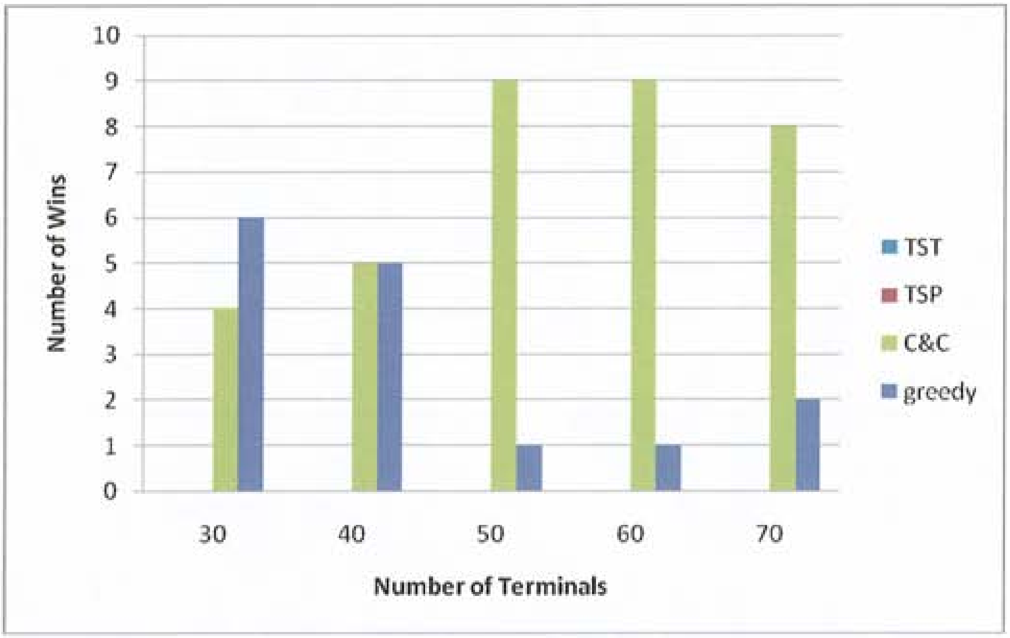

Figure 7. Number of Wins in graphs of order 300

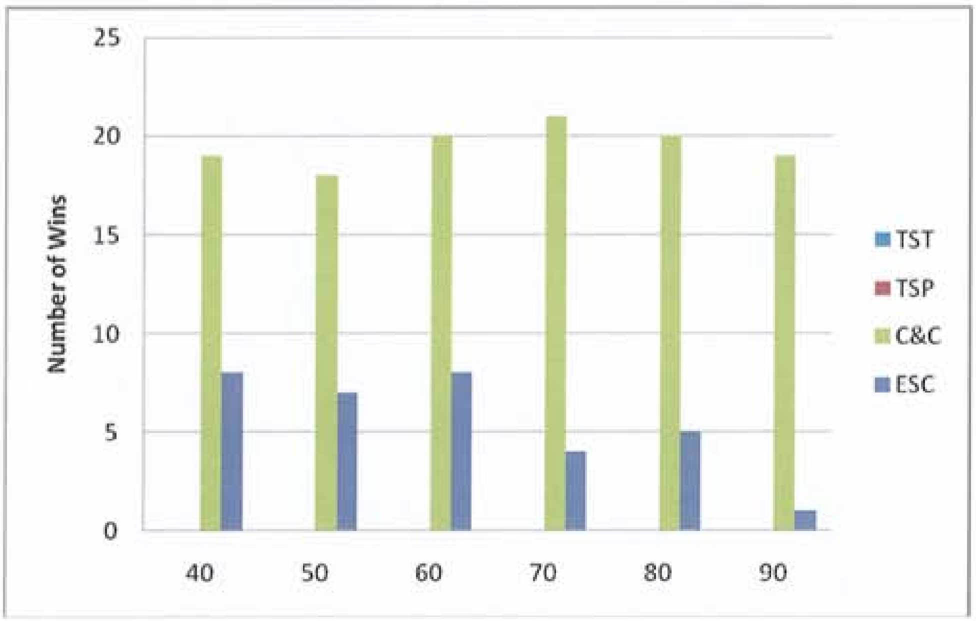

Figure 8. Number of wins in graphs of size 400 nodes

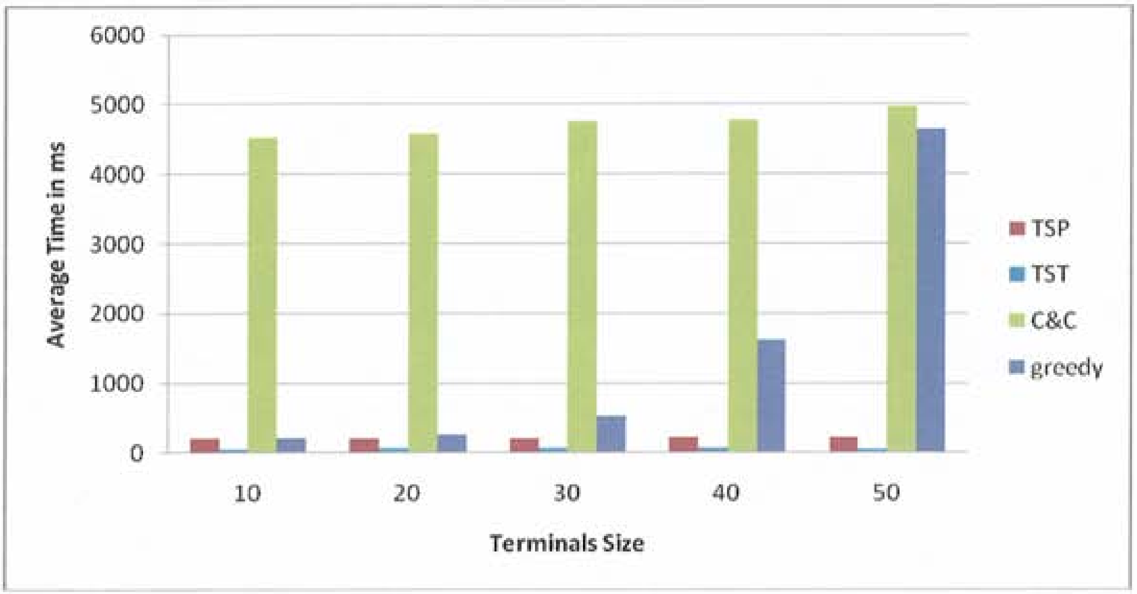

Figure 9. Average run time for graphs of order 200

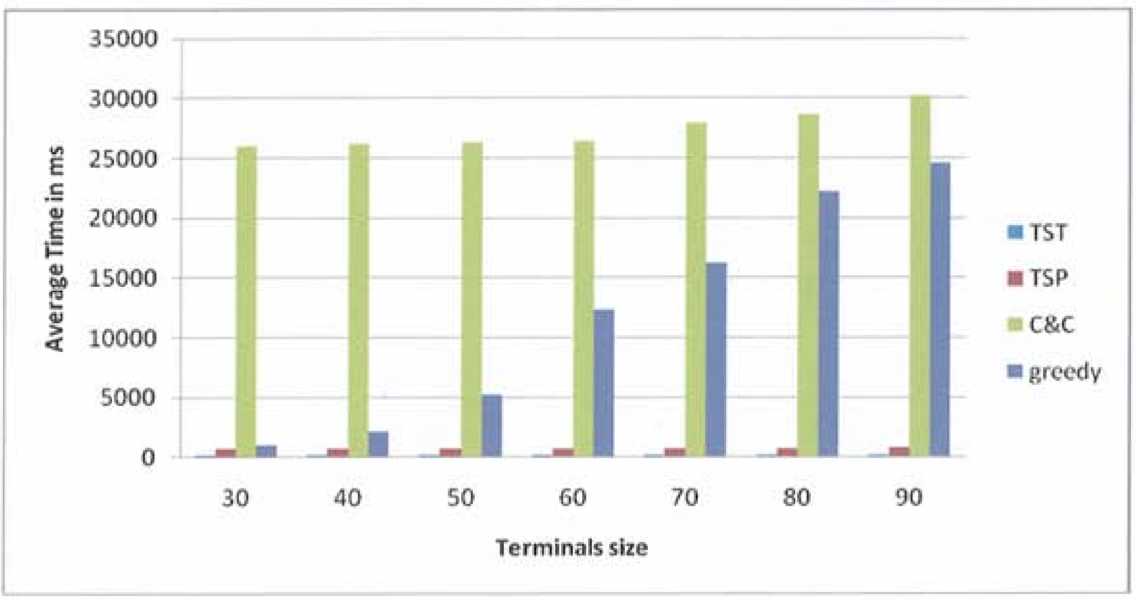

Figure 10. Average run time for graphs of order 300

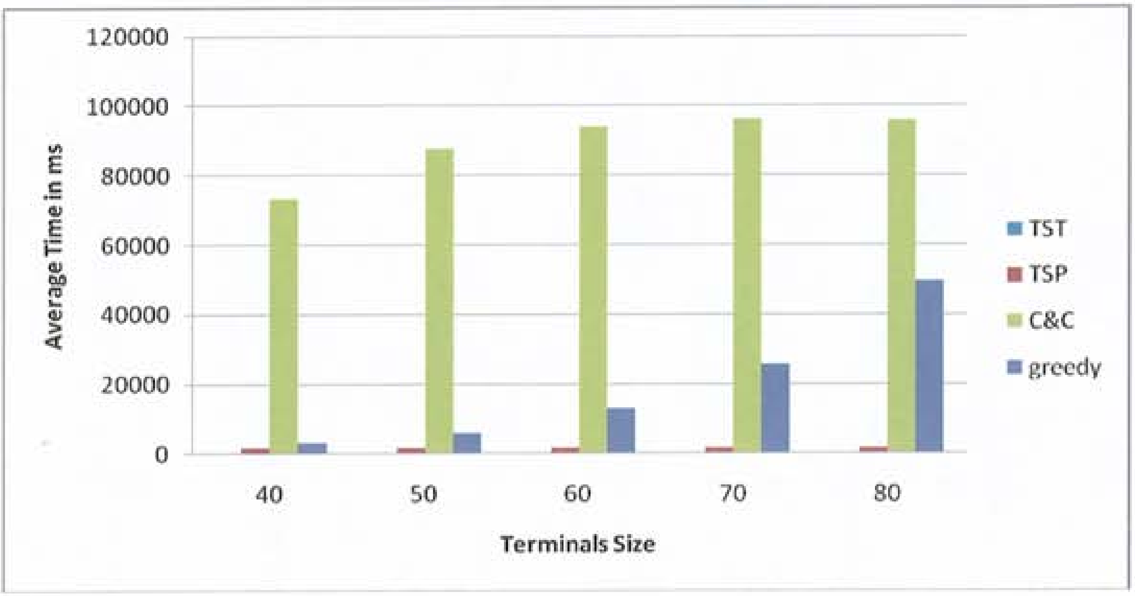

Figure 11. Average run time for graphs of order 400

Chapter 5

Conclusion and Future Work

In this thesis, we introduced an interesting problem that has not received attention from the computer science community thus far. The Euclidean Steiner Walk problem naturally arises in different practical applications. We defined the problem, proved that it is NP complete and presented four different heuristic approaches to the problem. We also developed a tool that allows users to find Steiner walks using any of the proposed approaches. Experimental results are summarized and compared.

A lot of work can be done in the future. One main track is to design exact algorithms for the Euclidean Steiner Walk problem. Another track is to design fixed parameter algorithms for the problem, which seems to be a good candidate for parameterized complexity. In this thesis we restricted our attention to the Euclidean version of Steiner Walk. We believe the general problem is also very important in many practical settings.

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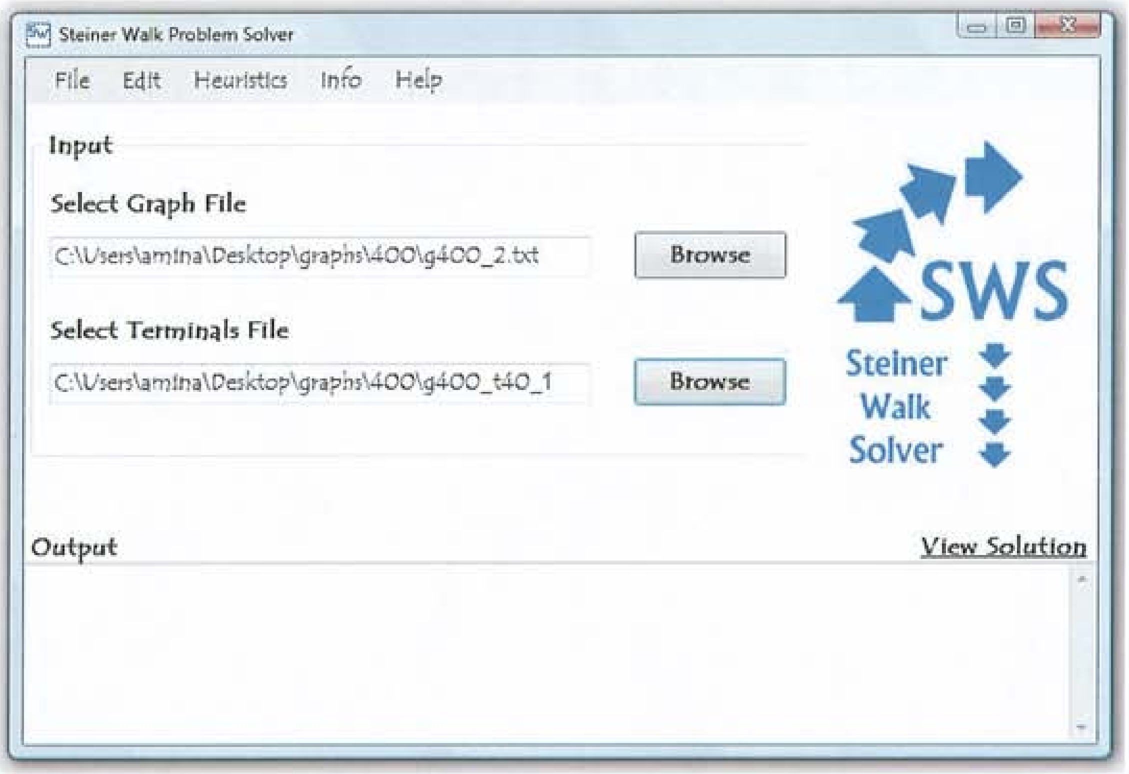
Appendix A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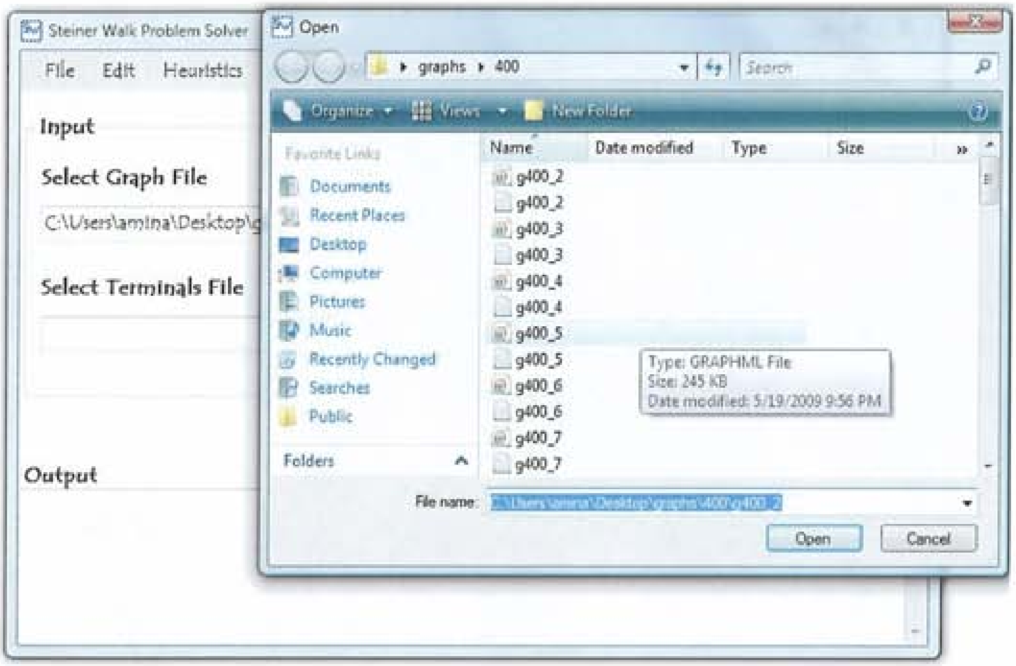
File Edit Heuristics Info Help
Input
Select Graph File
C:\Users\amina\Desktop\graphs\400\g400_2.txt
Browse
Select Terminals File
C:\Users\amina\Desktop\graphs\400\g400_t40_1
Browse
SWS
Steiner
Walk
Solver
Output
View Solution


Steiner Walk Problem Solver
File Edit Heuristics
Input
Select Graph File
Select Terminals File
Output
Open
graphs ▸ 400
Search
Organize
Views
New Folder
Favorite Links
Documents
Recent Places
Desktop
Computer
Pictures
Music
Recently Changed
Searches
Public
Folders
Name
Date modified
Type
Size
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g400_5
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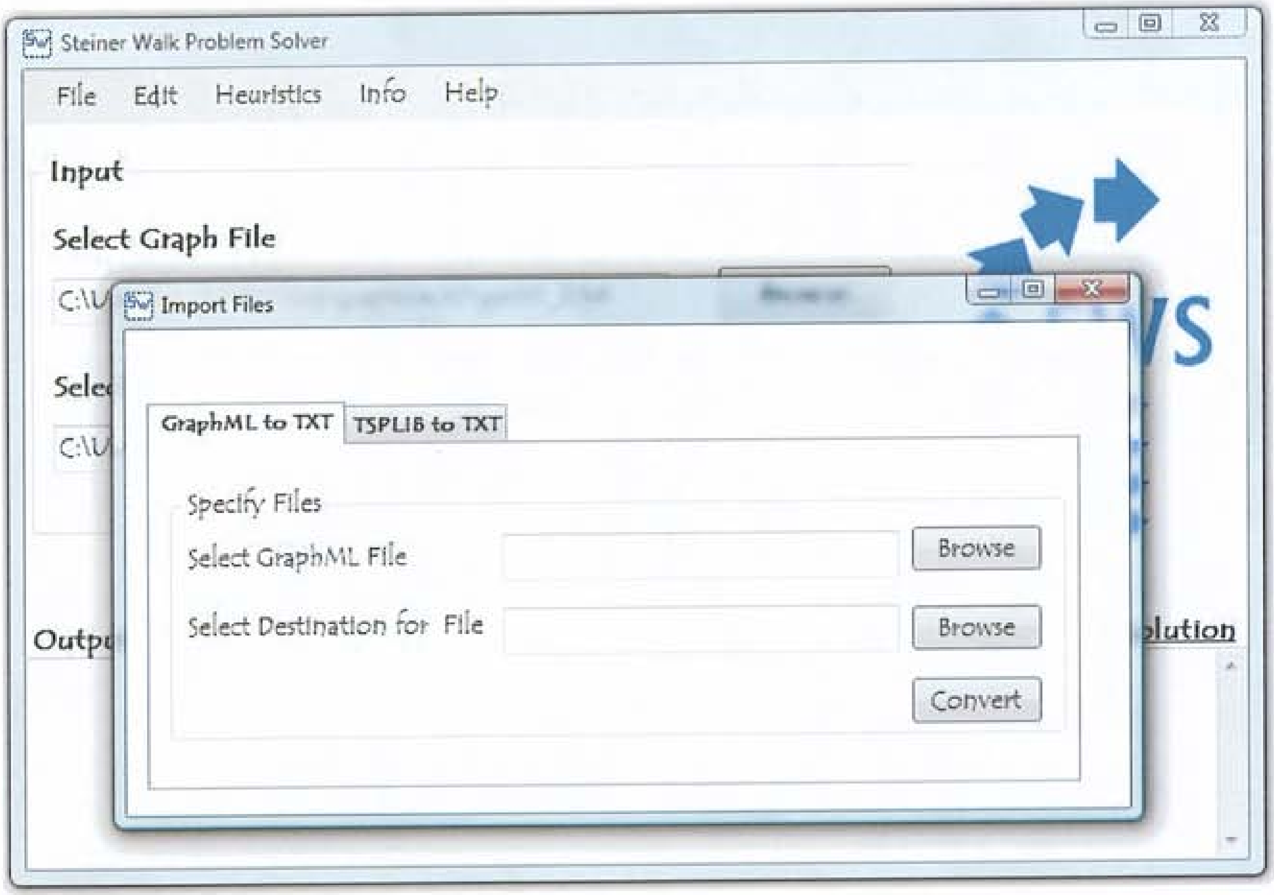
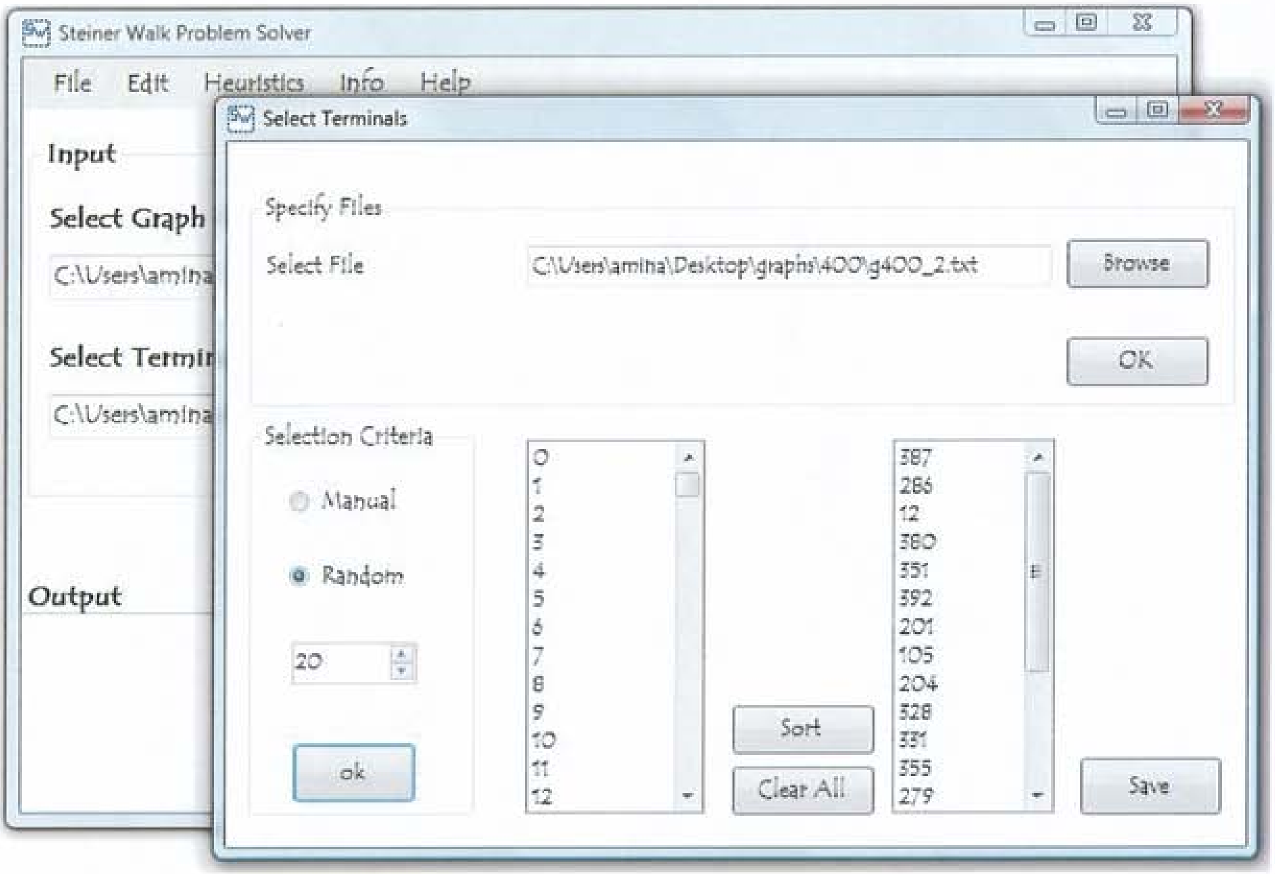
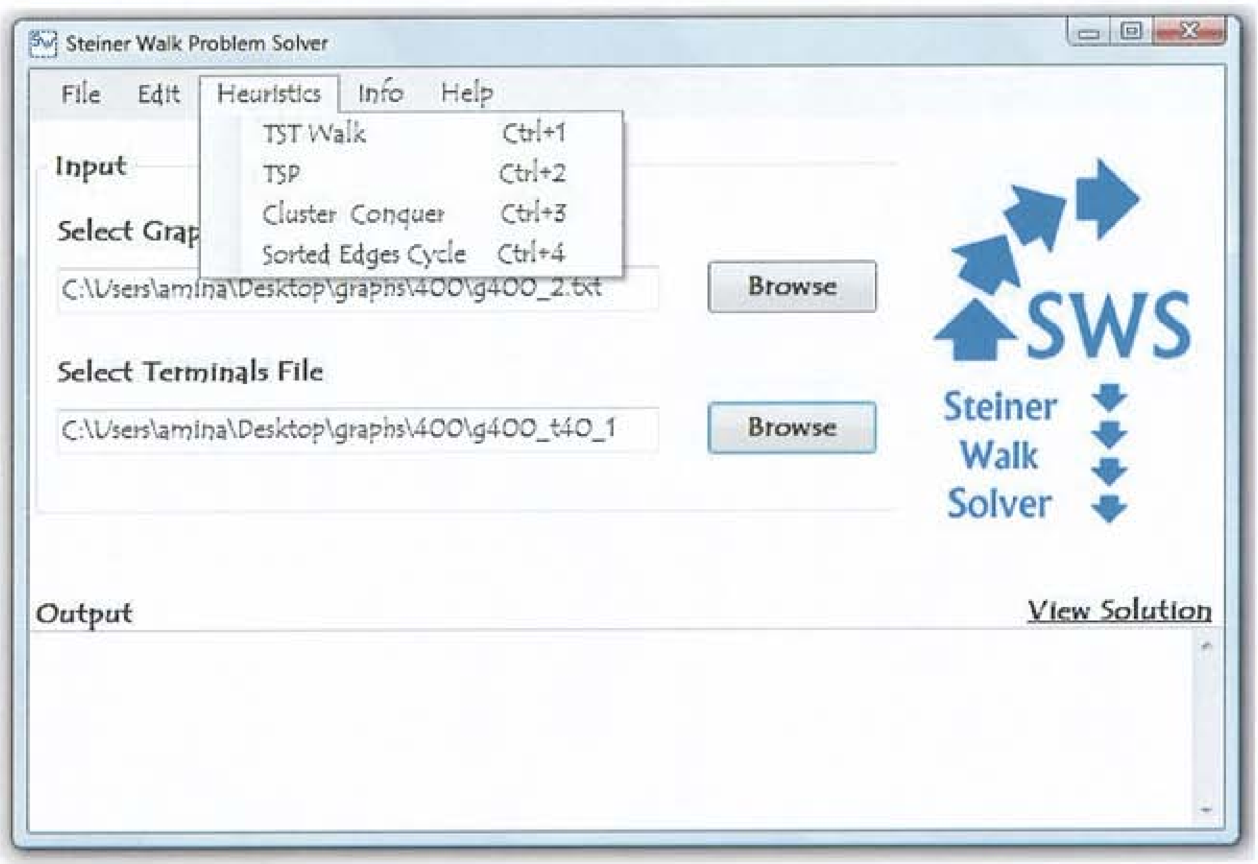
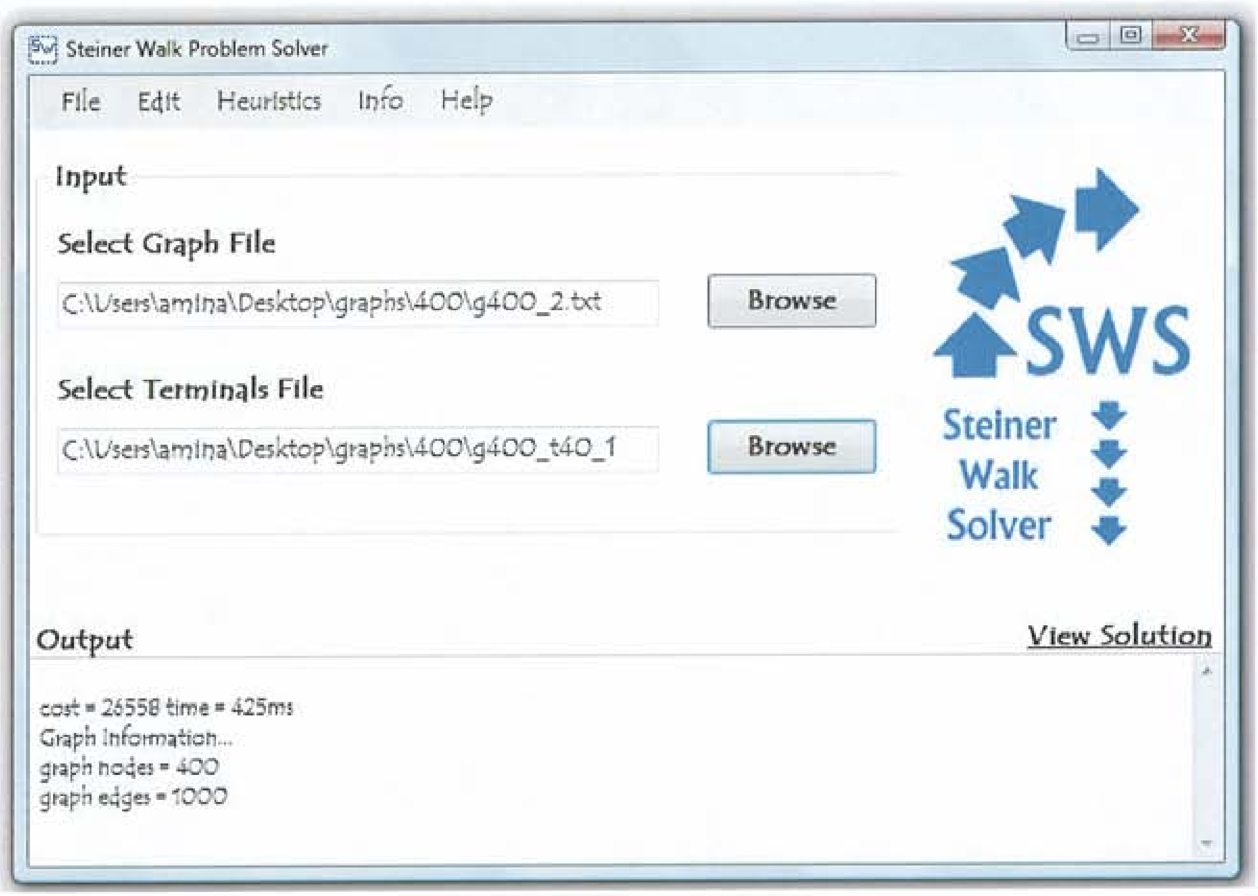
Open
Cancel


Steiner Walk Problem Solver
File
Edit
Heuristics
Info
Help
Input
Select Graph File
Import Files
GraphML to TXT
TSPLIB to TXT
Specify Files
Select GraphML File
Browse
Select Destination for File
Browse
Convert


Steiner Walk Problem Solver
File Edit Heuristics Info Help
Input
Select Graph
C:\Users\amina
Select Termi
C:\Users\amina
Output
Select Terminals
Specify Files
Select File
C:\Users\amina\Desktop\graphs\400\g400_2.txt
Browse
OK
Selection Criteria
Manual
Random
20
ok
0
1
2
3
4
5
6
7
8
9
10
11
12
Sort
Clear All
387
286
12
380
351
392
201
105
204
328
331
355
279
Save


Steiner Walk Problem Solver
File
Edit
Heuristics
Info
Help
TST Walk Ctrl+1
TSP Ctrl+2
Cluster Conquer Ctrl+3
Sorted Edges Cycle Ctrl+4
Input
Select Grap
C:\Users\amina\Desktop\graphs\400\g400_2.txt
Browse
Select Terminals File
C:\Users\amina\Desktop\graphs\400\g400_t40_1
Browse
SWS
Steiner
Walk
Solver
Output
View Solution


Steiner Walk Problem Solver
File Edit Heuristics Info Help
Input
Select Graph File
C:\Users\amina\Desktop\graphs\400\g400_2.txt
Browse
Select Terminals File
C:\Users\amina\Desktop\graphs\400\g400_t40_1
Browse
SWS
Steiner
Walk
Solver
Output
View Solution
cost = 26558 time = 425ms
Graph Information...
graph nodes = 400
graph edges = 1000

Appendix B

Graph ID	Graph Size (nodes)	No. of Terminals	Cost of Steiner walk given by			
			TST	TSP	CC	RPM
g_200_1	200	10	6707	5255	5108	5089
g_200_1	200	10	9862	8166	7344	6862
g_200_1	200	10	6841	6722	6877	6248
g_200_1	200	10	5026	5369	5188	4375
g_200_1	200	10	9472	6166	6111	5836
g_200_1	200	10	6591	5458	5061	5195
g_200_1	200	10	7485	4706	5638	4427
g_200_1	200	10	11498	8545	7287	6910
g_200_1	200	10	7295	5910	5199	5910
g_200_1	200	10	10120	7769	7022	6878
g_200_1	200	20	13477	10574	9656	8128
g_200_1	200	20	13065	8303	8498	7423
g_200_1	200	20	9833	10539	8500	7825
g_200_1	200	20	16238	10587	11122	8047
g_200_1	200	20	11762	10184	10024	8500
g_200_1	200	20	16856	12059	11009	11055
g_200_1	200	20	15463	10624	7689	8578
g_200_1	200	20	15247	8895	10227	7758
g_200_1	200	20	12865	8376	8193	7481
g_200_1	200	20	16390	11058	11042	9599
g_200_2	200	10	7091	5974	4844	5441
g_200_2	200	10	4780	5424	5415	4568
g_200_2	200	10	6129	6602	5455	4751
g_200_2	200	10	8044	6366	4272	5643

g_200_2	200	10	9996	6680	7630	5272
g_200_2	200	10	6531	5596	7078	5364
g_200_2	200	10	5999	7091	8785	6288
g_200_2	200	10	6701	6721	6468	5441
g_200_2	200	20	10140	7979	6273	8319
g_200_2	200	20	19531	10698	7465	8763
g_200_2	200	20	12732	9188	9416	7708
g_200_2	200	20	6770	7749	5435	6738
g_200_2	200	20	10767	7929	6192	7479
g_200_2	200	20	15268	9990	9080	7950
g_200_2	200	20	15852	11682	7817	9930
g_200_2	200	20	13036	10439	8067	9298
g_200_2	200	20	12117	9530	10151	8035
g_200_2	200	30	18487	12785	11992	10716
g_200_2	200	30	14553	12106	12208	10280
g_200_2	200	30	16407	13550	12026	11331
g_200_2	200	30	20078	12241	9689	10624
g_200_2	200	30	15474	11825	12869	10658
g_200_2	200	30	18908	12631	9945	10927
g_200_2	200	30	19199	11882	11155	10000
g_200_2	200	30	20509	13873	12330	11147
g_200_2	200	30	19946	12713	12872	10973
g_200_2	200	40	25217	16751	10653	13774
g_200_2	200	40	25010	15295	12456	13076
g_200_2	200	40	21152	15038	14318	12632
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g_200_2	200	50	22735	16631	10171	14708
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g_200_2	200	50	29341	18629	16340	16077
g_200_2	200	50	25947	16245	12411	14353

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g_400_2	400	70	50138	35224	30280	31701
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g_400_4	400	60	36398	21269	16693	18803
g_400_4	400	60	32777	20196	14374	17765
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g_400_4	400	60	36924	23979	14502	20982
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