COMPLEXITIES OF SPECIAL MATRIX MULTIPLICATION PROBLEMS

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Abstract—This paper develops optimal algorithms to multiply an \( n \times n \) symmetric tridiagonal matrix by:

(i) an arbitrary \( n \times m \) matrix using \( 2nm - m \) multiplications;

(ii) a symmetric tridiagonal matrix using \( 6n - 7 \) multiplications; and

(iii) a tridiagonal matrix using \( 7n - 8 \) multiplications. Efficient algorithms are also developed to multiply a tridiagonal matrix by an arbitrary matrix, and to multiply two tridiagonal matrices.

1. INTRODUCTION

Let \( K \) be a ring and let \( x = (x_1, \ldots, x_p)^T \) and \( y = (y_1, \ldots, y_q)^T \) be two column vectors of indeterminates. A system \( B = \{B_1, \ldots, B_m\} \) of bilinear forms is given by

\[
B_i = \sum_{j=1}^p \sum_{k=1}^q a_{jk}x_jy_k = x^T G_i y,
\]

where \( G_i \) is a \( p \times q \) matrix with elements in \( K \). The complexity \( \delta(B) \) of the set of bilinear forms is the smallest number of nonscalar multiplications required to compute the \( B_i \)s. One can show that \( \delta \) is equal to the smallest number of rank one matrices necessary to include the \( G_i \)s in their span [1].

The problem can be reformulated by introducing the characteristic matrix

\[
G(s) = \sum_{i=1}^m s_i G_i,
\]

where \( \{s_i\} \) is a set of new indeterminates. We denote by \( \delta(G(s)) \) the complexity of the associated set of bilinear forms. If \( P, Q \) and \( R \) are respectively \( p \times p, q \times q \) and \( m \times m \) nonsingular matrices, then \( \delta(PG(Rs)Q) = \delta(G(s)) \).

One of the main open problems in algebraic complexity theory is the matrix multiplication problem: given an \( m \times n \) matrix \( X = \{x_{ij}\} \) and an \( n \times p \) matrix \( Y = \{y_{jk}\} \), we want to compute their product, an \( m \times p \) matrix whose entries are the bilinear forms

\[
z_{ik} = \sum_{j=1}^n x_{ij}y_{jk}.
\]

Except for the trivial cases, the only known optimal algorithms correspond to the multiplication of \( 2 \times 2 \) by \( 2 \times n \) matrices using \( \lceil 7n/2 \rceil \) multiplications, over \( Z \) or \( Z_2 \) [2]. In this paper we attack several special cases. In the next section, we develop optimal algorithms to multiply a symmetric tridiagonal matrix respectively by an arbitrary matrix, a symmetric tridiagonal matrix and a tridiagonal matrix. In Section 3, we find elegant and efficient algorithms to multiply a tridiagonal matrix by an arbitrary matrix and to multiply two tridiagonal matrices.

Before closing this section we state one theorem from [1] which will be used frequently.

**Theorem 1.1** [1]

Let \( K = F \) be a field and let \( G(s) \) be a proper characteristic matrix. Then, for

\[
G(s) = [G_1(s) ; G_2(s)], \quad G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} \quad \text{or} \quad G(s) = G_1(u) + G_2(v),
\]

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we have respectively
\[ \delta[G_1(s) : G_2(s)] \geq \min_M \delta[G_1(s) + G_2(s)M] + \text{column rank } (G_2), \]  
(1)
\[ \delta \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} \geq \min_N \delta[G_1(s) + NG_2(s)] + \text{row rank } (G_2), \]  
(2)
\[ \delta(G_1(u) + G_2(v)) \geq \min_j \delta(G_1(u) + G_2(Tu)) + \text{dim } v. \]  
(3)

2. SYMMETRIC TRIDIAGONAL MATRIX MULTIPLICATION PROBLEM

Let \( X \) be an \( n \times n \) symmetric tridiagonal matrix, i.e.
\[
X = \begin{bmatrix}
  x_1 & x_2 & & \\
  x_2 & x_3 & x_4 & \\
  & x_4 & x_5 & \\
  & & \ddots & \ddots \\
  & & & x_{2n-4} & x_{2n-3} & x_{2n-2} \\
  & & & & x_{2n-2} & x_{2n-1}
\end{bmatrix}
\]

First, consider the problem of multiplying \( X \) by an arbitrary \( n \)-dimensional vector
\[
y = \begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
\]

One equivalent form of the characteristic matrix is \( X \) itself. Let \( E_{ij} \) be the \( n \times n \) matrix that consists of 1 in the \((i,j)\) entry and 0 elsewhere. Then
\[
G(x) = X = \sum_{i=1}^{n} x_{2i-1} E_{ii} + \sum_{i=1}^{n-1} x_{2i} (E_{i,i+1} + E_{i+1,i})
\]
\[
= \sum_{i=1}^{n} x_{2i-1} E_{ii} + \sum_{i=1}^{n-1} x_{2i} (E_{i,i+1} + E_{i+1,i} + E_{ii} + E_{i+1,i+1})
\]
\[
- \sum_{i=1}^{n-1} x_{2i} (E_{ii} + E_{i+1,i+1}) = D + \sum_{i=1}^{n-1} x_{2i} M_{ii},
\]
where \( M_{ii+1} = E_{ii+1} + E_{i+1,i} + E_{ii} + E_{i+1,i+1} \) is a rank one matrix and \( D \) is a diagonal matrix. Hence, \( \delta(G(x)) \leq n - 1 + \delta(D) \leq n - 1 + n = 2n - 1 \). On the other hand, \( \text{dim } x = 2n - 1 \). Hence we have the following lemma.

**Lemma 2.1**

The complexity of multiplying an \( n \times n \) symmetric tridiagonal matrix by an arbitrary vector is \( 2n - 1 \), over any ring.

**Corollary 2.1**

The complexity \( \delta_{nm} \) of multiplying an \( n \times n \) symmetric tridiagonal matrix by an arbitrary \( n \times m \) matrix satisfies
\[
\delta_{nm} \leq (2n - 1)m.
\]
Theorem 2.2

The complexity $\delta_{nm}$ of multiplying an $n \times n$ symmetric tridiagonal matrix by an arbitrary $n \times m$ matrix is equal to $(2n - 1)m$.

Proof. We have to show that $(2n - 1)m$ is a lower bound. The characteristic matrix is:

$$G_{nm}(s) = \begin{bmatrix}
S_1 & \cdots & S_m \\
S_{m+1} & \cdots & S_{2m} \\
S_{2m+1} & \cdots & S_{3m} \\
\vdots & & \ddots \\
S_{nm-m+1} & \cdots & S_{nm-2m} \\
S_{nm-2m+1} & \cdots & S_{nm-m} \\
S_{nm-m} & \cdots & S_{nm-m} \\
S_{nm-1} & \cdots & S_{nm}
\end{bmatrix}$$

Note that $G_{n-1,m}(s)$ is obtained from $G_{nm}(s)$ by deleting the last two rows and the last $m$ columns. Using Theorem 1.1, we get $\delta(G_{nm}(s)) \geq m + \delta(G(s))$ where $G(s)$ is obtained from $G_{nm}(s)$ by deleting the last $m$ columns and adding linear combinations of them to the remaining ones. In $G(s)$ replace $s_{nm-m+1}, s_{nm-m+2}, \ldots, s_{nm}$ by linear combinations of the remaining ones and delete the last 2 rows, we get $\delta(G(s)) \geq m + \delta(G_{n-1,m}(s))$. Therefore, $\delta(G_{nm}(s)) \geq 2m + \delta(G_{n-1,m}(s))$, and this is true $\forall n \geq 1$. Hence $\delta(G_{nm}(s)) \geq 2m(n - 1) + \delta(G_{1m}(s))$. But $G_{1m}(s) = [s_1, s_2, \ldots, s_m]$, therefore $\delta(G_{1m}(s)) = m$, hence $\delta(G_{nm}(s)) \geq (2n - 1)m$.

Corollary 2.2

The complexity of multiplying an $n \times n$ symmetric tridiagonal matrix by an arbitrary $n \times n$ matrix is

$$\delta(G(s)) = 2n^2 - n.$$

Secondly, consider now the problem of multiplying two $n \times n$ symmetric tridiagonal matrices. Let us start with the case when $n = 2$:

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix}.$$

The characteristic matrix is

$$G_2(s) = \begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 + s_4 \end{bmatrix}.$$

Lemma 2.3

$\delta(G_2(s)) = 5$, over any field of characteristic $\neq 2$.

Proof.

$$\delta(G_2(s)) \geq 1 + \delta \begin{bmatrix} s_1 & s_2 + \alpha s_3 \\ s_3 & s_4 + s_1 + \beta s_3 \end{bmatrix} = 1 + \delta \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}^{s_3} = 5.$$
For the upper bound, consider the following five rank-one matrices:

\[
\begin{align*}
\mathbf{n}_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, & \mathbf{n}_2 &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{n}_3 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \mathbf{n}_4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \\
\mathbf{n}_5 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.
\end{align*}
\]

Then

\[
G_2(s) = \frac{1}{2}(\mathbf{n}_2 + \mathbf{n}_3)s_1 + (\frac{1}{2}\mathbf{n}_3 - \frac{1}{2}\mathbf{n}_2 - \mathbf{n}_1)s_2 + (\frac{1}{2}\mathbf{n}_3 - \frac{1}{2}\mathbf{n}_4 + \mathbf{n}_1)s_3 + \frac{1}{2}(\mathbf{n}_4 + \mathbf{n}_5)s_4.
\]

Therefore \( \delta = 5 \).

Before we attack the general case, we need the following lemma.

**Lemma 2.4**

If

\[
H(s) = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_4 \\ s_3 & s_4 & s_5 \end{bmatrix}
\]

then \( \delta(H(s)) = 5 \).

**Proof.** \( \delta(H(s)) \geq 5 \) because \( \dim s = 5 \). For the upper bound consider the following five rank-one matrices:

\[
\begin{align*}
\mathbf{n}_0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, & \mathbf{n}_2 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{n}_3 &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}, & \mathbf{n}_4 &= \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \\
\mathbf{n}_5 &= \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.
\end{align*}
\]

Then

\[
H(s) = (\mathbf{n}_1 + \mathbf{n}_2 + \frac{1}{2}\mathbf{n}_3 + \frac{1}{2}\mathbf{n}_4)s_1 + (\mathbf{n}_1 + \mathbf{n}_2 + \frac{1}{2}\mathbf{n}_3 + \frac{1}{2}\mathbf{n}_5)s_2 + \mathbf{n}_2s_3 + (\mathbf{n}_1 + \mathbf{n}_2 + \frac{1}{2}\mathbf{n}_4 + \frac{1}{2}\mathbf{n}_5)s_4.
\]

Therefore \( \delta = 5 \).

**Theorem 2.5**

The complexity of multiplying two \( n \times n \) symmetric tridiagonal matrices is \( 6n - 7 \), over any field of characteristic \( \neq 2 \).
Proof. The characteristic matrix is:

\[
\begin{bmatrix}
\beta_1 & \beta_2 \\
\beta_4 & \beta_1 + \beta_5 & \beta_2 & \beta_3 \\
\beta_4 & \beta_5 & \beta_6 \\
\beta_8 & \beta_6 & \beta_1 + \beta_{10} & \beta_9 & \beta_7 \\
\beta_9 & \beta_{13} & & & \\
\end{bmatrix}
\]

\[
G_n(s) =
\begin{bmatrix}
\beta_5 - \beta_8 & \beta_9 & \beta_7 & \beta_6 & \beta_5 \\
\beta_8 & \beta_1 + \beta_{10} & \beta_9 & \beta_{13} & & \\
\beta_4 & \beta_5 & \beta_6 & & & \\
\beta_1 & \beta_2 & \beta_3 & & & \\
\end{bmatrix}
\]

The characteristic matrix is:

\[
\begin{bmatrix}
\delta_{15} & \delta_{10} & \delta_{13} \\
\delta_{11} & \delta_{10} & \delta_9 \\
\delta_8 & \delta_9 & \delta_{10} + \delta_6 & \delta_9 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\end{bmatrix}
\]

\[
G_n(s) =
\begin{bmatrix}
\delta_{15} & \delta_{10} & \delta_{13} \\
\delta_{11} & \delta_{10} & \delta_9 \\
\delta_8 & \delta_9 & \delta_{10} + \delta_6 & \delta_9 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\end{bmatrix}
\]

\[
H_n(s) =
\begin{bmatrix}
\delta_{15} & \delta_{10} & \delta_{13} \\
\delta_{11} & \delta_{10} & \delta_9 \\
\delta_8 & \delta_9 & \delta_{10} + \delta_6 & \delta_9 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\delta_7 & \delta_6 & \delta_5 & \delta_4 & \delta_3 & \delta_2 & \delta_1 \\
\end{bmatrix}
\]

\[
K_n(s) =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \delta_2 - \delta_4 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_9 - \delta_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{14} - \delta_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{14} - \delta_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{14} - \delta_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{14} - \delta_{11} \\
\end{bmatrix}
\]

n blocks = \{2 \times 2\} blocks and
n - 2 blocks.
Then $G_\nu(s) = H_\nu(s) + K_\nu(s)$. $K_\nu(s)$ has $n - 1$ nonzero rows, each of them can be generated by one rank-one matrix. Therefore $\delta(K_\nu(s)) = n - 1$.

On the other hand,

$$\delta(H_\nu(s)) \leq \delta \begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix} + \delta \begin{bmatrix} s_3 & s_4 & s_5 \\ s_4 & s_5 & s_6 \\ s_5 & s_6 & s_7 \end{bmatrix} + \cdots + \delta \begin{bmatrix} s_{5n-10} & s_{5n-11} & s_{5n-13} \\ s_{5n-11} & s_{5n-10} & s_{5n-9} \\ s_{5n-10} & s_{5n-9} & s_{5n-8} \end{bmatrix} + \begin{bmatrix} s_{5n-6} & s_{5n-7} \\ s_{5n-7} & s_{5n-6} \end{bmatrix},$$

$$\delta \begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix} = 2,$$

in fact,

$$\begin{bmatrix} s_1 & s_2 \\ s_2 & s_1 \end{bmatrix} = \frac{1}{2}(n_1 + n_2)s_1 + \frac{1}{2}(n_1 - n_2)s_2,$$

where

$$n_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } n_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Similarly,

$$\delta \begin{bmatrix} s_{5n-6} & s_{5n-7} \\ s_{5n-7} & s_{5n-6} \end{bmatrix} = 2.$$

By Lemma 2.4, the complexity of each of the remaining $3 \times 3$ blocks is equal to 5. Therefore, $\delta(H_\nu(s)) \leq 4 + 5(n - 2) = 5n - 6$. Hence $\delta(G_\nu(s)) \leq n - 1 + 5n - 6 = 6n - 7$.

For the lower bound, note that

$$G_\nu(s) = \begin{bmatrix} \ldots & 0 \\ \ldots & \cdots & \cdots \\ \ldots & \cdots & \ldots \\ 0 \ldots 0 & s_{5n-13} & s_{5n-12} \\ 0 \ldots 0 & 0 & s_{5n-9} \\ 0 \ldots 0 & 0 & 0 \end{bmatrix} G_{n-1}(s),$$

where $G_{n-1}(s)$ is the characteristic matrix of multiplying two $(n - 1) \times (n - 1)$ symmetric tridiagonal matrices, and

$$\mathcal{G} = s - \{s_{5n-13}, s_{5n-9}, s_{5n-8}, s_{5n-7}, s_{5n-6}\}.$$

Add linear combinations of the last column to the remaining ones, and use Theorem 1.1, then delete the last row, we get

$$\delta(G_\nu(s)) \geq 1 + \delta \begin{bmatrix} \ldots & 0 \\ \ldots & \cdots & \cdots \\ \ldots & \cdots & \ldots \\ 0 \ldots 0 & s_{5n-8} + \alpha_3 s_{5n-9} & s_{5n-7} + \alpha_2 s_{5n-9} \\ 0 \ldots 0 & 0 & s_{5n-9} \end{bmatrix} G_{n-1}(s),$$

Replace $s_{5n-6}, s_{5n-7}, s_{5n-8}, s_{5n-9}$ and $s_{5n-13}$ by linear combinations of the remaining ones, then delete the last row and last column, we get:

$$\delta(G_\nu(s)) \geq 6 + \delta(G_{n-1}(\mathcal{G})).$$

and this is true $\forall n \geq 2$. Therefore,

$$\delta(G_\nu(s)) \geq 6(n - 2) + \delta(G_{2}(s)).$$

By Lemma 2.3, $\delta(G_2(s)) = 5$, hence, $\delta(G_\nu(s)) \geq 6n - 7$. 
Thirdly, let $Y$ be a tridiagonal matrix, i.e.

$$Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \\ \vdots & \vdots & \vdots \\ y_{3n-6} & y_{3n-5} & y_{3n-4} \\ y_{3n-3} & y_{3n-2} \end{bmatrix}.$$ 

The characteristic matrix of multiplying the symmetric tridiagonal matrix $X$ by $Y$ is

$$G_n(s) = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & \cdots & s_{3n-15} & s_{3n-14} & s_{3n-13} \\ s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & \cdots & s_{3n-14} & s_{3n-13} & s_{3n-12} \\ s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & \cdots & s_{3n-12} & s_{3n-11} & s_{3n-10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{3n-6} & s_{3n-5} & s_{3n-4} & s_{3n-3} & s_{3n-2} & s_{3n-1} & s_{3n} & \cdots & s_{3n-13} & s_{3n-12} & s_{3n-11} \end{bmatrix}.$$ 

Lemma 2.6

If

$$H = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

then $\delta(H) = 3$.

Proof. Notice that $H = (n_1 + n_2)s_1 + (n_2 + n_3)s_2$, where

$$n_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad n_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad n_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Therefore, $\delta(H) \leq 3$. On the other hand, $\delta(H) \geq 3$ because row rank $(H) = 3$.

Lemma 2.7

If

$$H = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_2 & s_1 & s_4 \\ s_3 & s_4 & s_5 \end{bmatrix}$$

then $\delta(H) = 5$. 
Proof. Consider the following five rank-one matrices:

\[
\begin{align*}
  n_1 &= \begin{bmatrix}
  1 & 0 & 0 \\
  -1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}, &
  n_2 &= \begin{bmatrix}
  0 & 0 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}, \\
  n_3 &= \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 0
\end{bmatrix}, &
  n_4 &= \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}, \\
  n_5 &= \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix},
\end{align*}
\]

and

\[
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}.
\]

Then \( H = (n_1 + n_2)s_1 + (n_2 + n_3 + n_4)s_2 + (n_4 + n_5)s_3 \). Therefore \( \delta(H) \leq 5 \). But row rank \( \delta(H) = 5 \), therefore \( \delta(H) = 5 \).

Theorem 2.8

The complexity of multiplying a symmetric tridiagonal matrix by a tridiagonal matrix is \( 7n - 8 \) over any ring.

Proof. For the upper bound, notice that \( G_n(s) \) can be decomposed into:

(a) \( n - 2 \) blocks of the form

\[
\begin{bmatrix}
  s_2 \\
  s_5 & s_2 \\
  & s_5 \\
  & & s_9 & s_5 \\
  & & & s_9
\end{bmatrix},
\]

(b) the 2 blocks

\[
\begin{bmatrix}
  s_1 \\
  s_4 & s_1 \\
  s_4
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  s_{5n-9} \\
  s_{5n-6} & s_{5n-9} \\
  & s_{5n-6}
\end{bmatrix},
\]

(c) the \( 2(n - 2) \) \( 1 \times 1 \) blocks

\( s_3, s_7, s_9, \ldots, s_{5n - 13}, s_{5n - 12}, s_{5n - 8} \).

Using the previous two lemmas, we get

\[
\delta(G_n(s)) \leq 5(n - 2) + 6 + 2(n - 2) = 7n - 8.
\]
For the lower bound, notice that $G_{n-1}(s)$ is obtained from $G_n(s)$ by deleting the last two rows and the last three columns:

$$G_n(s) = \begin{bmatrix}
G_{n-1}(s) & s_{5n-13} & s_{5n-9} \\
& s_{5n-8} & s_{5n-7} & s_{5n-6} & s_{5n-10} & s_{5n-9} \\
& & s_{5n-7} & s_{5n-6}
\end{bmatrix},$$

Delete the last two columns and add linear combinations to the remaining ones, we get

$$\delta(G_n(s)) \geq 2 + \delta(G_{n-1}(s)).$$

Replace $s_{5n-13}, s_{5n-9}, s_{5n-8}, s_{5n-7}$ and $s_{5n-6}$ by linear combinations of the remaining indeterminates, then delete the last two rows and the last column of $\bar{G}_n(s)$, we get

$$\delta(G_n(s)) \geq 7 + \delta(G_{n-1}(s))$$

and this is true $\forall n \geq 3$. Hence $\delta(G_n(s)) \geq 7(n-2) + \delta(G_2(s))$, where

$$G_2(s) = \begin{bmatrix} s_1 & s_2 \\
& s_3 & s_4 & s_5 \\
& & s_5 & s_4 \\
\end{bmatrix}.$$

But

$$\delta(G_2(s)) \geq 2 + \delta\begin{bmatrix} s_1 \\
s_3 + \alpha_1 s_1 + \beta_1 s_2 \\
\alpha_1 s_3 + \beta_1 s_4 \\
\end{bmatrix} \geq 2 + \delta\begin{bmatrix} s_1 & s_2 \\
& s_3 & s_4 \\
\end{bmatrix} = 6.$$

Therefore

$$\delta(G_n(s)) \geq 7(n-2) + 6 = 7n - 8.$$

### 3. TRIDIAGONAL MATRIX MULTIPLICATION PROBLEM

Let $X$ be a tridiagonal matrix, i.e.

$$X = \begin{bmatrix} x_1 & x_3 \\
x_2 & x_4 & x_6 \\
x_5 & x_7 & x_9 \\
\cdot & \cdot & \cdot \\
x_{3m-5} & x_{3m-3} \\
x_{3m-4} & x_{3m-2}
\end{bmatrix}.$$
First consider the problem of multiplying $X$ by an arbitrary $n \times n$ matrix. Let us start with the case when $n = 4$. The characteristic matrix is

$$G_4(s) = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & s_8 \\ s_9 & s_{10} & s_{11} & s_{12} \\ s_{13} & s_{14} & s_{15} & s_{16} \end{bmatrix}$$

**Lemma 3.1**

If

$$H(s) = \begin{bmatrix} 0 & s_2 \\ s_3 & s_4 \\ s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}$$

then $\delta(H(s)) = 6$, over any ring.

*Proof.* Consider the following rank-one matrices:

$$n_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad n_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad n_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$n_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad n_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad n_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}.$$

Then

$$H(s) = n_1 s_1 + (n_1 + n_2 + n_3 + n_6) s_2 + (n_1 + n_3 + n_4 + n_6) s_3 + (n_1 + n_4 + n_5 + n_6) s_4.$$ 

Therefore $\delta \leq 6$. On the other hand,$$
\delta \geq 2 + \delta \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = 6.$$

**Lemma 3.2**

If

$$\hat{H}(s) = \begin{bmatrix} s_3 & s_4 + s_2 \\ s_2 & s_1 & s_2 \\ -s_2 & s_3 & s_4 \end{bmatrix}$$

then $\delta(\hat{H}(s)) = 5.$
Proof. From the proof of Lemma 3.1, one can see that
\[ \delta(H(s)) = \delta(H(s) - n_2 s_2). \]
But
\[ H(s) - n_2 s_2 = n_1 s_1 + (n_1 + n_5 + n_6) s_2 + (n_1 + n_3 + n_4 + n_6) s_3 + (n_1 + n_3 + n_5 + n_6) s_4. \]
Therefore, \( \delta(H(s)) = 5. \)

Lemma 3.3

If
\[ K(s) = \begin{bmatrix}
  s_1 & s_2 \\
  s_3 & s_4 \\
  s_1 & s_2 \\
  s_4 & s_7 \\
  s_6 & s_7 \\
  s_4 & s_5 \\
  s_6 & s_7
\end{bmatrix} \]
then \( \delta(K(s)) = 11. \)

Proof.

By the previous lemma, each block can be generated by five rank-one matrices. Therefore, \( \delta \leq 11. \)

Using Theorem 1.1, one can easily show that
\[ \delta \geq 3 + \delta \]
Replacing \( s_1, s_2, s_3, s_5, s_6, s_7 \) by multiples of \( s_4 \) and using Theorem 1.1, we get
\[ \delta \geq 9 + \delta \geq 11. \]

Theorem 3.4

The complexity of multiplying a \( 4 \times 4 \) tridiagonal matrix by an arbitrary \( 4 \times 4 \) matrix satisfies
\[ \delta(G_4(s)) \leq 34. \]
Proof. It is enough to notice that $G_s(s)$ can be decomposed into two blocks of the same form as in Lemma 3.3, and two blocks of the same form as in Lemma 3.1.

Corollary 3.4.1

If $n = 4m$ then the complexity of multiplying a $4 \times 4$ tridiagonal matrix by an arbitrary $4 \times n$ matrix satisfies

$$\delta(G_s(s)) \leq \frac{12}{7} n.$$ 

Corollary 3.4.2

Assume that $n = 4m$ and let

\[
\mathbf{X} = \begin{bmatrix}
    x_1 & x_3 \\
    x_2 & x_4 & x_6 \\
    x_5 & x_7 & x_9 \\
    x_8 & x_{10} & 0 \\
    0 & x_{13} & x_{15} \\
    x_{14} & x_{16} & x_{18} \\
    x_{17} & x_{19} & x_{21} \\
    x_{20} & x_{22} & 0 \\
    0 & \ldots & \ldots & \ldots
\end{bmatrix}
\]

Then the complexity of multiplying $\mathbf{X}$ by an arbitrary $n \times n$ matrix satisfies

$$\delta \leq \frac{12}{7} n^2.$$ 

Corollary 3.4.3

If $n = 4m$ then the complexity of multiplying an $n \times n$ tridiagonal matrix by an arbitrary $n \times n$ matrix satisfies

$$\delta(G_s(s)) \leq \frac{21}{8} n^2 - 2n.$$ 

Corollary 3.4.4

$$\delta(G_s(s)) \leq \frac{21}{8} n^2 + O(n).$$ 

Remark 1

By Theorem 2.2, we have

$$\delta(G_n(s)) \geq 2n^2 - n,$$

over any ring.

Remark 2

One can easily show that

$$\delta(G_n(s)) \geq 2n^2 - \left\lfloor \frac{n}{2} \right\rfloor,$$

over $\mathcal{F}$ or $\mathcal{F}_2$. 

Secondly, consider now the problem of multiplying two $n \times n$ tridiagonal matrices. The characteristic matrix is:

$$H_n(s) = \begin{bmatrix}
    s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
    s_8 & s_9 & s_{10} & s_1 & s_2 & s_3 \\
    s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\
    s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    s_{5n-13} & s_{5n-14} & s_{5n-15} & s_{5n-16} & s_{5n-17} & s_{5n-18} \\
    s_{5n-8} & s_{5n-9} & s_{5n-10} & s_{5n-11} & s_{5n-12} & s_{5n-13} \\
    s_{5n-3} & s_{5n-4} & s_{5n-5} & s_{5n-6} & s_{5n-7} & s_{5n-8} \\
    s_{5n-2} & s_{5n-3} & s_{5n-4} & s_{5n-5} & s_{5n-6} & s_{5n-7} \\
    s_{5n-1} & s_{5n-2} & s_{5n-3} & s_{5n-4} & s_{5n-5} & s_{5n-6} \\
    s_{5n} & s_{5n-1} & s_{5n-2} & s_{5n-3} & s_{5n-4} & s_{5n-5} \\
\end{bmatrix}$$

By Lemma 3.1, each of the $n - 1$ blocks can be generated by six rank-one matrices. The remaining terms can be generated by $2n - 3$ rank-one matrices. Therefore,

$$\delta(H_n(s)) \leq 6(n - 1) + 2n - 3 = 8n - 9.$$ 

Using the same arguments as in the proof of Theorem 2.8, one can show that

$$\delta(H_n(s)) \geq 7(n - 2) + \delta \begin{bmatrix}
    s_1 & s_2 \\
    s_4 & s_5 \\
    s_1 & s_2 \\
    s_4 & s_5 \\
\end{bmatrix} = 7n - 7.$$ 

Let us summarize the above results in the following theorem.

**Theorem 3.5**

The complexity of multiplying two $n \times n$ tridiagonal matrices satisfies

$$7n - 7 \leq \delta(H_n(s)) \leq 8n - 9,$$

over any ring.

**REFERENCES**