Forbidden subgraph characterization of (P_3-free, K_3-free)-colourable cographs

3 authors, including:

Faisal N. Abu-khzam
Lebanese American University
71 PUBLICATIONS **688** CITATIONS

Some of the authors of this publication are also working on these related projects:

Project Processing Big Graphs View project

Project Cluster Editing View project

Forbidden Subgraph Characterization of (P_3-free, K_3-free)-colourable Cographs

Carl Feghali[*1], Faisal N. Abu-Khzam[†2], and Haiko Müller[‡3]

[1]School of Engineering and Computing Sciences, Durham
University, Durham, U.K.
[2]Department of Computer Science and Mathematics, Lebanese
American University, Beirut, Lebanon.
[3]School of Computing, University of Leeds, Leeds, U.K.

March 26, 2014

Abstract

A (P_3-free, K_3-free)-colouring of a graph $G = (V, E)$ is a partition of $V = A \cup B$ such that $G[A]$ is P_3-free and $G[B]$ is K_3-free. This problem is known to be NP-complete even when restricted to planar graphs and perfect graphs. In this paper, we provide a finite list of 17 forbidden induced subgraphs for cographs with a (P_3-free, K_3-free)-colouring. This yields a linear time recognition algorithm.

1 Introduction

According to [3], given graph properties $\mathcal{P}$ and $\mathcal{Q}$, a $(\mathcal{P}, \mathcal{Q})$-colouring of a graph G is a partition of its vertex set into two sets A and B such that A

[*]carl.feghali@durham.ac.uk
[†]faisal.abukhzam@lau.edu.lb
[‡]h.muller@leeds.ac.uk

induces a subgraph that belongs to $\mathcal{P}$ and B induces a subgraph that belongs to $\mathcal{Q}$. A graph G is $(\mathcal{P}, \mathcal{Q})$-colourable if G admits a $(\mathcal{P}, \mathcal{Q})$-colouring.

In this paper, we investigate the computational complexity of deciding whether a graph G is (P_3-free, K_3-free)-colourable, that is, whether G admits a partition of its vertex into two sets A and B such that A induces a P_3-free graph (i.e., a disjoint union of cliques) and B induces a K_3-free graph (i.e., a graph with no triangle). This problem is known to be NP-complete on general graphs [3]. Recently, it was shown to remain NP-complete even when restricted to planar or perfect graphs [12]. We focus our attention on the class of cographs. It is known that the relation of being an induced subgraph is a well-quasi-ordering on cographs [11]. As the class of (P_3-free, K_3-free)-colourable cographs forms a subfamily of the class of cographs and is closed under induced subgraphs, it follows that (P_3-free, K_3-free)-colourable cographs have a finite list of forbidden induced subgraphs. Therefore, deciding (P_3-free, K_3-free)-colourability can be done in linear time. However, this proof of membership in P is non-constructive. In our next theorem, we provide a constructive proof.

Theorem 1.1. *A cograph G is (P_3-free, K_3-free)-colourable if and only if G does not contain the graphs $H_1, H_2, \ldots, H_{17}$ depicted in Figure 5.*

Section 2 introduces the terminology that will be used in the rest of this paper. Section 3 deals with the forbidden induced subgraph characterization of (P_3-free, K_3-free)-colourable cographs where our approach was to decompose the problem into several subproblems. Namely, subsections 3.1.1, 3.1.2, and 3.1.3 each treat one such subproblem, while Subsection 3.2 incorporates these results and proves Theorem 1.1.

2 Background

All graphs considered here are finite and have no multiple edges and no loops. For undefined graph terminology we refer the reader to Diestel [10]. Let $G = (V, E)$ be a graph and $V' \subseteq V$. The graph G' induced by deleting the vertices $V \setminus V'$ from G is denoted by $G' = G[V']$. The complement of a graph G, denoted by $\overline{G}$, has the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if they are non-adjacent in G. K_n, C_n, P_n denote a complete graph, a cycle, and a path on n vertices respectively. We say that G is H-free if it contains no subgraph isomorphic to some graph H. The graph

$G \setminus v$ is obtained from G by deleting the vertex v. We do not distinguish between isomorphic graphs. A vertex $v \in V$ is a universal vertex if for every $u \in V$ with $u \neq v$ uv is an edge. A vertex $v \in V$ is an isolated vertex if it has no neighbours, i.e. for every $u \in V$, $uv \notin E$. Clearly a vertex is universal in G if and only if it is isolated in $\overline{G}$. The join $P = G \oplus H$ of disjoint graphs G and H is such that for any $v \in V(G)$ and $u \in V(H)$, $uv \in E(P)$. The union $Q = G \cup H$ of graphs G and H is such that for any $v \in V(G)$ and $u \in V(H)$, $uv \notin E(Q)$. Given a disconnected graph G, it can be expressed as a union $G_1 \cup G_2 \cup \ldots \cup G_k$ of connected graphs. Furthermore, each G_i is said to be a (connected) component of G and each component is clearly a maximal connected subgraph of G. A graph G containing a graph H implies that H is an induced subgraph of G.

A cograph [8] (also known as complement reducible graph) is defined recursively as follows:

(i) K_1, the graph on a single vertex, is a cograph.

(ii) If $G_1, G_2, \ldots, G_k$ are cographs, then so is their union $G_1 \cup G_2 \cup \ldots \cup G_k$.

(iii) If G is a cograph, then so is its complement $\overline{G}$.

The class of P_4-free graphs is equivalent to the class of cographs [8]. It is well-known that a cograph or its complement is disconnected unless the cograph is K_1.

A graph is (s, k)-polar if there exists a partition $\{A, B\}$ of its vertex set such that A induces a union of k cliques, and B induces a join of s independent sets. A graph is monopolar if it is $(1, k)$-polar for some positive integer k. Clearly, the class of monopolar graphs forms a proper subclass of the class of (P_3-free, K_3-free)-colourable graphs. The complexity of polar and monopolar graphs has been investigated thoroughly and the reader is invited to look at [2, 4, 5, 6, 7] for some examples.

A graph G is (k, l)-partitionable if it can be partitioned in up to k cliques and l independent sets with $k + l \geq 1$. G is (∞, l)-partitionable if it can be partitioned in up to l independent sets and a union of cliques, and (k, ∞)-partitionable if it can be partitioned in up to k cliques and a join of stable sets. Table 1 contains trivial complexity results on (k, l)-partitionable problems in special classes of graphs. In [9] efficient algorithms are devised for solving the (k, l)-partition problem on cographs, where k and l are finite. In

3

k	l	graph class	recognition	forbidden cographs	forbiden others
0	1	edge-less	$\mathcal{O}(n)$	K_2	none
1	0	complete	$\mathcal{O}(n+m)$	$2K_1$	none
1	1	split	$\mathcal{O}(n+m)$	$2K_2, C_4$	C_5
0	2	bipartite	$\mathcal{O}(n+m)$	K_3	odd cycles
2	0	co-bipartite	$\mathcal{O}(n+m)$	$3K_1$	odd co-cycles

Table 1: Some trivial complexity results on (k,l)-partitionable problems

[1] a characterization of (k,l)-partitionable cographs by forbidden induced subgraphs is provided, where k and l are finite.

A P_3-free graph is a union of cliques. A $\overline{P_3}$-free graph, or equivalently a $(K_2 \cup K_1)$-free graph, is a join of stable sets. Split graphs are exactly the $(1,1)$-partitionable graphs. They are characterized by the absence of $2K_2$, C_4 and C_5. The intersection of cographs and split graphs are the threshold graphs, characterised by the absence of $2K_2$, C_4 and P_4. The diamond, paw, and butterfly graph can be written as $K_2 \oplus 2K_1$, $K_1 \oplus (K_1 \cup K_2)$ and $K_1 \oplus 2K_2$, respectively. The k-wheel graph is formed by a cycle C or order $k-1$ and a vertex not in C with $k-1$ neighbours in C. A 5-wheel can be written as $C_4 \oplus K_1$, or $P_3 \oplus 2K_1$.

Observation 2.1. *A cograph G is (P_3-free, K_3-free)-colourable if and only if G is $(\infty, 2)$-partitionable.*

Proof. It is well-known that a graph is bipartite if and only if it contains no odd cycle. Noting that a cograph contains no cycle of odd length at least 5 yields the desired result. $\square$

Thus, in the rest of this paper we say that G is $(\infty, 2)$-partitionable instead of (P_3-free, K_3-free)-colourable. Unless otherwise specified, let partitionable mean $(\infty, 2)$-partitionable, and let in-partitionable mean $(\infty, 2)$-in-partitionable.

3 Proof of Theorem 1.1

3.1 Subclasses of partitionable cographs

In this section, we characterize subclasses of partitionable cographs by forbidden induced subgraphs. These results will prove useful in establishing the

main theorem. First, a set of definitions and lemmas is needed.

Definition 3.1. A bi-threshold graph is a bipartite or threshold graph.

Definition 3.2. A monopolar graph is a $(\infty, 1)$-partitionable graph.

Definition 3.3. A monopolar nearly split graph is a $(\infty, 1)$-partitionable or $(1, 2)$-partitionable graph.

Lemma 3.1. *Let G be a cograph. If G contains P_3 and K_3, then G contains $F_1 = P_3 \cup K_3$, $F_2 = diamond$, or $F_3 = paw$.*

Proof. Consider the triangle. If there is a vertex with exactly one or two neighbours in the triangle we have F_3 or F_2 respectively. If two non-adjacent vertices with three neighbours in the triangle exist we have F_2. If none of these cases applies to any triangle in G, then all triangles form a clique with no neighbours in the rest of the graph. Consequently we find F_1. □

Lemma 3.2. *Let G be a cograph. If G contains P_3 and $2K_2$, then G contains $Q_1 = P_3 \cup K_2$, or $Q_2 = butterfly$.*

Proof. Consider the disjoint edges e_1 and e_2 in $2K_2$. Let G_1 be the component containing e_1. Suppose G_1 contains e_2. Let v be a vertex adjacent to some endpoint of e_1. Since G is a cograph, any induced path between two vertices in a component of G has length at most 2. As e_1 and e_2 have no edges between them every induced path between e_1 and e_2 has length 2. It follows that v must be adjacent to every vertex in e_1 and e_2, in which case we get Q_2. Otherwise, suppose G_1 does not contain e_2. If there is a vertex with one neighbour in e_1, we get Q_1. If this case does not apply to any vertex in G_1, then G_1 forms a clique with no neighbours in the rest of the graph, in which case we get Q_1. □

Lemma 3.3. *Let G be a cograph. If G is C_4-free and contains $P_3, 2K_2$ and K_3, then G contains $S_1 = F_1$, $S_2 = Q_2$, $S_3 = K_2 \cup paw$, or $S_4 = K_2 \cup diamond$.*

Proof. Consider the disjoint edges e_1 and e_2 in $2K_2$. Let G_1 be the component containing e_1. If G_1 contains e_2, then by a similar argument as in the proof of Lemma 3.2, we get S_2. Suppose G_1 does not contain e_2. If there exists two non-adjacent vertices with two neighbours in e_1, we get S_4. If there exists two non-adjacent vertices with one and two neighbours respectively in e_1,

we get S_3. If there exists two adjacent vertices with one and two neighbours respectively in e_1, we get S_4. If none of these cases applies to any edge in G_1, by considering the absence of P_4 and C_4 we get that G_1 either (i) forms a star graph with no neighbours in the rest of the graph, or (ii) forms a clique with no neighbours in the rest of the graph. In the case of (i), we get S_1. In the case of (ii) if G_1 contains a triangle, we get S_1, and if G_1 is a single edge, from Lemma 3.1, we get S_1, S_3 or S_4. □

Lemma 3.4. *Let G be a cograph. If G contains P_3 and $2K_3$, then G contains $W_1 = 2K_3 \cup P_3$, $W_2 = K_3 \cup diamond$, $W_3 = K_3 \cup paw$, or $W_4 = K_1 \oplus 2K_3$.*

Proof. Consider the disjoint triangles t_1 and t_2 in $2K_3$. If t_1 and t_2 share a neighbour, to avoid inducing P_4 we get W_4. Otherwise, by a similar argument than in Lemma 3.1 we get W_1, W_2, or W_3. □

3.1.1 Bi-threshold cographs

This section establishes the following theorem.

Theorem 3.1. *Let G be a cograph that is connected but not complete. Then G is bi-threshold if and only if G does not contain the graphs $B_1, \ldots, B_6$ illustrated in Figure 1.*

(1) $B_1 = butterfly$.
(2) $B_2 = C_4 \oplus K_1$.
(3) $B_3 = 2K_1 \oplus (K_2 \cup K_1)$.
(4) $B_4 = K_2 \cup diamond$.
(5) $B_5 = K_3 \cup P_3$.
(6) $B_6 = K_2 \cup paw$.

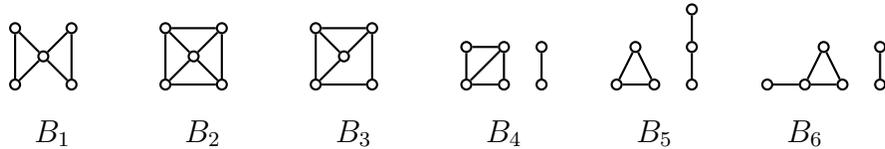$$B_1 \qquad B_2 \qquad B_3 \qquad B_4 \qquad B_5 \qquad B_6$$

Figure 1: The graphs B_1, B_2, B_3, B_4, B_5 and B_6

Proof. For the "only if" direction, a threshold graph and a bipartite graph are $(C_4, P_4, 2K_2)$-free and triangle-free, respectively. The graphs $B_1, \ldots, B_6$ each contain a triangle, and C_4 or $2K_2$.

6

For the "if" direction, let G be a connected cograph containing P_3 that is neither bipartite nor threshold and minimal. Then G contains K_3, and C_4 or $2K_2$. We consider two cases.

Case 1: G contains C_4.

Since G is connected and P_4-free the triangle and the quadrangle share an edge. The third vertex of the triangle has another neighbour in the quadrangle, otherwise there would be a P_4. Hence G contains B_2 or B_3.

Case 2: G contains $2K_2$.

We can assume G is C_4-free. By Lemma 3.3, G contains B_1, B_4, B_5 or B_6. This completes the proof.

$\square$

3.1.2 Monopolar cographs

In [2] a characterization of monopolar cographs, defined in the paper as (s, k)-polar cographs where $\min(s, k) \leq 1$, is presented. Essentially, the same proof shows the following:

Theorem 3.2. *For a connected cograph G, G is monopolar if and only if G has no induced subgraph isomorphic to the graphs $J_1, \ldots, J_4$ depicted in Figure 2.*

(1) $J_1 = $ 5-wheel.
(2) $J_2 = K_1 \oplus (P_3 \cup K_2)$.
(3) $J_3 = K_2 \oplus 2K_2$.
(4) $J_4 = (K_2 \cup K_1) \oplus (K_2 \cup K_1)$.

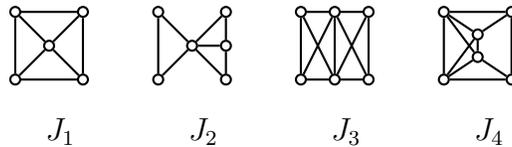$$J_1 \qquad J_2 \qquad J_3 \qquad J_4$$

Figure 2: The graphs J_1, J_2, J_3 and J_4

Proof. For the "only if" direction, recall that a monopolar graph is a graph that can be partitioned into an independent set and a union of cliques. Since

7

every J_i is not a union of cliques, it must contain a join of stable sets in any partition. It is routine to verify that there exists no partition of these graphs such that their join of stable sets in the partition is a stable set.

For the "if" direction, since G is connected it is the join of two subgraphs $G[A]$ and $G[B]$. Given that a threshold graph is a $(C_4, P_4, 2K_2)$-free graph, it is enough to consider the following cases.

Case 1: $G[A]$ is not a threshold graph.
Subcase 1: $G[A]$ contains C_4.
Since $G[B]$ is non-empty, G contains J_1.
Subcase 2: $G[A]$ contains $2K_2$.
If $G[B]$ contains K_2, G contains J_3. Thus, suppose $G[B]$ is a stable set. If $G[A]$ contains P_3, by Lemma 3.2 it contain Q_1 or Q_2. If $G[A]$ contains Q_2, then G contains $J_3 = Q_2 \oplus K_1$, and if $G[A]$ contains Q_1, then G contains $J_2 = Q_1 \oplus K_1$. Finally, if $G[A]$ is P_3-free, then $G = G[A] \oplus G[B]$ is a complete $(\infty, 1)$-partitionable graph and therefore monopolar.

We may assume by symmetry that both $G[A]$ and $G[B]$ do not contain C_4, $2K_2$ and P_4 and hence form threshold graphs.

Case 2: $G[A]$ and $G[B]$ are threshold graphs
Subcase 1: $G[A]$ contains a triangle
i) If $G[A]$ forms a single clique, then $G[B]$ being a threshold graph, G too is a threshold graph and therefore monopolar.

ii) Suppose $G[A]$ contains a paw. If $G[B]$ is not a clique, then as $P_3 \subset paw$ G contains $J_1 = P_3 \oplus 2K_1$, and if $G[B]$ is a clique, then G is a threshold graph.

iii) Suppose $G[A]$ contains a clique and at least one isolated vertex. If $G[B]$ contains P_3, then G contains $J_1 = P_3 \oplus 2K_1$. Thus, $G[B]$ forms a disjoint union of cliques. If $G[B]$ contains $K_2 \cup K_1$, then G contains $J_4 = (K_2 \cup K_1) \oplus (K_2 \cup K_1)$. If $G[B]$ is a non-trivial stable set, then G is $(\infty, 1)$-partitionable. Finally, if $G[B]$ is a clique, then G forms a threshold graph.

iv) Suppose $G[A]$ contains a diamond. If $G[B]$ is not a clique G contains $J_1 = P_3 \oplus 2K_1$ since $P_3 \subset diamond$, and if $G[B]$ forms a clique, G is a threshold graph.

Subcase 2: Both $G[A]$ and $G[B]$ are triangle-free
i) Suppose $G[A]$ contains P_3. If $G[B]$ has a non-edge, then G contains $J_1 = P_3 \oplus 2K_1$. If $G[B]$ is a clique, G is a threshold graph.

ii) By symmetry, suppose $G[A]$ and $G[B]$ are P_3-free. Suppose $G[A]$ contains $K_2 \cup K_1$. If $G[B]$ contains $K_2 \cup K_1$, then G contains $J_4 = (K_2 \cup$

8

$K_1) \oplus (K_2 \cup K_1)$. Suppose $G[B]$ does not contain $K_2 \cup K_1$. If $G[B]$ is a stable set, then G is a complete $(\infty, 1)$-partitionable graph. If $G[B]$ is a clique, then G is a threshold graph. If $G[A]$ is a clique, B being a threshold graph it follows that G is a threshold graph. If $G[A]$ is a stable set, $G[B]$ being P_3-free it follows that G is a complete $(\infty, 1)$-partitionable graph. $\square$

Observation 3.1. *the graphs J_1, J_2, J_3 and J_4 are $(1, 2)$-partitionable connected cographs.*

Proof. Let $i \in \{1, 2, 3, 4\}$ and let $C(J_i)$ be any maximum clique of J_i. It is easy to check that for every i, $J_i[V \setminus C(J_i)]$ is bipartite. $\square$

3.1.3 Monopolar nearly split cographs

$I_1 \qquad I_2 \qquad I_3 \qquad \overline{I_1} \qquad \overline{I_2} \qquad \overline{I_3}$
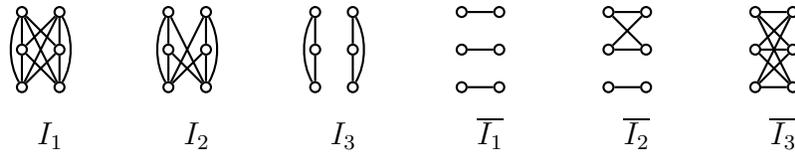
Figure 3: The graphs I_1, I_2, I_3 and their complements

Proposition 3.1 ([9]). *A cograph is $(2, 1)$-partitionable if and only if it does not contain the graphs $\overline{I_1}$, $\overline{I_2}$, $\overline{I_3}$ illustrated in Figure 3.*

Corollary 3.1. *A cograph is $(1, 2)$-partitionable if and only if it does not contain the graphs $I_1 = \overline{3K_2}, I_2 = 2K_2 \oplus 2K_1, I_3 = 2K_3$ depicted in Figure 3.*

We are now ready to prove the following theorem.

Theorem 3.3. *Let G be a connected cograph. Then G is a monopolar nearly split graph if and only if G does not contain the graphs $R_1, \ldots, R_8$ depicted in Figure 4.*

(1) $R_1 = 2K_1 \oplus 2K_1 \oplus 2K_1$.
(2) $R_2 = 2K_2 \oplus (K_2 \cup K_1)$.
(3) $R_3 = 2K_1 \oplus (P_3 \cup K_2)$.
(4) $R_4 = K_1 \oplus (2K_1 \oplus 2K_2)$.
(5) $R_5 = K_2 \oplus 2K_3$.
(5') $R_5 = K_1 \oplus (K_1 \oplus 2K_3)$.

(6) $R_6 = K_1 \oplus (P_3 \cup 2K_3)$.
(7) $R_7 = K_1 \oplus (K_3 \cup (P_3 \oplus K_1))$.
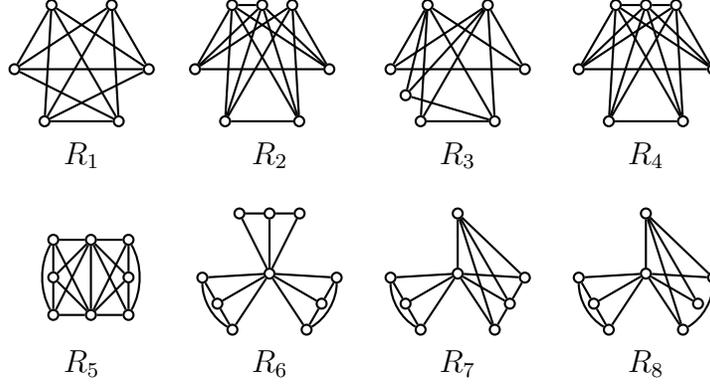(8) $R_8 = K_1 \oplus (K_3 \cup (K_1 \oplus (K_1 \cup K_2)))$.

Figure 4: The graphs $R_1, \ldots, R_8$

Proof. The "only if" direction can be proved by a careful case analysis.

For the "if" direction, suppose G is neither monopolar nor $(1,2)$-partitionable, and minimal. Since G is connected, let $\{A, B\}$ be a partition of the vertex set of G such that $G = G[A] \oplus G[B]$. By the minimality of G, $G[A]$ and $G[B]$ are either monopolar or $(1,2)$-partitionable. We consider several cases.

Case 1: $G[A]$ and $G[B]$ are $(K_2 \cup K_1)$-free.

That is, $G[A]$ and $G[B]$ are a join of stable sets. Thus, G is a join of stable sets. It follows that G either contains $R_1 = \overline{3K_2}$, or is $(1,2)$-partitionable.

Case 2: $G[A]$ and $G[B]$ contain $K_2 \cup K_1$.

i) If $G[A]$ contains C_4, then G contains $R_1 = C_4 \oplus 2K_1$. ii) If $G[A]$ contains $2K_2$, G contains $R_2 = 2K_2 \oplus (K_2 \cup K_1)$. iii) By symmetry, if $G[A]$ and $G[B]$ are threshold graphs, then G is $(1,2)$-partitionable.

Case 3: $G[A]$ is $(K_2 \cup K_1)$-free, and $G[B]$ contains $K_2 \cup K_1$.
Subcase 1: $G[A]$ is a clique.

If $G[B]$ is $(1,2)$-partitionable, G is $(1,2)$-partitionable. If $G[B]$ is $(1,2)$-in-partitionable, it must be monopolar. By Corollary 3.1 and given that $J_1 \subset I_1$, it follows that $G[B]$ contains I_2 or I_3.

10

i) If $G[B]$ contains I_2, G contains $R_4 = K_1 \oplus I_2$.

ii) Otherwise, suppose $G[B]$ contains I_3. If $G[A]$ has at least 2 vertices, G contains $R_5 = K_2 \oplus I_3$. Thus, suppose $G[A]$ is a single vertex. In this case, if $G[B]$ is P_3-free, G is monopolar. If $G[B]$ contains P_3, by Lemma 3.4, it contains W_1, W_2, W_3 or W_4, in which case G contains $R_6 = K_1 \oplus W_1$, $R_7 = K_1 \oplus W_2$, $R_8 = K_1 \oplus W_3$, or $R_9 = K_1 \oplus W_4$.

Subcase 2: $G[A]$ is an independent set.

The case where $G[A]$ is a single vertex is covered in Subcase 1. Hence assume $G[A]$ contains $2K_1$. If $G[B]$ is P_3-free, G is monopolar. If $G[B]$ is a threshold graph, G is $(1,2)$-partitionable. Otherwise, suppose $G[B]$ contains C_4, or P_3 and $2K_2$. i) If $G[B]$ contains C_4, G contains $R_1 = 2K_1 \oplus C_4$. If $G[B]$ contains P_3 and $2K_2$, by Lemma 3.2 $G[B]$ contains Q_1, or Q_2, in which case G contains $R_3 = 2K_1 \oplus Q_1$, or $R_4 = 2K_1 \oplus Q_2$.

Since $G[A]$ is a join of stable sets, we are left to consider the following subcases.

Subcase 3: $G[A]$ contains $2K_1 \oplus 2K_1$.

Since $G[B]$ contains $K_2 \cup K_1$, it follows that G contains $R_1 = 2K_1 \oplus 2K_1 \oplus 2K_1$.

Subcase 4: $G[A] = qK_1 \oplus K_r$ for some integers $q \geq 2$ and $r \geq 1$.

If $G[B]$ is a threshold graph, then G is $(1,2)$-partitionable. Otherwise, if $G[B]$ contains $2K_2$, then G contains R_4. Finally, if $G[B]$ contains $C_4 = 2K_1 \oplus 2K_1$, G contains R_1. This completes the proof. $\square$

3.2 Main Result

This section establishes Theorem 1.1. First the following two lemmas are required.

Lemma 3.5. *Minimal in-partitionable cographs are connected.*

Proof. By way of contradiction. Let $G = (V, E)$ be a cograph. Suppose that G is disconnected and, without loss of generality, minimal in-partitionable. Let $\{A, B\}$ be a partition of V such that $G = G[A] \cup G[B]$ (which exists by assumption). By the minimality of G, $G[A]$ and $G[B]$ are partitionable. Let C and D be a partition of $G[A]$, P and Q a partition of $G[B]$ such that $G[C]$, $G[P]$ are bipartite, and $G[D]$, $G[Q]$ are P_3-free. It follows that $G[C \cup P]$ is bipartite and $G[D \cup Q]$ is P_3-free, which is a partition of G, a contradiction. $\square$

Lemma 3.6. *Let $G = (V, E)$ be a cograph, and let $\{A, B\}$ be a partition of V such that $G = G[A] \oplus G[B]$. If both $G[A]$ and $G[B]$ are threshold graphs, then G is partitionable.*

Proof. Let $G' = G[A]$ and $G'' = G[B]$. Let $\{C, D\}$ be a partition of $V(G')$ such that C induces a clique and D induces a stable set. Similarly, let $\{F, P\}$ be a partition of $V(G'')$ such that F induces a clique and G induces a stable set. Because $G = G[A] \oplus G[B]$, it follows that $G[C \cup F] = G[C] \oplus G[F]$ is a clique and $G[D \cup P] = G[D] \oplus G[P]$ is a complete bipartite graph. $\square$
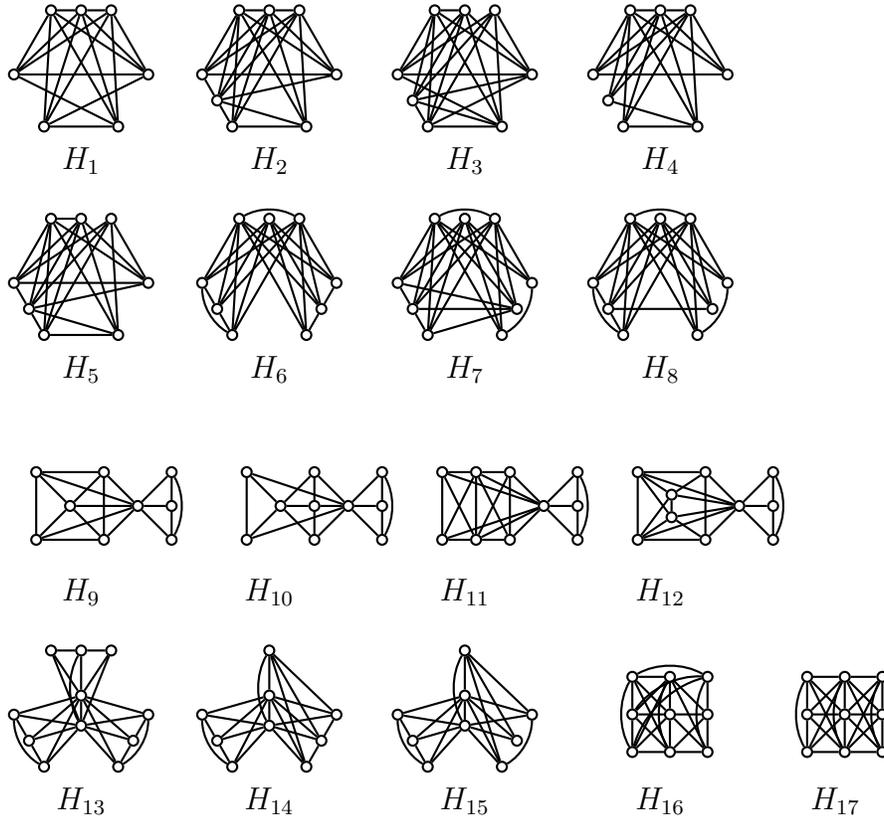

Figure 5: Forbidden subgraphs of partitionable cographs.

The following graphs depicted in Figure 5 will be used:

(1) $H_1 = 2K_1 \oplus 2K_1 \oplus 2K_1 \oplus K_1$

(2) $H_2 = P_3 \oplus K_1 \oplus 2K_2$

(3) $H_3 = 2K_1 \oplus (K_2 \cup K_1) \oplus (K_2 \cup K_1)$

(4) $H_4 = P_3 \oplus (K_2 \cup P_3)$

(5) $H_5 = (K_2 \cup K_1) \oplus K_1 \oplus 2K_2$

(6) $H_6 = (K_2 \cup K_1) \oplus (K_3 \cup P_3)$

(7) $H_7 = (K_2 \cup K_1) \oplus (K_2 \cup (P_3 \oplus K_1))$

(8) $H_8 = (K_2 \cup K_1) \oplus (K_2 \cup (K_1 \oplus (K_2 \cup K_1)))$

(9) $H_9 = K_1 \oplus (K_3 \cup (C_4 \oplus K_1))$

(10) $H_{10} = K_1 \oplus (K_3 \cup (K_1 \oplus (P_3 \cup K_2)))$

(11) $H_{11} = K_1 \oplus (K_3 \cup (K_2 \oplus 2K_2))$

(12) $H_{12} = K_1 \oplus (K_3 \cup ((K_2 \cup K_1) \oplus (K_2 \cup K_1)))$

(13) $H_{13} = K_2 \oplus (P_3 \cup 2K_3)$

(14) $H_{14} = K_2 \oplus (K_3 \cup (P_3 \oplus K_1))$

(15) $H_{15} = K_2 \oplus (K_3 \cup (K_1 \oplus (K_1 \cup K_2)))$

(16) $H_{16} = (K_3 \cup K_2) \oplus (K_3 \cup K_1)$

(17) $H_{17} = K_3 \oplus 2K_3$

Proof of Theorem 1.1*.* By a careful case analysis, the graphs $H_1, \ldots, H_{17}$ are minimal in-partitionable.

Conversely, suppose G is minimal in-partitionable. By Lemma 3.5, G is connected. We prove that G must contain one of the graphs $H_1, \ldots, H_{17}$.

Claim 1. *If G has no universal vertex, then G contains one of the graphs $H_1, \ldots, H_8, H_{16}$.*

Proof. Let $\{A, B\}$ be partition of the vertex set of G such that $G = G[A] \oplus G[B]$. Such a partition exists since G is connected. By the minimality of G, $G[A]$ and $G[B]$ are partitionable. Furthermore, because G has no universal vertex, $G[A]$ and $G[B]$ have no universal vertex, and it follows that $G[A]$ and $G[B]$ each contain $2K_1$. We consider two cases:

Case 1: $G[A]$ is P_3-free.
Then $G[A]$ is a union of at least two cliques C_1, C_2 because it contains $2K_1$.

Subcase 1: $G[B]$ is P_3-free.

Similarly, $G[B]$ is a union of at least two cliques C_3, C_4. If $G[B]$ or $G[A]$ is bipartite, then G is partitionable. Thus, without loss of generality, let $|C_1|, |C_3| \geq 3$. Furthermore, C_2 or C_4 contains K_2, for otherwise $G[A]$ and $G[B]$ form threshold graphs and G is partitionable by Lemma 3.6. It follows that G contains $H_{16} = (K_3 \cup K_2) \oplus (K_3 \cup K_1)$.

Subcase 2: $G[B]$ contains P_3.

(1) $G[A]$ is a stable set of order at least two.

If $G[B]$ is $(\infty, 1)$-partitionable, then G is partitionable. Otherwise, by Theorem 3.2 $G[B]$ contains the graph J_1, J_2, J_3 or J_4. It follows that G contains $H_1 = 2K_1 \oplus J_1$, $H_2 = 2K_1 \oplus J_3$, $H_3 = 2K_1 \oplus J_4$, or $H_4 = 2K_1 \oplus J_2$.

(2) $G[A]$ contains exactly one clique with more than one vertex.

If $G[B]$ is a threshold graph (i.e., $(1,1)$-partitionable), then G is $(1,2)$-partitionable. If $G[B]$ is bipartite, then G is $(\infty, 2)$-partitionable. Thus, suppose $G[B]$ is neither bipartite nor threshold, i.e., it contains K_3, and C_4 or $2K_2$. By Theorem 3.1, $G[B]$ contains the graph B_1, B_2, B_3, B_4, B_5 or B_6. It follows that G contains $H_5 = (K_2 \cup K_1) \oplus B_1$, $H_1 = 2K_1 \oplus B_2$, $H_3 = (K_2 \cup K_1) \oplus B_3$, $H_7 = (K_2 \cup K_1) \oplus B_4$, $H_6 = (K_2 \cup K_1) \oplus B_5$, or $H_8 = (K_2 \cup K_1) \oplus B_6$.

(3) $G[A]$ contains $2K_2$.

If $G[B]$ is bipartite, then G is partitionable. If $G[B]$ is not bipartite, it contains K_3. Given that $G[B]$ contains P_3, by Lemma 3.1 $G[B]$ contains the graph F_1, F_2 or F_3. If $G[B]$ contains F_1, then G contains $H_6 = (K_2 \cup K_1) \oplus F_1$. If $G[B]$ contains F_2, then G contains $H_2 = 2K_2 \oplus F_2$. If $G[B]$ contains F_3, then G contains $H_5 = 2K_2 \oplus F_3$. This completes the treatment of Case 1.

Case 2: $G[A]$ and $G[B]$ contain P_3.

Since G is a cograph, it has no induced C_5. Together with the fact that a threshold graph is a $(C_4, P_4, 2K_2)$-free graph, it is sufficient to consider the following cases.

SubCase 1: $G[A]$ contains C_4.

G contains $H_1 = C_4 \oplus P_3$.

SubCase 2: $G[A]$ contains $2K_2$.

By Lemma 3.2 $G[A]$ contains Q_1 or Q_2. It follows that G contains $H_4 = P_3 \oplus Q_1$ or $H_2 = P_3 \oplus Q_2$.

SubCase 3: $G[A]$ and $G[B]$ are threshold graphs.
By Lemma 3.6, G is partitionable. This completes the treatment of Case 2 and the proof of Claim 1. ∎

Claim 2. *If G has a universal vertex v such that $G' = G \setminus v$ is disconnected, then G contains one of the graphs $H_9, H_{10}, H_{11}, H_{12}$.*

Proof. Let $\mathcal{G} = \{G_1, \ldots, G_k\}$, where $k \geq 2$, be the set of components of G'. By the minimality of G, for every $G_i \in \mathcal{G}$, the graphs G_i and $G_i' = v \oplus G_i$ are partitionable. From this, it is easy to see that if G_i is (k, l)-partitionable (where the k cliques are disjoint) with $\min(k, l) \geq 2$, then G_i' is in-partitionable. Thus, each G_i is either $(1, 2)$-partitionable or $(\infty, 1)$-partitionable. Suppose each G_i is $(\infty, 1)$-partitionable. Then every G_i' admits a partition where v is in the bipartite part. Consequently, as the G_i's are disjoint, G admits a partition where v is in the bipartite part, a contradiction. Thus, there exists a $G_j \in \mathcal{G}$ that is $(\infty, 1)$-in-partitionable and $(1, 2)$-partitionable. By Theorem 3.2 and Observation 3.1, G_j contains one of the graphs J_1, J_2, J_3 or J_4. Suppose, for the sake of contradiction, that there exists no $p \neq j$ such that G_p contains K_3. Let $C(G_j)$ and $S(G_j)$ denote the partition of G_j into a clique and a bipartite graph respectively, which exists by assumption. It is easy to see that $\{A, B\}$, where $A = v \cup C(G_j)$, and $B = S(G_j) \cup \bigcup_{p \neq j} G_p$ is a partition of V where $G[A]$ is P_3-free and $G[B]$ is bipartite, a contradiction. It follows that G contains $H_9 = v \oplus (K_3 \cup J_1)$, $H_{10} = v \oplus (K_3 \cup J_2)$, $H_{11} = v \oplus (K_3 \cup J_3)$ or $H_{12} = v \oplus (K_3 \cup J_4)$. This completes the proof of Claim 2. ∎

Claim 3. *If G has a universal vertex v such that $G' = G \setminus v$ is connected, then G contains one of the graphs $H_1, H_2, H_4, H_5, H_{13}, H_{14}, H_{15}, H_{17}$.*

Proof. By the minimality of G, G' is partitionable. In particular, G' is neither $(\infty, 1)$-partitionable nor $(1, 2)$-partitionable, for otherwise $G = G' \oplus v$ is partitionable. Thus, by Theorem 3.3, G' contains one of the graphs $R_1, \ldots, R_8$. It follows that G contains $H_1 = v \oplus R_1$, $H_5 = v \oplus R_2$, $H_4 = v \oplus R_3$, $H_2 = v \oplus R_4$, $H_{17} = v \oplus R_5$, $H_{13} = v \oplus R_6$, $H_{14} = v \oplus R_7$, or $H_{15} = v \oplus R_8$. This completes the proofs of Claim 3 and Theorem 1.1.

∎

15

□

Acknowledgments

References

[1] S. Francisco, S. Klein, and L.T. Nogueira, Characterizing (k, l)-partitionable cographs. *Electronic Notes in Discrete Mathematics* **22** (2005) 277–280.

[2] T. Ekim, N.V.R. Mahadev, and D. Werra, Polar cographs. *Discrete Applied Mathematics* **156** (2008) 1652–1660.

[3] A. Farrugia, Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard. *The Electronic Journal of Combinatorics* **11** (2004) R46.

[4] T. Ekim, P. Hell, J. Stacho, and D. Werra, Polarity of chordal graphs. *Discrete Applied Mathematics* **156** (2008) 2469–2479.

[5] T. Ekim, P. Heggernes, and D. Meister, Polar permutation graphs are polynomial time recognizable. *European J. Comb.* **34** (2013) 576–592.

[6] J. Huang, and B. Xu, A forbidden subgraph characterization of line-polar bipartite graphs. *Discrete Applied Mathematics* **158** (2010) 666–680.

[7] R. Churchley, and J. Huang, List monopolar partitions of claw-free graphs. *Discrete Mathematics* **312** (2012) 2545–2549.

[8] D.G. Corneil, H. Lerchs, and L. Stewart, Complement reducible graphs. *Discrete Applied Mathematics* **3** (1981) 163–174.

[9] M. Demange, T. Ekim, and D. Werra, Partitioning cographs into cliques and stable sets. *Discrete Optimization* **2** (2005) 145–153.

[10] R. Diestel, Graph theory, Third edition. *Springer* (2006).

[11] P. Damaschke, Induced subgraphs and well-quasi-ordering. *Journal of Graph Theory* **14** (1990) 427–435.

[12] F.N. Abu-Khzam, C. Feghali, and H. Müller, NP-hardness results for partitioning graphs into disjoint cliques and a triangle-free subgraph. *arXiv:1403.5248* (2014)