Smooth and nonsmooth duality for free time problem

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Summary. The main result of this paper contains a representation of the minimum cost of a free time control problem in terms of the upper envelope of generalized semisolutions of the Hamilton-Jacobi equation. A corollary generalizes a similar result due to Vinter using smooth subsolutions.

Key words: free time problems, duality, Hamilton-Jacobi equations, viscosity solutions, proximal analysis, nonsmooth analysis

1 Introduction

The following optimal control problem is considered in Vinter [15]:

$$(Q)\ \begin{cases} \text{Minimize } \ell(T, x(T)), \\ T \in [0,1], \\ \dot{x}(t) \in F(t, x(t)) \ \ a.e.\ t \in [0, T], \\ x(0) = x_0, \\ (t, x(t)) \in A \subset [0,1] \times \mathrm{I\!R}^n \ \ \forall t \in [0, T], \\ (T, x(T)) \in C \subset [0,1] \times \mathrm{I\!R}^n, \end{cases}$$

where the given data is a point x_0, the function $\ell : \mathrm{I\!R} \times \mathrm{I\!R}^n \longrightarrow \mathrm{I\!R} \cup \{+\infty\}$, the multivalued function $F : \mathrm{I\!R} \times \mathrm{I\!R}^n \longrightarrow \mathrm{I\!R}^n$, and the sets A and C. Vinter in addition formulated a convex optimization problem (W), associated with (Q), namely, the minimization of a linear functional under linear constraints of equality type on the set $\mathcal{W}$ of generalized flows, a weak*-convex compact set of a space of Radon measures also associated with problem (Q). Based on the apparatus of convex analysis and, in particular, on convex duality, he established a very close interconnection between problems (Q) and (W). He proved that the set $\mathcal{W}$ is the convex closure of the set of admissible arcs of the original problem (Q), and also that both problems are solvable and that, moreover, their values coincide. This makes it possible to prove a necessary and sufficient condition for optimality for problem (Q) related to well-known sufficient conditions, referred to as verification theorems, in dynamic optimization, see [4] and [6]. Simultaneously, Vinter gives a "smooth duality" for the problem (Q); that is, the value of problem (Q) is represented in terms of the upper envelope of smooth subsolutions of the Hamilton-Jacobi equation. This so-called "convex duality" method was first introduced by Vinter and Lewis [17], [18]. For more information about the possibility of approaching control problems via duality theory in abstract spaces, see ([7], [8], [11], [12], [15], [17] and [18]).

We remark that the problem (Q) treated by Vinter is an optimal control problem with finite horizon ($T \in [0,1]$), and he has affirmed [16] that his generalized flows approach does not extend to free time problems with infinite horizon ($T \in [0,+\infty[$) and do not leads an upper envelope characterization of the minimum cost, in term of smooth solutions of the *autonomous* Hamilton-Jacobi inequality.

In this article, we consider the following free time problem

$$(P) \begin{cases} \text{Minimize } T + \ell(x(T)), \\ T \in [0,+\infty[, \\ \dot{x}(t) \in F(x(t)) \;\; a.e.\ t \in [0,+\infty[, \\ x(0) = x_0, \\ x(t) \in A \;\; \forall t \in [0,T], \\ x(T) \in C. \end{cases}$$

Our main result is a "nonsmooth duality" for the problem (P); that is, a representation of the minimum cost of (P) in terms of the upper envelope of generalized semisolutions of the Hamilton-Jacobi equation. This type of duality is well studied in the literature with several techniques and particularly for fixed time problems, see for example [1], [2], [4, Chapter 4], [9], [10], [13], [14] and [19]. We use the *proximal subdifferential* to define our generalized semisolutions. This concept of solution appeared in Clarke and Ledyaev [3], where the various concepts were also unified. Using our nonsmooth duality we extend Vinter's smooth duality for free time problems with infinite horizon. Let us enter into the details.

We assume in the problem (P) that the set A is closed, that C is compact, and that the extended-valued function $\ell : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and bounded below by a constant ω. As for the multivalued function F, we assume that it takes nonempty compact convex values, has closed graph, and satisfies a linear growth condition: for some positive constants γ and c, and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c.$$

Finally, we assume that (P) is nontrivial in the sense that there is at least one admissible trajectory for which the cost is finite.

We associate with F the following function h, the (lower) Hamiltonian:

$$h(x,p) := \min\{\langle p, v\rangle : v \in F(x)\}.$$

The augmented Hamiltonian $\bar{h}$ is defined by

$$\bar{h}(x,\theta,\zeta) := \theta + h(x,\zeta).$$

Given a lower semicontinuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ and a point $x \in \mathrm{dom} f := \{x' \in \mathbb{R}^n : f(x') < +\infty\}$, we say that $\xi \in \mathbb{R}^n$ is a *proximal subgradient* of f at x if and only if there exists $\sigma \geq 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \xi, y - x\rangle,$$

for all y in a neighborhood of x. The set (which could be empty) of all proximal subgradients of f at x is denoted by $\partial_P f(x)$, and is referred to as the proximal subdifferential. The Proximal Density Theorem asserts that $\partial_P f(x) \neq \emptyset$ for all x in a dense subset of $\mathrm{dom} f$. We also define the *limiting subdifferential* of f at x by

$$\partial_L f(x) := \{\lim \xi_i : \xi_i \in \partial_P f(x_i), x_i \longrightarrow x \text{ and } f(x_i) \longrightarrow f(x)\}.$$

We refer the reader to [4] for full account of proximal and nonsmooth analysis. Now we define Ψ to be the set of all locally Lipschitz functions ψ on $\mathbb{R}^n$ that satisfy the proximal Hamilton-Jacobi inequality

$$\bar{h}(x, \partial_L \psi(x)) \geq 0 \ \ \forall x \in A$$

as well as the boundary condition

$$\psi(x) \leq \ell(x) \ \ \forall x \in C.$$

The following nonsmooth duality is the main result.

Theorem 1.

$$\min(P) = \sup_{\psi \in \Psi} \psi(x_0).$$

This leads directly to the following optimality conditions.

Corollary 1. *Let $(\bar{T}, \bar{x}(\cdot))$ be an admissible trajectory for (P). Then $(\bar{T}, \bar{x}(\cdot))$ is a minimizer for (P) iff there exists a sequence of functions $\{\psi_i\}$ in Ψ such that*

$$\lim_{i \longrightarrow +\infty} \psi_i(x_0) = \bar{T} + \ell(\bar{x}(\bar{T})).$$

Our theorem, whose proof is self-contained modulo some basic facts from proximal analysis, is new with respect to its very mild regularity hypotheses on F (which need not even be continuous), as well as the presence of a unilateral state constraint. Moreover, the fact that locally Lipschitz functions and limiting subgradients figure in our duality also gives easy access to smooth duality of the type found by Vinter. We extend his result by obtaining a duality in which feature only smooth solutions of an *autonomous* Hamilton-Jacobi inequality.

We note that using our methods we can also prove Vinter's duality presented in [15] and extend it for fixed time problems, but due to space restriction we only treat here the free time with infinite horizon case and we only sketch the proofs. For complete details, see [5].

This article is organized as follows. In the next section we sketch the proof of the above theorem. Section 3 is devoted to the generalization of Vinter's smooth duality.

2 Proof of Theorem 1

First we note that under our hypotheses on F, any trajectory can be extend indefinitely both forward and backward, so all trajectories can be considered as being defined on $]-\infty,+\infty[$. By the compactness property of trajectories and since ℓ is bounded below, it is easy to prove that the problem (P) admits a solution. For all $k \in \mathbb{N}^*$, we consider the function ℓ_k defined by

$$\ell_k(x) := \inf_{y \in \mathbb{R}^n} \{\ell(y) + k\|x-y\|^2\}. \tag{1}$$

The sequence $(\ell_k)_k$ is the quadratic inf-convolution sequence of ℓ. The following lemma gives some properties of ℓ_k.

Lemma 1. *For all $k \in \mathbb{N}^*$, we have:*

1. *$\ell_k(\cdot) \leq \ell(\cdot)$ and the set of minimizing points y in (1) is nonempty.*
2. *ℓ_k is locally Lipschitz and bounded below by ω.*
3. *For all $x \in \mathbb{R}^n$,*
$$\lim_{k \longrightarrow +\infty} \ell_k(x) = \ell(x).$$

We also consider a locally Lipschitz approximation for the multifunction F. By [4, Proposition 4.4.4] there exists a sequence of locally Lipschitz multifunctions $\{F_k\}$ also satisfying the hypotheses of F such that:

- For each $k \in \mathbb{N}$, for every $x \in \mathbb{R}^n$,
$$F(x) \subseteq F_{k+1}(x) \subseteq F_k(x) \subseteq \overline{co}\, F(x + 3^{-k+1}B).$$
- $\bigcap_{k\geq 1} F_k(x) = F(x)\ \forall x \in \mathbb{R}^n$.

A standard method of approximating the terminally constrained problem (P) by a problem free of such constraints involves the imposition of a penalty term, the inf-convolution technique, and the preceding approximation of F. We consider for all $k \geq 1$ the following optimal control problem:

$$(P_k)\ \begin{cases} \text{Minimize } T + \ell_k(x(T)) + kd_C(x(T)) + k\int_0^T d_A(x(t))\,dt, \\ T \geq 0, \\ \dot{x}(t) \in F_k(x(t)) \ \ a.e.\ t \in [0,+\infty[, \\ x(0) = x_0. \end{cases}$$

Lemma 2. *There exists a sequence λ_n strictly increasing in $\mathbb{N}^*$ such that:*

$$\lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) = \min(P)$$

We continue the proof and remark that the problem (P_{λ_n}) is exactly the following problem:

$$\begin{cases} \text{Minimize } T + \hat{\ell}_{\lambda_n}(z(T)), \\ T \geq 0, \\ \dot{z}(t) \in \hat{F}_{\lambda_n}(z(t)) \ \ a.e.\ t \in [0,+\infty[, \\ z(0) = (0, x_0), \end{cases}$$

where $\hat{F}_{\lambda_n}$ is the augmented locally Lipschitz multivalued function defined by $\hat{F}_{\lambda_n}(y,x) := \{\lambda_n d_A(x)\} \times F_{\lambda_n}(x)$, $\forall (y,x) \in \mathbb{R} \times \mathbb{R}^n$ and $\hat{\ell}_{\lambda_n}$ is the locally Lipschitz function defined by $\hat{\ell}_{\lambda_n}(y,x) = \ell_{\lambda_n}(x) + \lambda_n d_C(x) + |y|$, $\forall (y,x) \in \mathbb{R} \times \mathbb{R}^n$.
Let $\hat{V}_{\lambda_n} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be the value function of the problem (P_{λ_n}); that is, for every $(\tau, \beta, \alpha) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$, $\hat{V}_{\lambda_n}(\tau, \beta, \alpha)$ is the minimum of the following problem:

$$\begin{cases} \text{Minimize } T + \hat{\ell}_{\lambda_n}(z(T)), \\ T \geq \tau, \\ \dot{z}(t) \in \hat{F}_{\lambda_n}(z(t)) \ \ a.e.\ t \in [\tau,+\infty[, \\ z(\tau) = (\beta, \alpha). \end{cases}$$

Lemma 3. *The value function $\hat{V}_{\lambda_n}$ satisfies the following:*

1. $\hat{V}_{\lambda_n}$ is locally Lipschitz on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$.
2. $\hat{V}_{\lambda_n}(\tau,\beta,\alpha) \leq \tau + \hat{\ell}_{\lambda_n}(\beta,\alpha),\ \ \forall (\tau,\beta,\alpha) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$.
3. $\forall (\tau,\beta,\alpha) \in \mathbb{R} \times [0,+\infty[\times \mathbb{R}^n$ we have

$$\hat{V}_{\lambda_n}(\tau,\beta,\alpha) = \tau + \hat{V}_{\lambda_n}(0,0,\alpha) + \beta.$$

4. $\forall (t,y,x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n,\ \ \forall (\theta,\xi,\zeta) \in \partial_P \hat{V}_{\lambda_n}(t,y,x)$ we have

$$\theta + \lambda_n d_A(x)\xi + h_{\lambda_n}(x,\zeta) \geq 0.^1$$

[1] This Hamilton-Jacobi inequality follows since the system $(\hat{V}_{\lambda_n}, \hat{F}_{\lambda_n})$ is *strongly increasing* on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ (the function $\hat{V}_{\lambda_n}(\cdot, z(\cdot))$ is increasing on $[a,b]$ whenever z is a trajectory of $\hat{F}_{\lambda_n}$ on some interval $[a,b]$), and using [4, Proposition 4.6.5] which gives a proximal characterization for the strong increase property. We note that h_{λ_n} is the lower Hamiltonian corresponding to F_{λ_n}.

Now let $\psi_{\lambda_n} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the function defined by

$$\psi_{\lambda_n}(x) := \hat{V}_{\lambda_n}(0,0,x), \ \ \forall x \in \mathbb{R}^n.$$

Using Lemma 3 and the definition of ∂_L we get:

Lemma 4. $\psi_{\lambda_n} \in \Psi$.

We continue the proof and remark that

$$\psi_{\lambda_n}(x_0) = \hat{V}_{\lambda_n}(0,0,x_0) = \min(P_{\lambda_n}),$$

then

$$\sup_{\psi \in \Psi} \psi(x_0) \geq \psi_{\lambda_n}(x_0) = \min(P_{\lambda_n}).$$

Therefore

$$\min(P) = \lim_{n \longrightarrow +\infty} \min(P_{\lambda_n}) \leq \sup_{\psi \in \Psi} \psi(x_0).$$

Now we show the reverse inequality by considering $\psi \in \Psi$ making the *temporary hypothesis* that F is locally Lipschitz. Then by reasoning by the absurd and using the definition of ∂_L we have the following lemma.

Lemma 5. *For all open and bounded subset $S \subset \mathbb{R}^n$, for all $\varepsilon > 0$, there exists a neighborhood U of A such that*

$$1 + \bar{h}(x, \partial_P \psi(x)) \geq -\varepsilon \quad \forall x \in S \cap U.$$

Let $(\bar{T}, \bar{x}(\cdot))$ be a solution of the problem (P). By *Gronwall's Lemma* (see [4, Proposition 4.1.4]) there exists $\rho > 0$ such that $\bar{x}(t) \in B(0;\rho)$, $\forall t \in [0, \bar{T}]$. We apply the preceding lemma for $S = B(0;\rho)$ and for $\varepsilon > 0$, we get the existence of a neighborhood U_ε of A such that

$$1 + \bar{h}(x, \partial_P \psi(x)) \geq -\varepsilon \quad \forall x \in S \cap U_\varepsilon.$$

Then by [4, Proposition 4.6.5] we get that

$$\psi(x_0) \leq \varepsilon \bar{T} + \psi(\bar{x}(\bar{T})) \leq \varepsilon \bar{T} + \ell(\bar{x}(\bar{T})) = \varepsilon \bar{T} + \min(P)$$

hence by taking $\varepsilon \longrightarrow 0$ we get

$$\psi(x_0) \leq \min(P)$$

therefore

$$\min(P) \geq \sup_{\psi \in \Psi} \psi(x_0).$$

To remove the need for the locally Lipschitz hypothesis on F it is sufficient to use the following lemma (follows by reasoning by the absurd and using a convergence property of ∂_L, see [4, Exercise 1.11.21]), and then continue as in the Lipschitz case.

Lemma 6. *For all $n \in \mathbb{N}$ there exists $k_n \geq n$ such that*

$$1 + \bar{h}_{k_n}(x, \partial_L \psi(x)) \geq \frac{-1}{n} \quad \forall x \in A \cap \bar{B}(0;\rho).$$

The proof of Theorem 1 is achieved. $\square$

3 Smooth duality

An important application of our main result is the smooth duality studied by Vinter in [15]. Using Theorem 1 we show the following theorem which extends the Vinter's smooth duality for our problem (P).

Corollary 2.

$$\min(P) = \sup_{\varphi \in \Phi} \varphi(x_0)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v \rangle \geq 0$, $\forall x \in A$, $\forall v \in F(x)$,
- $\varphi(x) \leq \ell_0(x)$ $\forall x \in C$.

Proof. Since for all $\varphi \in \Phi$ we have $\partial_L \varphi(t, x) = \{\varphi'(t, x)\}$, we get that $\Phi \subset \Psi$. Then by Theorem 1 we have

$$\min(P) = \sup_{\psi \in \Psi} \psi(0, x_0) \geq \sup_{\varphi \in \Phi} \varphi(0, x_0).$$

For the reverse inequality, let $\psi \in \Psi$. Using the fact that if ψ is differentiable at $\alpha \in \mathbb{R}^n$ then $\psi'(\alpha) \in \partial_L \psi(\alpha)$, we have the following lemma.

Lemma 7. *Let $\alpha \in A$ such that ψ is differentiable at α. Then*

$$1 + \langle \psi'(\alpha), v \rangle \geq 0, \ \forall v \in F(\alpha).$$

Since ψ is locally Lipschitz and by Rademacher's theorem we have that ψ is differentiable a.e $\alpha \in \mathbb{R}^n$. Using the sequence F_k and the penalization term $k \int_0^T d_A(x(t))\, dt$ (as in Lemma 2), we can assume that F is Lipschitz and $A = \mathbb{R}^n$. Then by Lemma 7 and by a standard mollification technique (convolution with mollifier sequence), we have the following lemma.

Lemma 8. *There exists a sequence $\delta_i \longrightarrow 0$ such that for all $\varepsilon > 0$ there exist $i_0 \in \mathbb{N}$ and a sequence of functions $(\psi_\varepsilon^i)_i$ which satisfy: for $i \geq i_0$ we have $\psi_\varepsilon^i \in \Phi$ and*

$$\psi_\varepsilon^i(x_0) \geq \frac{\psi(x_0) - M|\delta_i| - \varepsilon}{1 + |\delta_i|},$$

where $M := \max_{x \in C} -\psi(x)$.

Clearly the preceding lemma gives the desired inequality. □

It is clear that Corollary 2 leads to a version of the necessary and sufficient conditions of Corollary 1 in which only smooth semisolutions are used.

A well-known and special case of the present framework involves the minimal time function associated to the target C and under the state constraint A:

$$T_C^A(\alpha) := \begin{cases} \text{Inf } T \geq 0, \\ \dot{x}(t) \in F(x(t)) \;\; a.e. \; t \in [0, +\infty[, \\ x(0) = x_0, \\ x(t) \in A \;\; \forall t \in [0, T], \\ x(T) \in C. \end{cases}$$

By Corollary 2 ($\ell_0 = 0$) we have the following characterization of T_C^A, which appear to be new at a technical level:

$$T_C^A(\alpha) = \sup_{\varphi \in \Phi} \varphi(\alpha)$$

where Φ is the set of all functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ which satisfy:

- $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$,
- $1 + \langle \varphi'(x), v \rangle \geq 0$, $\forall x \in A \; \forall v \in F(x)$,
- $\varphi(x) \leq 0 \; \forall x \in C$.

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