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Interior Sphere Condition for the Graph of a Multifunction

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Abstract For a given multifunction, we provide proximal differentiability condition under which the equivalence between the interior sphere condition of each value of the multifunction and the interior sphere condition of its graph holds.

Keywords Interior sphere condition · pseudo-differentiability of multifunctions · Proximal analysis · Nonsmooth analysis

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1 Introduction

Let S be a nonempty and closed subset of $\mathbb{R}^n$. We recall that S is said to satisfy a *uniform interior sphere condition* if there exists $r > 0$ such that for any boundary point x of S, one can find a closed ball in S of radius r containing x. This condition is a well known one in control theory, and is important especially in deriving regularity properties of the *minimal time function* associated to a given target set. Indeed, Cannarsa and Sinestrari proved in [1], after finding a connection between the dynamic and the target, that if the dynamic is smooth enough and the target satisfies a uniform interior sphere condition, then the minimal time function is *semiconcave*. Recall that semiconcavity is an intermediate property between Lipschitz continuity and continuous differentiability. More precisely, semiconcave functions are essentially a C^2-perturbation of concave functions and therefore inherit several regularity properties from convexity; see [2] where several features of semiconcavity were thoroughly studied. In [3,4], Cannarsa, Frankowska and Sinestrari, generalized the semiconcavity result of [1] for the case where the dynamics satisfies the uniform interior sphere condition and not necessarily the target set. Moreover, these regularity results are also generalized to the non-Lipschitz case (that is, when the *Petrov condition* is not necessarily satisfied) in [7–9].

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In [12], Nour and Stern studied the semiconcavity of the *bilateral minimal time function* (a version of the minimal time function in which both initial and end points are variable, see [11]) for a linear control system. More precisely, Nour and Stern considered the following linear differential inclusion:

$$\dot{x}(t) \in F(x(t)) := Ax(t) + U \;\; a.e. \; t \in [0,T], \; x(0) = x_0, \tag{1}$$

where A is an $n \times n$ matrix and U is a convex and compact subset of $\mathbb{R}^n$. They proved that if $F(x)$ satisfies a uniform interior sphere condition for all $x \in \mathbb{R}^n$ (which is equivalent to the uniform interior sphere condition of U), then the bilateral minimal time function is semiconcave if and only if it is Lipschitz. One of the key proof of this result is the fact that in the linear case, if $F(x)$ satisfies a uniform interior sphere condition for all $x \in \mathbb{R}^n$ (this is what we call *pointwise interior sphere condition of* F) then the graph of F (the set of all points of the form (x,v) where $v \in F(x)$) also satisfies a uniform interior sphere condition. Indeed, this follows directly from the formula

$$\operatorname{Gr} F = M.(\mathbb{R}^n \times U),$$

where $\operatorname{Gr} F$ denotes the graph of F and M is the nonsingular matrix $M := \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$.

The main goal of the present article is the generalization of the preceding result to the nonlinear case, that is, for a nonlinear multifunction F. A simple example, see Example 4, proves that an extra condition on the multifunction F is needed. After introducing a new differentiability property for multifunctions, which turns out to be a *proximal* version of the differentiability property defined in [10], we prove that under this property, the equivalence between the pointwise interior sphere condition of the multifunction F and the interior sphere condition of its graph holds. This equivalence will open the floor to a possible generalization of the semiconcavity result of [12] to the nonlinear case, which remains until now an open question, see [12, Section 3]. This will be a topic for future research.

The layout of the article is as follows. In the next section, we present some preliminaries from proximal analysis and introduce a new concept of differentiability, called proximal-pseudo-differentiability, for multifunctions. Section 3 is devoted to study the equivalence between the pointwise interior sphere condition of a multifunction F and the interior sphere condition of its graph.

2 Preliminaries

2.1 General Notations

We denote by $\|\cdot\|$, $\langle \cdot,\cdot \rangle$, B and $\overline{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ the open ρ-ball centered at x and $\overline{B}(x;\rho) := x + \rho \overline{B}$ the closed ρ-ball centered at x. For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, $\operatorname{bdry} S$ and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of S, respectively. We also denote by $\widetilde{S}$ the complement of the interior of S, that is, $\tilde{S} := (\operatorname{int} S)^c = \operatorname{cl}(S^c)$. The distance from a point x to a set S is denoted by $d_S(x)$. We also denote by $\operatorname{proj}_S(x)$ the set of closest points in S to x, that is, the set of points $s \in S$ which satisfy $d_S(x) = \|s - x\|$. The hypograph and the epigraph of a function $f : U \longrightarrow \mathbb{R}$ are denoted respectively by

- $\operatorname{hypo} f := \{(x,r) : x \in U, r \in \mathbb{R} \text{ and } r \leq f(x)\}$, and
- $\operatorname{epi} f := \{(x,r) : x \in U, r \in \mathbb{R} \text{ and } f(x) \leq r\}$.

2.2 Nonsmooth Analysis

Now we provide certain definitions from proximal analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [5]; see also Penot [16] and Rockafellar and Wets [17]. Let S be a nonempty closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, s-x \rangle \leq \sigma \|s-x\|^2 \quad \forall s \in S. \tag{2}$$

Relation (2) is commonly referred to as the *proximal normal inequality*. This proximal normal inequality can be taken to be a local one, that is, only for s near x in S as proved in [5, Proposition 1.5]. We note that no nonzero vector ζ satisfying (2) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \operatorname{bdry} S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0,0)$. For such points, the only proximal normal is $\zeta = 0$. In view of (2), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \operatorname{bdry} S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{3}$$

Then ζ is proximal normal to S at x and we say that ζ is *realized by an r-ball*. Note that ζ is then also realized by an r'-ball for any $0 < r' < r$. One can show that ζ being realized by an r-ball is equivalent to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s-x \right\rangle \leq \frac{1}{2r}\|s-x\|^2 \quad \forall s \in S. \tag{4}$$

If S is convex and $x \in \operatorname{bdry} S$, then we can easily see that a vector $0 \neq \zeta \in \mathbb{R}^n$ belongs to $N_S^P(S)$ if and only if it is realized by an r-ball for all $r > 0$. Therefore, the proximal normal inequality (2) becomes

$$\langle \zeta, s-x \rangle \leq 0 \quad \forall s \in S.$$

This proves that in the convex case, the proximal normal cone to S at x coincides with the well known convex normal cone to S at x, denoted here by $N_S(x)$. Now we recall from [5] that for a function $f : U \longrightarrow \mathbb{R}$, a vector $\zeta \in \mathbb{R}^n$ is a *proximal subgradient* of a lower semicontinuous function f at x provided that

$$(\zeta, -1) \in N^P_{\operatorname{epi} f}(x, f(x)),$$

and this is equivalent to the existence of $\sigma = \sigma(x,\zeta) \geq 0$ and $\delta > 0$ such that the following *proximal subgradient inequality* holds:

$$f(y) - f(x) + \sigma\|y-x\|^2 \geq \langle \zeta, y-x \rangle \quad \forall y \in B(x;\delta) \cap U.$$

The *proximal subdifferential* of f at x, denoted $\partial_P f(x)$, is the set of all the proximal subgradients at x. The *proximal superdifferential* of an upper semicontinuous function f at a point x, denoted by $\partial^P f(x)$, is defined by $\partial^P f(x) := -\partial_P(-f)(x)$.

We proceed to define the uniform interior sphere condition. For more information about this property, we invite the reader to see [14, 15] where the equivalence between this property and the union of closed balls property was established. The regularity of functions with epigraphs satisfying an interior sphere condition was studied in [13] for Lipschitz continuous functions, and in [9] for continuous functions with application to the minimal time function.

Definition 1 Let S be a nonempty and closed subset of $\mathbb{R}^n$ and let $S' \subset \operatorname{bdry} S$. We say that S satisfies a uniform interior sphere condition on S' if there exists $r > 0$ such that for all $x \in S'$, one can find a closed r-ball in S containing x. If S' coincides with $\operatorname{bdry} S$ then we say that S satisfies a uniform interior sphere condition of radius r.

Remark 1 Clearly the following assertions are equivalent:

1. S satisfies a uniform interior sphere condition on S'
2. There exists $r > 0$ such that for all $x \in S'$ one can find a unit vector $\zeta_x \in N^P_{\widetilde{S}}(x)$ realized by an r-ball.
3. There exists $r > 0$ such that for all $x \in S'$ one can find a unit vector ζ_x satisfying

$$\langle \zeta_x, s - x \rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in \widetilde{S}.$$

2.3 Multifunctions and Differentiability

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction mapping $\mathbb{R}^n$ to nonempty subsets of $\mathbb{R}^n$. We denote by $\operatorname{Gr} F$ the graph of F, that is,

$$\operatorname{Gr} F := \{(x, v) : x \in \mathbb{R}^n \text{ and } v \in F(x)\}.$$

We also denote by $\widetilde{F}$ the multifunction, called the complement of F, defined by

$$\widetilde{F}(x) := \widetilde{F(x)} \quad \forall x \in \mathbb{R}^n.$$

We assume throughout this paper that $F(x)$ is closed for all $x \in \mathbb{R}^n$. We also assume that $\operatorname{Gr} F$ and $\operatorname{Gr} \widetilde{F}$ are closed subsets of $\mathbb{R}^{2n}$. Under these hypotheses, one can easily verify that

$$\bigcup_{x \in \mathbb{R}^n} \{x\} \times \operatorname{bdry}(F(x)) = \operatorname{bdry}(\operatorname{Gr} F). \tag{5}$$

For an open set $O \subset \mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$, we define $O_{F(x)}$ to be the set of all boundary points v of $F(x)$ such that (x, v) belongs to O, that is,

$$O_{F(x)} := \{v \in \operatorname{bdry} F(x) : (x, v) \in O\}.$$

Remark 2 For a multifunction F with closed values, a sufficient condition for the closedness of $\operatorname{Gr} F$ and $\operatorname{Gr} \widetilde{F}$, is the continuity of F with the convexity of its values. We note that we cannot drop the convexity assumption as can be shown by the following example: On $\mathbb{R}$, we consider $F(x) := [-|x| - 1, -|x|] \cup [|x|, |x| + 1]$. Clearly F is continuous and $\operatorname{Gr} F$ is closed but $\operatorname{Gr} \widetilde{F}$ is not closed. On the other hand, a multifunction F that has closed values and closed $\operatorname{Gr} F$ and $\operatorname{Gr} \widetilde{F}$, is not necessarily continuous. Indeed, if we consider on $\mathbb{R}$, $F(x) := \{0, |x|^{-1}\}$ if $x \neq 0$ and $\{0\}$ if $x = 0$, then we can easily verify that F is not continuous but $\operatorname{Gr} F$ and $\operatorname{Gr} \widetilde{F}$ are both closed.

Now we introduce the proximal pseudo-differentiability for multifunctions. We note that this notion is an outer one in the sense that it imposes a control on the dilation, and not on the shrinking, of the multifunction F near a given point x_0.

Definition 2 Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction mapping $\mathbb{R}^n$ to nonempty and closed subsets of $\mathbb{R}^n$ and let $(x_0, v_0) \in \mathrm{Gr}\, F$. We say that F is proximally-pseudo-differentiable at x_0 for v_0 if F is lower semicontinuous at (x_0, v_0) and there exist an $n \times n$ matrix A, a constant $c \geq 0$, a neighborhood V of x_0 and a neighborhood W of v_0 such that

$$F(x) \cap W \subset F(x_0) + A.(x - x_0) + c\|x - x_0\|^2 \overline{B} \quad \forall x \in V. \tag{6}$$

The matrix A is called proximal derivative of F at x_0 for v_0. Clearly inclusion (6) can be replaced by the following (for a positive constant ρ)

$$F(x) \cap B(v_0; \rho) \subset F(x_0) + A.(x - x_0) + c\|x - x_0\|^2 \overline{B} \quad \forall x \in B(x_0; \rho). \tag{7}$$

The multifunction F is said to be proximally-pseudo-differentiable at x_0 if it is proximally-pseudo-differentiable at x_0 for v, for all $v \in F(x_0)$. It is proximally-pseudo-differentiable on an open set $O \subset \mathbb{R}^n$ if it is proximally-pseudo-differentiable at x for all $x \in O$.

Remark 3 If we replace in (7) the quadratic term $c\|x - x_0\|^2$ by $\mu(\|x - x_0\|)\|x - x_0\|$ where $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ is a nondecreasing function continuous at 0 with $\mu(0) = 0$ (called modulus), then we obtain the pseudo-differentiability defined by Nachi and Penot in [10]. Therefore our proximal-pseudo-differentiability is a special form of the pseudo-differentiability of [10] obtained by taking a linear modulus $\mu(t) = ct$.

Definition 3 Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction mapping $\mathbb{R}^n$ to nonempty and closed subsets of $\mathbb{R}^n$ and let $O \subset \mathbb{R}^{2n}$ be an open set. We say that F is uniformly proximally-pseudo-differentiable on O if for all $(x, v) \in O \cap \mathrm{Gr}\, F$, F is proximally pseudo-differentiable at x for v with constants c and ρ chosen independently of (x, v). More precisely, F is uniformly proximally-pseudo-differentiable on O if there exist a constant $c \geq 0$ and a constant $\rho > 0$ such that for all $(x, v) \in O \cap \mathrm{Gr}\, F$, F is lower semicontinuous at (x, v) and one can find an $n \times n$ matrix A such that

$$F(y) \cap B(v; \rho) \subset F(x) + A.(y - x) + c\|y - x\|^2 \overline{B} \quad \forall y \in B(x; \rho). \tag{8}$$

If, moreover, there exists $k \geq 0$ such that the derivative A of inclusion (8) can be chosen to be bounded by k, then we say that F is uniformly proximally-pseudo-differentiable on O with k-bounded derivative. The multifunction is uniformly proximally-pseudo-differentiable near a point $(x_0, v_0) \in \mathrm{Gr}\, F$ if it is uniformly proximally-pseudo-differentiable on an open set containing (x_0, v_0).

Let us illustrate Definitions 2 and 3 with some examples.

Example 1 The following are basic examples:

- If F is linear, that is, $F(x) := Ax + U$ where A is an $n \times n$ matrix and $U \subset \mathbb{R}^n$ is a nonempty and closed set, then F is proximally-pseudo-differentiable at x for v, for all $(x, v) \in \mathrm{Gr}\, F$, with A as derivative. Moreover, F is uniformly proximally-pseudo-differentiable on $\mathbb{R}^{2n}$ with $\|A\|$-bounded derivative and with $c = 0$ and ρ any positive number.
- Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function such that $f \in C^{1,1}(\mathbb{R}^n)$. For $U \subset \mathbb{R}^n$ a nonempty and closed set, we define the semilinear multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by $F(x) := f(x) + U$ for all $x \in \mathbb{R}^n$. We can easily verify that F is proximally-pseudo-differentiable on $\mathbb{R}^n$. Moreover, if $f \in C^2(\mathbb{R}^n)$, then F becomes uniformly proximally-pseudo-differentiable on any bounded and open set $O \subset \mathbb{R}^{2n}$ with k-bounded derivative.

- Let F be of the form $F(x) := [f(x), g(x)]$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions with $f(x) < g(x)$ for all $x \in \mathbb{R}$. Then for $x_0 \in \mathbb{R}$, we have:
  - F is proximally-pseudo-differentiable at x_0 for $g(x_0)$ if and only if $\partial^P g(x_0) \neq \emptyset$.
  - F is proximally-pseudo-differentiable at x_0 for $f(x_0)$ if and only if $\partial_P f(x_0) \neq \emptyset$.
  - $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $g(x_0)$ if and only if $\partial_P g(x_0) \neq \emptyset$.
  - $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $f(x_0)$ if and only if $\partial^P f(x_0) \neq \emptyset$.

3 Main Results

The goal of this section is to study the equivalence between the interior sphere condition of the graph of a multifunction F and its pointwise interior sphere condition. We begin by the first direction. More precisely, the following proposition proves that under the proximal pseudo-differentiability of $\widetilde{F}$, if the graph of F satisfies the interior sphere condition at a boundary point (x_0, v_0), then $F(x_0)$ satisfies the interior sphere condition at v_0, with a formula relating the two radii.

Proposition 1 *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction and let $(x_0, v_0) \in \operatorname{bdry}(\operatorname{Gr} F)$. Suppose that:*

(i) $N^P_{\widetilde{\operatorname{Gr} F}}(x_0, v_0) \neq \{0\}$, that is, there exist $r > 0$ and a unit vector $(\zeta, \theta) \in N^P_{\widetilde{\operatorname{Gr} F}}(x_0, v_0)$ realized by an r-ball.
(ii) $\widetilde{F}$ is uniformly proximally-pseudo-differentiable near (x_0, v_0) with k-bounded derivative.

Then the vector $\theta \in N^P_{\widetilde{F}(x_0)}(v_0)$ and it is realized by an $\frac{r}{\sqrt{1+k^2}}$-ball.

Proof By (i) we have that

$$\langle (\zeta, \theta), (x, v) - (x_0, v_0) \rangle \leq \frac{1}{2r} \|(x, v) - (x_0, v_0)\|^2 \quad \forall (x, v) \in \widetilde{\operatorname{Gr} F}. \tag{9}$$

One can easily verify that $v \in \widetilde{F}(x_0)$ implies that $(x_0, v) \in \widetilde{\operatorname{Gr} F}$. Then by (9) we deduce that

$$\langle \theta, v - v_0 \rangle \leq \frac{1}{2r} \|v - v_0\|^2 \quad \forall v \in \widetilde{F}(x_0). \tag{10}$$

If $\|\zeta\| = 0$ then θ becomes a unit vector and inequality (10) yields

$$\left\langle \frac{\theta}{\|\theta\|}, v - v_0 \right\rangle \leq \frac{1}{2r} \|v - v_0\|^2 \leq \frac{1}{2\frac{r}{\sqrt{1+k^2}}} \|v - v_0\|^2 \quad \forall v \in \widetilde{F}(x_0),$$

which is equivalent to θ being a vector in $\in N^P_{\widetilde{F}(x_0)}(v_0)$ realized by an $\frac{r}{\sqrt{1+k^2}}$-ball.
Now we assume that $\|\zeta\| \neq 0$ and we consider the sequence

$$x_i = x_0 + \frac{1}{i} \frac{\zeta}{\|\zeta\|} \quad \forall i \geq 1.$$

Using the lower semicontinuity of $\widetilde{F}$ at (x_0, v_0), we get the existence of a subsequence (we do not relabel) of x_i and a sequence $v_i \in \widetilde{F}(x_i)$ such that $v_i \longrightarrow v_0$. By (ii), there exists a

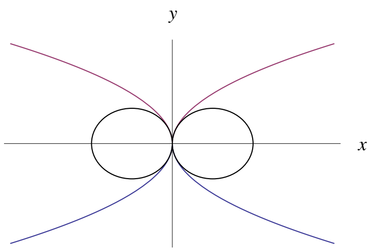

Fig. 1 Example 2

constant $c \geq 0$ and a constant $\rho > 0$ such that for i sufficiently large, one can find an $n \times n$ matrix A_i bounded by k such that

$$\widetilde{F}(x_0) \cap B(v_i;\rho) \subset \widetilde{F}(x_i) + A_i.(x_0 - x_i) + c\|x_0 - x_i\|^2 \overline{B}.$$

This gives the existence of two sequences $v_i' \in \widetilde{F}(x_i)$ and $u_i \in \overline{B}$ such that

$$v_0 = v_i' - \frac{1}{i\|\zeta\|} A_i.\zeta + \frac{c}{i^2} u_i.$$

By applying (9) on (x_i, v_i') we obtain that

$$\frac{\|\zeta\|}{i} + \frac{1}{i\|\zeta\|}\langle\theta, A_i.\zeta\rangle - \frac{c}{i^2}\langle\theta, u_i\rangle \leq \frac{1}{2r}\left(\frac{1}{i^2} + \left(\frac{k}{i} + \frac{c}{i^2}\right)^2\right),$$

and then

$$\|\zeta\| + \frac{1}{\|\zeta\|}\langle\theta, A_i.\zeta\rangle - \frac{c}{i}\langle\theta, u_i\rangle \leq \frac{1}{2ri}\left(1 + \left(k + \frac{c}{i}\right)^2\right).$$

Now since $u_i \in \overline{B}$ and $\|A_i\| \leq k$, we get (after taking $i \longrightarrow \infty$) the existence of a matrix A_0 bounded by k such that

$$\|\zeta\| + \frac{1}{\|\zeta\|}\langle\theta, A_0.\zeta\rangle \leq 0.$$

Hence

$$\|\zeta\| \leq \frac{1}{\|\zeta\|}\langle -\theta, A_0.\zeta\rangle \leq k\|\theta\|.$$

Now using the fact the (ζ, θ) is a unit vector, we obtain that

$$\|\theta\| \geq \frac{1}{\sqrt{1+k^2}}.$$

This inequality combined with (10) will give that

$$\left\langle \frac{\theta}{\|\theta\|}, v - v_0 \right\rangle \leq \frac{1}{2\frac{r}{\sqrt{1+k^2}}}\|v - v_0\|^2 \quad \forall v \in \widetilde{F}(x_0),$$

which terminates the proof. □

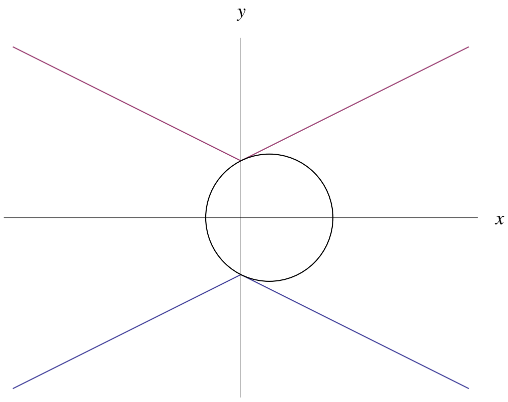

Fig. 2 Examples 3

Example 2 In this example, we will prove that replacing the uniform proximal-pseudo-differentiability of $\widetilde{F}$ near (x_0, v_0) in the preceding proposition, by the proximal-pseudo-differentiability of $\widetilde{F}$ only at x_0 for v_0, will not guarantee that $N^P_{\widetilde{F}(x_0)}(v_0) \neq \{0\}$. In fact, if F is the multifunction defined by (see Fig. 1):

$$F(x) := \left[-\sqrt{|x|}, \sqrt{|x|}\right] \text{ for all } x \in \mathbb{R},$$

then one can easily verify that:

- $N^P_{\widetilde{\mathrm{Gr}F}}(0,0) = \{(t,0) : t \in \mathbb{R}\} \neq \{0\}$.
- $\tilde{F}$ is proximally-pseudo-differentiable at 0 for 0, but it is not uniformly proximally-pseudo-differentiable near $(0,0)$ with k-bounded derivative.
- $N^P_{\widetilde{F}(0)}(0) = \{0\}$.

Example 3 Now we will show that the radius $r/\sqrt{1+k^2}$, obtained in Proposition 1, can be the best radius for a given point $(x_0, v_0) \in \mathrm{bdry}\,(\mathrm{Gr}F)$. Indeed, for $a \geq 0$ and $b > 0$ let $F(x) := [-a|x| - b, a|x| + b]$ for all $x \in \mathbb{R}$, see Fig. 2. For $x_0 \geq 0$ we have:

- $(a, -1) \in N^P_{\widetilde{\mathrm{Gr}F}}(x_0, ax_0 + b)$ and it is realized by an r-ball where $r = (ax_0 + b)\sqrt{1+a^2}$.
- The multifunction $\tilde{F}$ is uniformly proximally-pseudo-differentiable near $(x_0, ax_0 + b)$ with a-bounded derivative.
- By Proposition 1, $-1 \in N^P_{\widetilde{F}(x_0)}(ax_0 + b)$ and it is realized by an $r/\sqrt{1+a^2}$-ball, where $r/\sqrt{1+a^2} = ax_0 + b$ is the maximum possible radius.

A direct consequence of Proposition 1 is the following local result.

Theorem 1 *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multifunction and let $O \subset \mathbb{R}^{2n}$ be an open set. Suppose that:*

(i) There exists $r > 0$ such that $\mathrm{Gr}F$ *satisfies a uniform interior sphere condition of radius r on $O \cap$* $\mathrm{bdry}\,(\mathrm{Gr}F)$.
(ii) $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on O with k-bounded derivative.

Then for all $x \in \mathbb{R}^n$*,* $F(x)$ *satisfies a uniform interior sphere condition of radius* $\frac{r}{\sqrt{1+k^2}}$ *on* $O_{F(x)}$*.*

Proof Let $x_0 \in \mathbb{R}^n$ and let $v_0 \in O_{F(x_0)}$. We have $(x_0, v_0) \in O \cap \operatorname{bdry}(\operatorname{Gr} F)$ and hence by applying (i), we get the existence of a unit vector (ζ, θ) in $N^P_{\widetilde{\operatorname{Gr} F}}(x_0, v_0)$ realized by an r-ball. On the other hand, (ii) gives that $\widetilde{F}$ is uniformly proximally-pseudo-differentiable near (x_0, v_0) with k-bounded derivative. Then the conditions of the preceding proposition are satisfied and this gives that $\theta \in N^P_{\widetilde{F}(x_0)}(v_0)$ and it is realized by an $r/\sqrt{1+k^2}$-ball. □

We proceed to study the converse implication. The following proposition proves that also under the proximal pseudo-differentiability of $\widetilde{F}$, if $F(x_0)$ satisfies the interior sphere condition at a boundary point v_0 then the graph of F satisfies the interior sphere condition at (x_0, v_0) with formulas relating the two normals and the two radii. Note that for an $n \times n$ matrix A, A^T will denote the transpose of A.

Proposition 2 *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be a multifunction and let* $x_0, v_0 \in \mathbb{R}^n$ *with* $v_0 \in \operatorname{bdry} F(x_0)$*. Suppose that:*

(i) $N^P_{\widetilde{F}(x_0)}(v_0) \neq \{0\}$*, that is, there exist* $r > 0$ *and a unit vector* $\zeta \in N^P_{\widetilde{F}(x_0)}(v_0)$ *realized by an* r*-ball.*
(ii) $\widetilde{F}$ *is proximally-pseudo-differentiable at* x_0 *for* v_0*, that is,* $\widetilde{F}$ *is lower semicontinuous at* (x_0, v_0) *and there exist an* $n \times n$ *matrix* A*, a constant* $c \geq 0$*, and a constant* $\rho > 0$ *such that*

$$\widetilde{F}(x) \cap B(v_0; \rho) \subset \widetilde{F}(x_0) + A.(x - x_0) + c\|x - x_0\|^2 \overline{B} \quad \forall x \in B(x_0; \rho).$$

Then the vector $(-A^T.\zeta, \zeta) \in N^P_{\widetilde{\operatorname{Gr} F}}(x_0, v_0)$ *and it is realized by an* r'*-ball with*

$$r' := \begin{cases} \frac{1}{2}\min\{\rho, 2r\} & \text{if } \|A\| = c = 0, \\ \frac{1}{2}\min\{\rho, t_0, r\sqrt{1 + \|A^T.\zeta\|^2}\} & \text{if } \|A\| \neq 0 \text{ or } c \neq 0, \end{cases}$$

where t_0 *is the unique positive real root of the equation:*

$$t = \frac{r\sqrt{1 + \|A^T.\zeta\|^2}}{cr + (\|A\| + ct)^2}.$$

Proof By Remark 1, it is sufficient to prove that

$$\left\langle \frac{(-A^T.\zeta, \zeta)}{\|(-A^T.\zeta, \zeta)\|}, (x, v) - (x_0, v_0) \right\rangle \leq \frac{1}{2r'}\|(x, v) - (x_0, v_0)\|^2 \quad \forall (x, v) \in (\operatorname{Gr} F)^c.$$

Let $(x, v) \in (\operatorname{Gr} F)^c$. Then $v \in F(x)^c \subset \widetilde{F}(x)$.

Case 1: $(x, v) \notin B((x_0, v_0); 2r')$.

Then

$$\left\langle \frac{(-A^T.\zeta, \zeta)}{\|(-A^T.\zeta, \zeta)\|}, (x, v) - (x_0, v_0) \right\rangle \leq \|(x, v) - (x_0, v_0)\| \leq \frac{1}{2r'}\|(x, v) - (x_0, v_0)\|^2.$$

Case 2: $(x,v) \in B((x_0,v_0);2r')$.

Since $2r' \le \rho$, we get that $x \in B(x_0;\rho)$ and $v \in \widetilde{F}(x) \cap B(v_0;\rho)$. Hence by (ii), there exists $v' \in \widetilde{F}(x_0)$ and $u \in \bar{B}$ such that

$$v = v' + A.(x-x_0) + c\|x-x_0\|^2 u.$$

From (i) we have that

$$\langle \zeta, w-v_0 \rangle \le \frac{1}{2r}\|w-v_0\|^2 \quad \forall w \in \widetilde{F}(x_0).$$

Then since $v' \in \widetilde{F}(x_0)$ we get that

$$\langle \zeta, v'-v_0 \rangle \le \frac{1}{2r}\|v'-v_0\|^2.$$

We have

$$\begin{aligned}
\big\langle (-A^T.\zeta,\zeta),(x,v)-(x_0,v_0)\big\rangle &= \\
\big\langle -A^T.\zeta, x-x_0\rangle + \langle \zeta, v-v_0\big\rangle &= \\
\langle -\zeta, A.(x-x_0)\rangle + \langle \zeta, v'+A.(x-x_0)+c\|x-x_0\|^2u - v_0\rangle &= \\
\langle \zeta, v'+c\|x-x_0\|^2u-v_0\rangle &= \\
\langle \zeta, v'-v_0\rangle + c\|x-x_0\|^2\langle \zeta,u\rangle &\le \\
\frac{1}{2r}\|v'-v_0\|^2 + c\|x-x_0\|^2 &\le \qquad (11)\\
\frac{1}{r}\|v'-v\|^2 + \frac{1}{r}\|v-v_0\|^2 + c\|x-x_0\|^2 &= \\
\frac{1}{r}\|A.(x-x_0)+c\|x-x_0\|^2u\|^2 + \frac{1}{r}\|v-v_0\|^2 + c\|x-x_0\|^2 &\le \\
\frac{1}{r}\left((\|A\|+c\|x-x_0\|)^2 + rc\right)\|x-x_0\|^2 + \frac{1}{r}\|v-v_0\|^2. & \qquad (12)
\end{aligned}$$

Case 2.1: $\|A\| = c = 0$.

Then $v' = v$ and hence inequality (11) yields

$$\begin{aligned}
\big\langle (-A^T.\zeta,\zeta),(x,v)-(x_0,v_0)\big\rangle &\le \frac{1}{2r}\|v-v_0\|^2 \\
&\le \frac{1}{2r}(\|x-x_0\|^2 + \|v-v_0\|^2).
\end{aligned}$$

Now since $\|((-A^T.\zeta,\zeta)\| = \|(0,\zeta)\| = 1$, we obtain that

$$\begin{aligned}
\left\langle \frac{(-A^T.\zeta,\zeta)}{\|(-A^T.\zeta,\zeta)\|},(x,v)-(x_0,v_0)\right\rangle &\le \frac{1}{2r}(\|x-x_0\|^2+\|v-v_0\|^2) \\
&\le \frac{1}{2r'}\|(x,v)-(x_0,v_0)\|^2.
\end{aligned}$$

Case 2.2: $\|A\| \neq 0$ or $c \neq 0$.

Then $\|x-x_0\| \leq \|(x-x_0, v-v_0)\| \leq 2r' \leq t_0$. Now using (12) we get that

$$\begin{aligned}\left\langle(-A^T.\zeta,\zeta),(x,v)-(x_0,v_0)\right\rangle \leq \\ \frac{1}{r}\left((\|A\|+ct_0)^2+rc\right)\|x-x_0\|^2+\frac{1}{r}\|v-v_0\|^2 = \\ \frac{\sqrt{1+\|A^T.\zeta\|^2}}{t_0}\|x-x_0\|^2+\frac{1}{r}\|v-v_0\|^2 \leq \\ \frac{\sqrt{1+\|A^T.\zeta\|^2}}{2r'}\|(x,v)-(x_0,v_0)\|^2.\end{aligned}$$

Since $\|((-A^T.\zeta,\zeta)\| = \sqrt{1+\|A^T.\zeta\|^2}$, we obtain that

$$\left\langle \frac{(-A^T.\zeta,\zeta)}{\|(-A^T.\zeta,\zeta)\|},(x,v)-(x_0,v_0)\right\rangle \leq \frac{1}{2r'}\|(x,v)-(x_0,v_0)\|^2.$$

This completes the proof. □

Example 4 We will prove in this example that without the proximal-pseudo-differentiability of $\widetilde{F}$ at x_0 for v_0, the proximal normal cone to $\widetilde{\mathrm{Gr}F}$ at (x_0,v_0) can be equal to $\{0\}$. Indeed, for $F(x) := [-|x|,-|x|+2]$ for all $x \in \mathrm{I\!R}$, we have:

- $N^P_{\widetilde{F}(0)}(2) = \{t : t \leq 0\} \neq \{0\}$.
- $\widetilde{F}$ is not proximally-pseudo-differentiable at 0 for 2.
- $N^P_{\widetilde{\mathrm{Gr}F}}(0,2) = \{0\}$.

The preceding proposition yields to the following local result.

Theorem 2 *Let $F : \mathrm{I\!R}^n \rightrightarrows \mathrm{I\!R}^n$ be a multifunction and let $O \subset \mathrm{I\!R}^{2n}$ be an open set. Suppose that:*

(i) There exists $r > 0$ such that for all $x \in \mathrm{I\!R}^n$, $F(x)$ satisfies a uniform interior sphere condition of radius r on $O_{F(x)}$.
(ii) $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on O with k-bounded derivative.

Then $\mathrm{Gr}F$ *satisfies a uniform interior sphere condition on* $O \cap \mathrm{bdry}\,(\mathrm{Gr}F)$.

Proof Let $(x_0,v_0) \in O \cap \mathrm{bdry}\,(\mathrm{Gr}F)$. Then $x_0 \in \mathrm{I\!R}^n$ and $v_0 \in O_{F(x_0)}$. By (i), we get the existence of a unit vector $\zeta \in \mathrm{I\!R}^n$ such that:

$$\langle \zeta, v-v_0\rangle \leq \frac{1}{2r}\|v-v_0\|^2 \quad \forall v \in \widetilde{F}(x_0).$$

On the other hand, (ii) gives the existence of a constant $c \geq 0$, a constant $k \geq 0$ and a constant $\rho > 0$ such that for all $(x,v) \in O \cap \mathrm{Gr}\widetilde{F}$, $\widetilde{F}$ is lower semicontinuous at (x,v) and one can find an $n \times n$ matrix A bounded by k such that

$$\widetilde{F}(y) \cap B(v;\rho) \subset \widetilde{F}(x) + A.(y-x) + c\|y-x\|^2\overline{B} \quad \forall y \in B(x;\rho).$$

Then $\widetilde{F}$ is lower semicontinuous at (x_0,v_0) and there exists an $n \times n$ matrix A bounded by k such that

$$\widetilde{F}(y) \cap B(v_0;\rho) \subset \widetilde{F}(x_0) + A.(y-x_0) + c\|y-x_0\|^2\overline{B} \quad \forall y \in B(x_0;\rho).$$

Now by applying Proposition 2 we get that $(-A^T.\zeta,\zeta)\in N^P_{\widetilde{\mathrm{Gr}F}}(x_0,v_0)$ and it is realized by an r'-ball with

$$r' := \begin{cases} \frac{1}{2}\min\{\rho,2r\} & \text{if } \|A\| = c = 0,\\ \frac{1}{2}\min\{\rho,t_0,r\sqrt{1+\|A^T.\zeta\|^2}\} & \text{if } \|A\|\neq 0 \;\; or \;\; c\neq 0,\end{cases}$$

where t_0 is the unique positive real root of the equation:

$$t=\frac{r\sqrt{1+\|A^T.\zeta\|^2}}{cr+(\|A\|+ct)^2}.$$

Let t_1 be the unique positive real root of the equation:

$$t=\frac{r}{cr+(k+ct)^2}.$$

Using the facts that $\sqrt{1+\|A^T.\zeta\|^2}\geq 1$ and $\|A\|\leq k$, we can easily prove, by contradiction, that $t_1\leq t_0$. Then for

$$r'' := \begin{cases} \frac{1}{2}\min\{\rho,2r\} & \text{if } k = c = 0,\\ \frac{1}{2}\min\{\rho,t_1,r\} & \text{if } k\neq 0 \text{ or } c\neq 0,\end{cases}$$

we have that $r''\leq r'$. This gives that the vector $(-A^T.\zeta,\zeta)$ is also realized by an r''-ball. Since the constants r, ρ, c and k are independent from the choice of (x_0,v_0), we get that $\mathrm{Gr}F$ satisfies a uniform interior sphere condition on $O\cap\mathrm{bdry}\,(\mathrm{Gr}F)$ of radius r''. □

Now we combine Theorem 1 and Theorem 2 to obtain the following theorem which asserts that under the proximal pseudo-differentiability of $\widetilde{F}$ there is an equivalence between the pointwise interior sphere condition of F and the interior sphere condition of its graph.

Theorem 3 *Let $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be a multifunction and let $O\subset\mathbb{R}^{2n}$ be an open set. Suppose that $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on O with k-bounded derivative. Then $\mathrm{Gr}F$ satisfies a uniform interior sphere condition on $O\cap\mathrm{bdry}\,(\mathrm{Gr}F)$ if and only if there exists $r>0$ such that for all $x\in\mathbb{R}^n$, $F(x)$ satisfies an interior sphere condition of radius r on $O_{F(x)}$.*

We terminate this section by providing some examples to which Proposition 2 and Theorem 2 can be applied, in order to deduce the interior sphere condition of the graph of a multifunction with a calculation of the radius r'.

Example 5

1. Let $F(x):=Ax+U$ where where A is an $n\times n$ matrix and $U\subset\mathbb{R}^n$ is a nonempty and closed set satisfying a uniform interior sphere condition of radius r. Then $F(x)$ satisfies an interior sphere condition of radius r for all $x\in\mathbb{R}^n$. Since $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on $\mathbb{R}^{2n}$ with $\|A\|$-bounded derivative and with $c=0$ and ρ any positive number, we get from the preceding theorem (and its proof) that $\mathrm{Gr}F$ satisfies a uniform interior sphere condition of radius

$$r' = \begin{cases} r & \text{if } A = 0,\\ \frac{1}{2}\min\left\{\frac{r}{\|A\|^2},r\right\} & \text{if } A\neq 0.\end{cases}$$

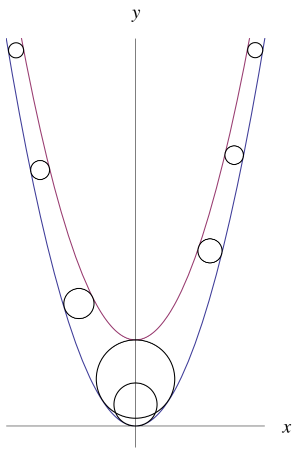

Fig. 3 Example 5.4

2. Let $F(x) := f(x) + U$ for all $x \in \mathbb{R}^n$, where f is in $C^2(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ is a nonempty and closed set satisfying a uniform interior sphere condition of radius r. Then $F(x)$ satisfies an interior sphere condition of radius r, for all $x \in \mathbb{R}^n$. Since $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on any bounded and open set $O \subset \mathbb{R}^{2n}$ with k-bounded derivative, we deduce from Theorem 2 that for any bounded and open set $O \subset \mathbb{R}^{2n}$, $\mathrm{Gr}\,F$ satisfies a uniform interior sphere condition on $O \cap \mathrm{bdry}\,(\mathrm{Gr}\,F)$ (in this case we say that $\widetilde{\mathrm{Gr}\,F}$ is *φ-convex*, that is, $\mathrm{Gr}\,F$ satisfies an interior sphere condition with locally bounded radius, see [6, 13]).
3. For $k \neq 0$, let $F(x) := \sin kx + [0,2]$ for all $x \in \mathbb{R}$. Clearly $F(x)$ satisfies an interior sphere condition of radius 1, for all $x \in \mathbb{R}$. On the other hand, $\widetilde{F}$ is uniformly proximally-pseudo-differentiable on $\mathbb{R}^2$ with k-bounded derivative and with $c = k^2/2$ and ρ any positive number. Then by the preceding theorem, $\mathrm{Gr}\,F$ satisfies a uniform interior sphere condition of radius r', where $2r'$ is the unique positive root of the third order equation

$$k^4 t^3 + 4k^3 t^2 + 6k^2 t - 4 = 0.$$

4. Let $F(x) := x^2 + [0,2]$ for all $x \in \mathbb{R}$. Clearly $F(x)$ satisfies an interior sphere condition of radius 1, for all $x \in \mathbb{R}$. We fix $x_0 \in \mathbb{R}$.
   - $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $v_0 := x_0^2 + 2$ with $A = 2x_0$, $c = 0$ and ρ any positive number for which the ball $B(v_0), \rho)$ is contained in the epigraph of the function $f(x) = x^2$ over $B(x_0, \rho)$. The maximum value of ρ is

$$\rho = \frac{1}{2}\left(\sqrt{(2|x_0|+1)^2 + 8} - 2|x_0| - 1\right).$$

   On the other hand, $-1 \in N^P_{\widetilde{F}(x_0)}(v_0)$ and it is realized by a 1-ball. By applying Proposition 2, we get that the vector $(2x_0, -1) \in N^P_{\widetilde{\mathrm{Gr}\,F}}(x_0, v_0)$ and it is realized by an r'-ball with

$$r' = \begin{cases} \dfrac{1}{2} & \text{if } x_0 = 0, \\[2ex] \dfrac{1}{2}\min\left\{\rho, \dfrac{\sqrt{1+4x_0^2}}{4x_0^2}\right\} & \text{if } x_0 \neq 0. \end{cases}$$

- $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $v_0 := x_0^2$ with $A = 2x_0$, $c = 1$ and ρ any positive number. On the other hand, $1 \in N^P_{\widetilde{F}(x_0)}(v_0)$ and it is realized by a 1-ball. By applying Proposition 2, we get that the vector $(-2x_0, 1) \in N^P_{\mathrm{Gr}\widetilde{F}}(x_0, v_0)$ and it is realized by an r'-ball with
$$r' = \frac{t_0}{2},$$
where t_0 is the unique positive root of the following equation:
$$t = \frac{\sqrt{1+4x_0^2}}{1+(2|x_0|+t)^2}.$$

From both formulas of r', we have that $\lim_{x_0 \longrightarrow \pm\infty} r' = 0$. Then the graph of F does not satisfy a uniform interior sphere condition. In fact, it satisfies an interior sphere condition with locally bounded radius. This can be predicted graphically, see Fig. 3.

5. We consider the multifunction:
$$F(x) := \frac{x^3}{3} + \left[0, x^2 + x + 2\right] \text{ for all } x \in \mathbb{R}.$$
Clearly $F(x)$ satisfies an interior sphere condition of radius $\frac{x^2+x+2}{2}$, for all $x \in \mathbb{R}$. Since $\frac{x^2+x+2}{2} \geq \frac{7}{8}$, we get that $F(x)$ satisfies an interior sphere condition of radius $\frac{7}{8}$, for all $x \in \mathbb{R}$. We fix $x_0 \in \mathbb{R}$.
- $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $v_0 := \frac{x_0^3}{3}$ with
$$\rho = 1,\ A = x_0^2 \text{ and } c = \max\{0, x_0 + 1\},$$
where ρ is chosen in such a way that the ball $B(v_0), \rho)$ is contained in the hypograph of the function $g(x) = \frac{x^3}{3} + x^2 + x + 2$ over $B(x_0, \rho)$. On the other hand, $1 \in N^P_{\widetilde{F}(x_0)}(v_0)$ and it is realized by a $\frac{1}{2}(x_0^2 + x_0 + 2)$-ball. By applying Proposition 2, we get that the vector $(-x_0^2, 1) \in N^P_{\mathrm{Gr}\widetilde{F}}(x_0, v_0)$ and it is realized by an r'-ball with
$$r' = \frac{1}{2}\min\left\{1, t_0, \left(\frac{x_0^2 + x_0 + 2}{2}\right)\sqrt{1+x_0^4}\right\}, \tag{13}$$
where t_0 is the unique positive real root of the equation:
$$t = \frac{(x_0^2 + x_0 + 2)\sqrt{1+x_0^4}}{c(x_0^2 + x_0 + 2) + 2(x_0^2 + ct)^2}.$$

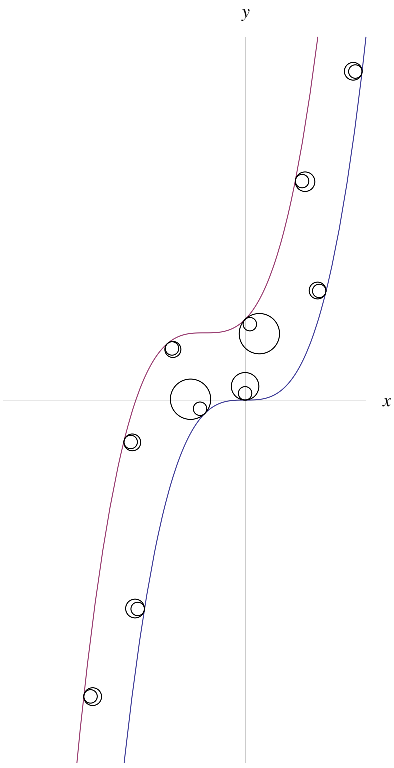

Fig. 4 Example 5.5

Case 1: $x_0 \leq -1$. Then $c = 0$ and $t_0 = \frac{(x_0^2+x_0+2)\sqrt{1+x_0^4}}{2x_0^4}$. By (13) we get that

$$r' = \frac{1}{2}\min\{1, t_0\}.$$

We can easily prove that $r' \geq \frac{1}{6}$.

Case 2: $x_0 > -1$. Then $c = x_0 + 1$ and t_0 becomes the unique positive real root of the equation:

$$t = \frac{(x_0^2+x_0+2)\sqrt{1+x_0^4}}{(x_0+1)(x_0^2+x_0+2)+2(x_0^2+(x_0+1)t)^2}.$$

We can easily prove that $t_0 \geq \frac{1}{3}$. Moreover,

$$\left(\frac{x_0^2+x_0+2}{2}\right)\sqrt{1+x_0^4} \geq \frac{x_0^2+x_0+2}{\sqrt{2}} \geq \frac{7}{4\sqrt{2}}.$$

Then $r' \geq \frac{1}{6}$.

- $\widetilde{F}$ is proximally-pseudo-differentiable at x_0 for $v_0 := \frac{x_0^3}{3} + x_0^2 + x_0 + 2$ with

$\rho = 1$, $A = x_0^2 + 2x_0 + 1 = (x_0+1)^2$ and $c = -\min\{0, (x_0-\rho)+1)\} = -\min\{0, x_0\}$, where ρ is chosen in such a way that the ball $B(v_0),\rho)$ is contained in the epigraph of the function $f(x)=\frac{x^3}{3}$ over $B(x_0,\rho)$. On the other hand, $-1\in N^P_{\tilde{F}(x_0)}(v_0)$ and it is realized by an $\frac{1}{2}(x_0^2+x_0+2)$-ball. Proposition 2 gives that the vector $(x_0^2+2x_0+1,-1)$ belongs to $N^P_{\widetilde{\mathrm{Gr}F}}(x_0,v_0)$ and it is realized by an r'-ball with

$$r'=\frac{1}{2}\min\left\{1,t_0,\left(\frac{x_0^2+x_0+2}{2}\right)\sqrt{1+(x_0+1)^4}\right\},$$

where t_0 is the unique positive real root of the equation:

$$t=\frac{(x_0^2+x_0+2)\sqrt{1+(x_0+1)^4}}{c(x_0^2+x_0+2)+2((x_0+1)^2+ct)^2}.$$

As above and using two cases ($x_0<0$ and $x_0\geq 0$), we can prove that $r'\geq\frac{1}{6}$.

Therefore, $\mathrm{Gr}F$ satisfies a uniform interior sphere condition of radius $\frac{1}{6}$. This can be verified in Fig. 4 where the largest balls are of radius r' and the smallest balls are of radius $\frac{1}{6}$.

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