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Central schemes for Ideal magnetohydrodynamics

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We present second-order accurate central finite volume methods adapted to three-dimensional ideal magnetohydrodynamics problems. These methods alternate between two staggered grids, thus leading to Riemann solver-free algorithms with relatively favorable computing times. The div·$\mathbf{B} = 0$ constraint on the magnetic field is enforced with a suitable adaptation of the constrained transport method to our central schemes. Numerical experiments show the feasibility of the proposed methods and our results are in good agreement with existing results in the recent literature.

1 Introduction

In this work we present a three-dimensional non-oscillatory central scheme along with a constrained transport method for solving ideal Magnetohydrodynamic problems. The numerical scheme we consider is an extension of our previous methods for structured Cartesian and unstructured triangular grids [1]. These methods alternate between two staggered grids, thus leading to Riemann solver-free algorithms with relatively favorable computing times. To ensure a physically admissible numerical solution, we construct a constrained transport method that treats the magnetic components in the numerical solution and guarantees the div·$\mathbf{B} = 0$ property of the magnetic field. Numerical experiments show the potential of these methods and our results are in good agreement with existing results in the literature.

2 Central schemes with a constrained transport method for ideal MHD

The conservation form of the ideal MHD equations is:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \rho e\\ \mathbf{B}\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\mathbf{v} + I(p+\frac{\mathbf{B}\cdot\mathbf{B}}{2}) - \mathbf{B}\mathbf{B}\\ (\rho e + p + \frac{\mathbf{B}\cdot\mathbf{B}}{2})\mathbf{v} - (\mathbf{v}\cdot\mathbf{B})\mathbf{B}\\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}\end{bmatrix} = 0 \tag{1}$$

Here ρ denotes the mass density, $\mathbf{v}$ is the velocity of the flow, $\mathbf{B}$ is the magnetic field, and e is the total energy per unit mass. The thermal pressure is computed from an ideal gas equation of state. The ideal MHD system (1) forms a hyperbolic system. We consider for our computational domain a uniform parallelepiped-shaped grid. Starting from the original Cartesian grid with cells $C_{i,j,k} \equiv [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}] \times [z_{k-1/2}, z_{k+1/2}]$ (solid-line cubes centered at nodes (x_i, y_j, z_k)), at time t^n, we alternate (at time t^{n+1}) to the diamond-shaped dual staggered cell $D_{i+1/2,j,k}$, $D_{i,j+1/2,k}$, and $D_{i,j,k+1/2}$ (dashed-line double pyramids cubes centered at the nodes $(x_{i+1/2}, y_j, z_k)$, $(x_i, y_{j+1/2}, z_k)$, and $(x_i, y_j, z_{k+1/2})$, respectively. The staggered dual cells we consider are obtained by joining the centers of two adjacent Cartesian cells $C_{i,j,k}$ and $C_{i+1,j,k}$ to their common interface. The solution will be computed on the original cells $C_{i,j,k}$ at time t^{n+2} as shown in Fig.1.
Usually the magnetic field in the numerical solution doesn't satisfy the divergence-free property and additional treatment is required. In this work we adapt Evans and Hawley's Constrained Transport (CT) approach [4] to the central schemes. The method consists of discretizing the induction equation $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$ on the computational cells in an appropriate way to ensure a zero divergence magnetic field. The x-component of the magnetic field on the cell $\mathcal{D}_{i+1/2,j,k}$ is treated using the following formula:

$$\mathbf{B}^{n+1,x}_{i+1/2,j,k} = \frac{1}{2}(B^{n,x}_{i,j,k} + B^{n,x}_{i+1,j,k}) - \Delta t\frac{\Omega^{n+1/2,z}_{i+1/2,j+1,k} - \Omega^{n+1/2,z}_{i+1/2,j-1,k}}{2\Delta y} + \Delta t\frac{\Omega^{n+1/2,y}_{i+1/2,j,k+1} - \Omega^{n+1/2,y}_{i+1/2,j,k-1}}{2\Delta z}, \tag{2}$$

and similar expressions for the y and z components [7].

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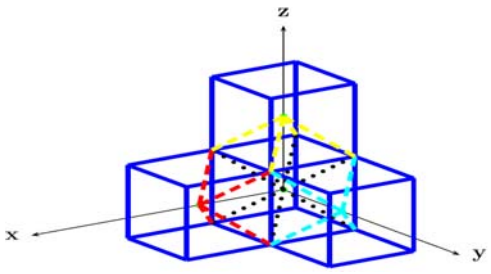

Fig. 1 Original Cartesian cells (solid-line cubes) and dual diamond cells (dashed-lines)

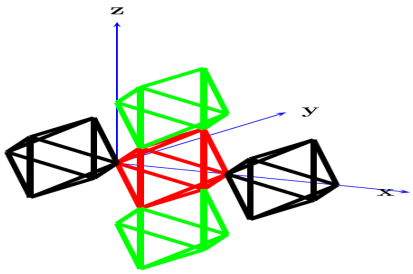

Fig. 2 Six (of the nine) dual cells (along the x and z) directions are used to compute $\nabla \cdot \mathcal{B}^{n+1}_{i+1/2,j,k}$

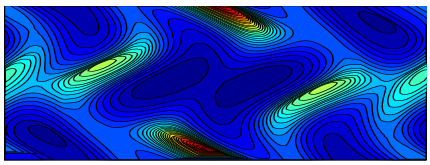

Fig. 3 Contour lines of the mass density in the plane $z = \pi$ for the Orszag-Tang problem at time t=0.5.

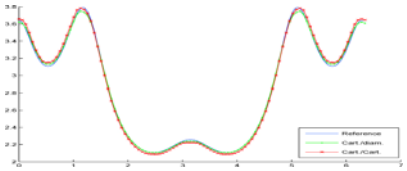

Fig. 4 Plot of the mass density for the Orszag-Tang vortex problem.

3 Numerical experiment: The Orszag-Tang problem

We present the classical three-dimensional Orszag-Tang MHD problem. The initial data for this problem are the following: $\rho(x,y,z) = \rho_0,\ p(x,y,z) = p_0, \mathbf{u}(x,y,z) = -\sin y\,\mathbf{i} + \sin x\,\mathbf{j},\ \mathbf{B}(x,y,z) = -\sin y\,\mathbf{i} + \sin(2x)\,\mathbf{j}$, with $0 \le x, y \le 2\pi$, $\rho_0 = 25/36$ and $p_0 = 5/3$. $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions.
We have computed the numerical solution on a $100 \times 100 \times 100$ grid at time t=0.5 using both Cartesian and diamond dual cell schemes along with the corresponding CTCS approach to maintain a divergence-free magnetic field. Fig.3 shows the contours of the mass density in the plane $z = \pi$; this plot compares very well with the corresponding results for the two-dimensional problem we previously considered in [2] as well as with those appearing in several recent papers in the literature [8].

Fig.4 shows two plots of the mass density along the line $y = \pi$ of the plane $z = \pi$ obtained using both Cartesian (x-line) and diamond (dotted line) dual cell schemes at time $t = 0.5$; the reference solution (solid line) is obtained using the two-dimensional diamond dual cell scheme on 400^2 gridpoints. Both methods yield almost undistinguishable results.

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