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TITLE PAGE

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Ring Foundation on Elastic Sub-grade: an Analytical Solution for Computer Modelling using the Lagrangian Multiplier Method

ABSTRACT

In the practice of geotechnical engineering, the case of a ring footing carrying a set of concentrated point loads is a common problem. At times, the induced vertical and angular displacements for the ring footing need to be evaluated at a relatively precise level. By making use of the governing set of equations derived for the case of a general curved beam, expressions that can be easily implemented in modern computing software are derived for the vertical and angular displacements of a ring footing of rectangular cross section as functions of the radial position. The loading case considered is a vertical point load, and the soil is modelled as elastic. Estimates of the displacements have been shown for a common range of practical applications. The behavior for a set of concentrated loads may be evaluated using the derived equations through direct superposition. Nonlinear finite-element analysis is used to evaluate the vertical deflection and angular twist of the ring foundation. Numerical analysis performed for three ring foundations with different radii and cross-sections is reported to validate the accuracy of the derived analytical solution.

Keywords: Analytical Techniques, Finite Element Modelling, Numerical Analysis, Substructures, Elastic Foundations, Geotechnical Mechanics, Deflection.

1. Introduction and Background

In the practice of geotechnical engineering the problem of supporting superstructures such as—among others—elevated tanks, containers, and dish shaped radar antennas by means of a ring footing frequently arises. In addition to uniform distributed loading, the ring may be subject to concentrated column loads as in the case of elevated circular tanks and antenna towers. The problem of curved train tracks under moving wheel loads is similarly formulated and is of interest to engineers. At times when the ring foundation is supporting critical infrastructure facilities, the magnitudes of its vertical displacements and angular twist need to be predicted with a more rigorous approach than is usually done.

Lee et al. [1] studied the dynamic behavior of an Euler beam traversed by a moving concentrated mass; he analyzed it for the general case of a mass moving with a varying speed. The equation of motion, in a matrix form, was formulated using the Lagrangian approach and the as assumed mode method. Amiri and Onyango [2] obtained the response of a simply supported beam on elastic foundation to repeated moving concentrated loads by means of the Fourier sine transformation. The study examined the cases of the response of the beam to loads of different and equal magnitudes.

Amiri and Esmaeily [3] obtained the response of a resilient- end-supported beam on an elastic sub-grade to a moving concentrated load using the method of the harmonic series analysis. The study used numerical examples in order to demonstrate the effects of various parameters on the dynamic response of the beam. From the solutions obtained, within the ranges of parameters considered, it was concluded that for the case of simply-supported beams the series solutions converge very rapidly. It was also shown that an increase in the elastic foundation parameter results in the decrease of beam deflections—with the maximum dynamic deflections occurring under the load.

Amiri and Esmaeily [4] investigated the traveling wave response of an unlimited beam on an elastic sub-grade to a moving body with a spatial distribution of mass. The study considered the two cases of neglecting the axial and friction forces; and including a constant axial force while neglecting the friction force. The pertinent differential equations of motion were derived, and the solutions obtained using the traveling wave phenomenon. The deflection curves and deflections under the load, as well as dynamic coefficients and the effects of neglecting various acceleration components on the deflection curves were presented. In addition, the study explored the effects of the critical velocity of the beam on the aforesaid cases and parameters.

The first attempt at analyzing the deflection of circular beams resting on elastic foundation and loaded by symmetric concentrated forces acting in a plane perpendicular to the plane of the original curvature of the beam was done by Volterra [5]. The study demonstrated that the foundation reacts following the classical Winkler and Zimmerman hypothesis, i.e., the reaction forces due to the foundation are proportional at every point to the deflection of the beam at that point. Arici et al. [6] performed Hamiltonian structural analysis of curved beams with or without generalized two-parameter foundation. A solution was presented for the curved Timoshenko beams with or without generalized two-parameter elastic foundation. The solution was obtained by the Hamiltonian structural analysis method, based on an energetic approach, solving a mixed canonical Hamiltonian system of twelve differential equations, leading to the fundamental matrix. The solution was numerically expressed and is shown to be fast and simple to implement on a computer.

Arici and Granata [7] found a solution of space curved bars with generalized Winkler soil found by means of the Transfer Matrix Method. Distributed loads, concentrated loads, and imposed strains were applied to the beam as well as considering rigid or elastic boundaries at the ends. The proposed approach gives the analytical and numerical exact solutions for circular beams and rings loaded in the plane or perpendicular to it. Veletsos and Tang [8,9] studied the vertical vibration of rigid ring foundations under dynamic loads.

Since late nineteenth century, geotechnical engineers and researchers have been challenged by the problem of modelling foundation-ground interaction. Because of the complexity of soil behavior, subgrade in soil-foundation interaction problems is replaced by a much simpler system called subgrade model. One of the most common and simple models developed and used in this context is the Winkler hypothesis. Winkler idealization represents the soil medium as a system of identical but mutually independent, closely spaced, discrete and linearly elastic springs. The ratio between contact pressure, P, at any given point and settlement, y, produced by it at that point, is given by the coefficient of subgrade reaction, k_s [10]. Also known as the 'elastic foundation beam method', the Winkler method proposed in 1867, is still however very much used in engineering design for its simplicity. For a beam of finite length on an elastic foundation with point load at one end, the analytical solution can be deduced based on the Winkler method. These analytical solutions can be used to analyze the response of a foundation with constant subgrade reactions [11].

Yang et al. [12] derived an analytical solution for a horizontal curved beam subjected to vertical and horizontal moving loads. The horizontal moving loads were simulating the effect of the centrifugal forces generated by vehicles moving along a curved path. Both a single moving load and a series of equidistant moving loads were considered. The case of uniform distributed loads commonly present in ring foundation due to the uniformly-applied soil pressure however remained unexamined. More recently Kharazi et al. [13] and Ovesy et al. [14] used a similar analytical derivation using the Rayleigh–Ritz method with displacements represented by a series of polynomials to predict the buckling and post buckling response of beams.

The forgoing literature is by no means an exhaustive survey of published work on this topic. It however shows some of the important studies conducted on this subject matter. The objective of this study is to obtain an analytical solution for the problem of ring foundations resting on elastic Sub-grades using the Lagrangian multiplier method. The results of the derived analytical solution are validated numerically using finite element analysis (FEA). The comparison shows good agreement between the results obtained using the two different approaches.

2. General Analytical Evaluation of a Curved Beam

The general representation of a curved beam of constant cross section is shown in Figure 1. The depicted beam is subjected to a distributed load (q) acting perpendicular to the centroidal axis of the beam, and a distributed torsional (twisting) moment (t) acting in the plane normal to the centroidal axis of the beam. Denoting vertical displacements by $w(s)$ and angular displacements by $\varphi(s)$, a set of prescribed vertical and angular displacements along the centroidal axis will be referred to as $\overline{w}(s)$ and $\overline{\varphi}(s)$, respectively.

In this study, the term of 'subgrade reaction' refers to the pressure distribution which is the result or 'reaction' of the 'subgrade' to a load imposed upon the top of the foundation structure. In practice, the foundation structure is commonly a reinforced concrete one, and the 'subgrade' usually refers to the soil or rock upon which the structure is constructed. In the present work, Winkler model was used for conceptualizing the subgrade reaction, as seen in Figure 1 (a).

[Figure 1]

Now, considering the minimum complementary potential energy for the particular system: for a prescribed set of $\overline{w}(s)$ and $\overline{\varphi}(s)$, the true combination of M (bending moment), T (torsional moment), V (shear), t (acting distributed torsional moment) and q (acting distributed vertical load) is the one which satisfies the equilibrium and minimizes the potential energy. Writing the complementary potential energy, π and considering the equilibrium conditions:

a) In the vertical direction,

$$\begin{aligned} qds + (V + dV) &= V \\ qds + dV &= 0 \end{aligned} \tag{1}$$

b) Taking moments about the axis $1-1$ in Figure 1

$$M-(M+dM)+T\frac{ds}{R}+Vds=0\rightarrow dM-Vds-T\frac{ds}{R}=0$$
$$\rightarrow\frac{dM}{ds}-V-\frac{T}{R}=0 \tag{2}$$

where $\frac{ds}{R}=d\theta$

In Equation 2, the moments due to t and q are of a higher order and are therefore neglected.

c) Taking moments about the axis $2-2$ in Figure 1

$$T-(T+dT)-tds-Md\theta=0\rightarrow dT+tds+M\frac{ds}{R}=0$$
$$\frac{dT}{ds}+t+\frac{M}{R}=0 \tag{3}$$

In Equation 3, the moment due to q is of a higher order and is therefore neglected. Eliminating V in Equations 1 and 2 given that it does not appear in the complementary potential energy expression, and differentiating Equation 2 with respect to s:

$$M''-\left(\frac{T}{R}\right)'-V'=0$$

Substituting through Equation 1

$$M''-\left(\frac{T}{R}\right)'+q=0 \tag{4}$$

Then Equations 4 and 3 state the constraint conditions X_1 and X_2, respectively. The complementary potential energy will now be minimized subjected to the conditions. Considering $\phi = f-\lambda g$, and introducing the Lagrangian multipliers, λ_1 and λ_2:

$$\phi=\frac{1}{2}\int\left(\frac{M^2}{EI}+\frac{T^2}{GJ}\right)ds-\int\left(q\overline{w}+t\overline{\varphi}\right)ds-\int\left(\lambda_1X_1+\lambda_2X_2\right)ds \tag{5}$$

In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints. Minimizing with respect to M:

$$\frac{M}{EI}-\frac{\lambda_2}{R}+\frac{d^2}{ds^2}(-\lambda_1)=0 \tag{6}$$

Minimizing with respect to T, Equation 7 is obtained:

$$\frac{T}{GJ} - \frac{\lambda_1}{R^2} - \frac{d}{ds}\left(\frac{\lambda_1}{R} - \lambda_2\right) = 0 \tag{7}$$

Minimizing with respect to q, Equation 8 is obtained:

$$-w - \lambda_1 = 0, \quad w = -\lambda_1 \tag{8}$$

Minimizing with respect to t:

$$-\phi - \lambda_2 = 0, \quad \phi = -\lambda_2 \tag{9}$$

Substituting the above into Equation 6:

$$\frac{M}{EI} = -\left(\frac{\varphi}{R} + w''\right) \tag{10}$$

And further substituting into Equation 7:

$$\frac{T}{GJ} = -\frac{w'}{R} + \phi' \tag{11}$$

Now, substituting Equation10 and Equation11 into Equation 3 and Equation 4:

$$\left\{-EI\left(\frac{\varphi}{R} + w''\right)\right\}'' - \left\{\frac{GJ}{R}\left(-\frac{w'}{R} + \varphi'\right)\right\}' + q = 0 \tag{12}$$

$$\left\{GJ\left(-\frac{w'}{R} + \varphi'\right)\right\}' - \frac{EI}{R}\left(w'' + \frac{\varphi}{R}\right) + t = 0 \tag{13}$$

where Equations 12 and 13 are the set of equations for the case of a general curved beam.

3. Analytical Evaluation of a Ring Footing

A ring footing which is subjected to a concentrated force i.e., point load acting normal to the centroidal axis of the footing will now be considered as shown in Figure 2. This particular ring footing has a constant radius of R.

[Figure 2]

Let $w(\theta)$ and $\varphi(\theta)$ be expressed in terms of the Fourier cosine series, which satisfies both symmetry and boundary conditions:

$$w = \sum_{n=0}^{\infty} a_n \cos n\theta \qquad (14)$$

$$\varphi = \sum_{n=0}^{\infty} b_n \cos n\theta \qquad (15)$$

In developing the general equations of a curved beam (Equations 12 and 13), the external loading has been considered as “distributed” whereas in this particular problem a “point” load is treated. In addition, in the consideration of external loading, the reaction of the elastic subgrade is to be included. In converting the point load F into a mathematically compatible equivalent distributed loading, use of the Dirac Delta Function will be made. Thus, writing for F :

$$F = \int_{-\pi}^{\pi} P\delta(\theta) R d\theta \qquad (16)$$

where:

$\delta(\theta) = 0$ for $\theta \neq 0$

$\delta(\theta) = \infty$ for $\theta = 0$ and,

$$\int_{-\pi}^{\pi} \delta(\theta) d\theta = 1$$

which also implies:

$$F = PR \text{ or } P = \frac{F}{R} \qquad (16a)$$

Establishing $q(\theta)$ for a unit length of the circular strip footing:

$$q(\theta) = -kw(\theta) + P\delta(\theta) \qquad (17)$$

where k is the elastic modulus of the subgrade reaction:

Now, considering $\delta(\theta)$ in terms of a Fourier series:

$\delta(\theta) = \sum_{n=0}^{\infty} C_n \cos n\theta$, and making use of the properties of the Fourier series:

$$\cos m\theta.\delta(\theta) = \sum_{n=0}^{\infty} C_n \cos n\theta.\cos m\theta$$

$$\int_{-\pi}^{\pi} \cos m\theta . \delta(\theta) d\theta = \sum_{n=0}^{\infty} C_n \int_{-\pi}^{\pi} \cos n\theta . \cos m\theta . d\theta \qquad \text{(A)}$$

Now, if $n \neq m$

$$\sum_{n=0}^{\infty} C_n \int_{-\pi}^{\pi} \cos n\theta . \cos m\theta d\theta = \sum_{n=0}^{\infty} \frac{C_n}{2} \left[\frac{\sin(m+n)\theta}{(m+n)} + \frac{\sin(m-n)}{(m-n)} \right]_{-\pi}^{\pi}$$

which is equal to zero, and if $m = n$

$$\sum_{n=0}^{\infty} C_m \int_{-\pi}^{\pi} (\cos m\theta)^2 d\theta = C_m \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2m\theta \right) d\theta, \qquad \text{(B)}$$

or $\sum_{n=0}^{\infty} C_m \int_{-\pi}^{\pi} (\cos m\theta)^2 d\theta = C_m . \pi$ for $m \neq 0$

Thus,

$$C_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos m\theta . \delta(\theta) d\theta$$

which implies:

$C_m = \frac{1}{\pi} \cos m(0)$, since $\int_{-\pi}^{\pi} f(\theta) . \delta(\theta) d\theta = f(0)$

Therefore

$C_m = \frac{1}{\pi}$, for all m except $m = 0$

For $m = 0$ from Equations (A) and (B),

$\int_{-\pi}^{\pi} \cos(0.\theta) \delta(\theta) d\theta = C_0 \int_{-\pi}^{\pi} d\theta = 2C_0 \pi$, or

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\theta) d\theta = \frac{1}{2\pi} \qquad \text{(C)}$$

summarizing:

For $n = m$ and $m \neq 0$, $C_m = \frac{1}{\pi}$, and

For $n = m$ and $m = 0, C_0 = \frac{1}{2\pi}$

Hence:

$$\delta(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi} \cos n\theta \qquad \text{(D)}$$

With Equation 14 incorporated, Equation 17 can now be rewritten as:

$$q(\theta) = \left(-ka_0 + \frac{P}{2\pi}\right) + \sum_{n=1}^{\infty} \left(-ka_n + \frac{P}{\pi}\right) \cos n\theta \qquad (18)$$

Recalling that Equations 12 and 13 have been developed in terms of $w(s)$ and $\varphi(s)$ rather than $w(\theta)$ and $\varphi(\theta)$:

$$ds = Rd\theta, \quad \frac{d}{ds} = \frac{1}{R}\frac{d}{d\theta}, \quad \frac{d^2}{ds^2} = \frac{1}{R^2}\frac{d^2}{d\theta^2}$$

Therefore

$$w(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta \qquad (14)$$

$$w'(s) = \frac{1}{R} \sum_{n=0}^{\infty} (-na_n \sin n\theta) \qquad (14.1)$$

$$w''(s) = \frac{1}{R^2} \sum_{n=0}^{\infty} (-n^2 a_n \cos n\theta) \qquad (14.2)$$

and similarly:

$$\varphi(\theta) = \sum_{n=0}^{\infty} b_n \cos n\theta \qquad (15)$$

$$\varphi'(s) = \frac{1}{R} \sum_{n=0}^{\infty} (-nb_n \sin n\theta) \qquad (15.1)$$

$$\varphi''(s) = \frac{1}{R^2} \sum_{n=0}^{\infty} (-n^2 b_n \cos n\theta) \qquad (15.2)$$

Substituting Equations 14 (14.1, 14.2), 15 (15.1, 15.2) and 18 into Equation 12, Equation 19 can be obtained.

$$-EI\left[\sum_{n=0}^{\infty}\left(\frac{b_n}{R}-\frac{n^2a_n}{R^2}\right)\cos n\theta\right]'' -\frac{GJ}{R}\left[\sum_{n=0}^{\infty}\left(\frac{na_n}{R^2}-\frac{b_n n}{R}\right)\sin n\theta\right]' +\sum_{n=0}^{\infty}\left(-ka_n+PC_n\right)\cos n\theta=0 \quad (19)$$

Note that in substitutingEquation18 into Equation12, it is considered that the terms ka_0 and $\left(\frac{P}{2\pi}\right)$ are of the same magnitude, and thus $\left(-ka_0+\frac{P}{2\pi}=0\right)$. This will be verified at a later stage in Equation 23.1.

Performing simple mathematical operations, Equation 19 can be rewritten as:

$$\frac{EI}{R^2}\sum_{n=0}^{\infty}n^2\left(\frac{b_n}{R}-\frac{n^2a_n}{R^2}\right)\cos n\theta-\frac{GJ}{R}\sum_{n=0}^{\infty}n\left(\frac{na_n}{R^2}-\frac{nb_n}{R}\right)\cos n\theta+\sum_{n=0}^{\infty}(-ka_n+PC_n)\cos n\theta=0 \quad (19.1)$$

$$\left[\frac{EI}{R^3}n^2+\frac{GJ}{R^3}n^2\right]b_n-\left[n^4\frac{EI}{R^4}+\frac{GJ}{R^4}n^2+k\right]a_n+PC_n=0 \quad (19.2)$$

Now, considering: $t=k'\varphi(\theta)$, where (t) is the elastic reaction of the subgrade against the twisting of the circular strip footing, and k' is the elastic modulus of the subgrade reaction for this mode of displacement. For the strip footing of a rectangular cross section, k and k' are related as:

$k'=k.\frac{b^3}{12}$, where (b) is the width of the cross section. Simultaneously to Equation 15, it can be written that:

$$t=k'\sum_{n=0}^{\infty}b_n\cos n\theta \quad (20)$$

Substituting Equations 14, (14.1, 14.2), 15 (15.1, 15.2) and 20 into Equation 13:

$$GJ\left[\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{na_n}{R}-nb_n\right)\sin n\theta\right]' -\frac{EI}{R}\left[\frac{1}{R}\sum_{n=0}^{\infty}\left(-\frac{n^2a_n}{R}+b_n\right)\cos n\theta\right]+k'\sum_{n=0}^{\infty}b_n\cos n\theta=0 \quad (21)$$

Performing mathematical operations, Equation 20 can now be expressed as:

$$\frac{GJ}{R^2}\sum_{n=0}^{\infty}n^2\left(\frac{a_n}{R}-b_n\right)\cos n\theta+\frac{EI}{R^2}\left(\frac{a_n n^2}{R}-b_n\right)\cos n\theta+k'\sum_{n=0}^{\infty}b_n\cos n\theta=0, \quad (21.1)$$

and grouping terms:

$$\left(\frac{GJ}{R^2}n^2+\frac{EI}{R^2}-k'\right)b_n-\left(\frac{GJn^2}{R^3}+\frac{EI}{R^3}n^2\right)a_n=0 \quad (21.2)$$

Solving Equations 19.2, 21.2 for a_n and b_n, Equations 21 and 22 are obtained:

$$b_n = \frac{(GJ + EI)n^2}{R\left(GJn^2 + EI - R^2k'\right)} a_n, \tag{22}$$

and

$$a_n = \frac{PC_nR^4}{\left(n^4EI + GJn^2 + kR^4\right) - \frac{(EI + GJ)^2 n^4}{\left(GJn^2 + EI - R^2k'\right)}} \tag{23}$$

where, for

$$n = 0,\ C_n = \frac{1}{2\pi} \text{ and } a_0 = \frac{P}{2nk} \tag{23.1}$$

and for $n \geq 1, C_n = \dfrac{1}{\pi}$ and:

$$a_n = \frac{R^4P\left(\frac{1}{\pi}\right)}{R^4k + \frac{n^2\left(n^2 - 1\right)GEJI - n^2R^2k'\left(n^2EI - GJ\right)}{\left(GJn^2 + EI - R^2k'\right)}} \tag{23.2}$$

Then:

$$w = \frac{P}{2\pi k} + \frac{R^4P}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\theta}{R^4k + \frac{n^2\left(n^2 - 1\right)GJEI - n^2R^2k'\left(n^2EI - GJ\right)}{\left(GJn^2 + EI - R^2k'\right)}} \tag{24}$$

which gives the vertical displacement as a function of radial position of the point load, θ.

Now, substituting Equations 23 (23.1, 23.2) into Equation 22:

for $n = 0, b_n = 0$,

for $n \geq 1$

$$b_n = \frac{\left(\frac{1}{\pi}\right)PR^3(GJ + EI)n^2}{R^4k\left(GJn^2 + EI - R^2k'\right) + n^2\left(n^2 - 1\right)GJEI - n^2R^2k'\left(n^2EI - GJ\right)} \tag{22.1}$$

$$\phi = \frac{PR^3}{\pi}(GJ + EI)\sum_{n=1}^{\infty} \frac{n^2 \cos n\theta}{R^4k\left(GJn^2 + EI - R^2k'\right) + n^2\left(n^2 - 1\right)GJEI - n^2R^2k'\left(n^2EI - GJ\right)} \tag{25}$$

which gives the rotational displacement as a function of radial position of the point load θ.

Equations24 and 25 will be evaluated numerically for three design sections and radii, which likely cover a large range of practical applications.

4. Validation Model

A numerical model of the ring foundation was constructed using finite elements in order to provide a structural analysis solution of the problem in continuum mechanics. It is this approximated solution, by the analysis of an assemblage of finite elements that are interconnected at a finite number of nodal points, which will be compared to, and used to validate, the analytically derived solution.

To investigate the behavior of the ring foundation under a concentrated load, three different cases were considered. The finite element structural analysis of the ring foundation under a concentrated load was done using general purpose FEA software. The analysis is carefully performed by using quadratic tetrahedral elements for modeling the concrete elements, as shown in Figure 3. The number of elements was selected for the model through a sensitivity analysis of the mesh such that additional refinement did not result in any changes in the solution.

[Figure 3]

A comparison of the vertical displacements and angular twists in the ring foundation was made between the numerical finite element model and the analytical results obtained by the use of the equations derived earlier. Three different footings, Case 1, Case 2, and Case 3 were analysed with radii of 305, 610, and 915 centimetres, respectively. The radii, depths, and widths of the footings analysed are listed in Table 1. The cross-sectional width (b) and depth (h) of the ring footing are visually explained in Figure 2, section A-A. The ring foundation is subjected to a 1 metric ton (1000 kg) point load at various radial positions. The concrete modulus of elasticity and Poisson's ratio used were 20.7 MPa (3000 psi) and 0.35, respectively. The modulus of the subgrade reaction (k) for vertical displacement was taken to be 1.7 MPa (250 psi) and $k' = k\left(\frac{b^3}{12}\right)$, as discussed by Terzaghi [15].

[Table 1]

The soil meduim underneath the ring foundation was idealized and modeled using the Winkler subgrade model where the soil behaves as a system of identical but mutually independent, closely spaced, discrete, linearly elastic springs with a stiffness K. The stiffness of the subgrade is independent of the footing radius, width, and depth; exactly as the beam on elastic foundation assumption that was solved by Timoshenko [16]. The soil surrounding the ring foundation was assumed to be of a homogeneous spectrum, therefore all the springs held constant values all through the thickness and perimeter of the foundation.

5. **Results and Discussion**

Figure 4 shows, as an example, the vertical deflection under the ring foundation for Case 1 using FEA, and with a radial position of the point load of 10 KN at $\theta = 0$.

[Figure 4]

Figures 5 and 6 show the vertical and rotational displacements of the ring foundation under investigation as functions of the radial position of a 1 metric ton (2200 lbs) point load of. Higher load intensities or a set of point loads can be treated in a similar manner and through a process of load and displacement superposition and compatibility. From a design point of view, internal bending moments and torsions as well as shears may be readily evaluated using Equations 10 and 11. The solid lines show the analytical results and the dash lines show the numerical results using FEA modelling.

Figures 5(a) and 6(a) compare the vertical and rotational displacements of the ring foundation for Cases 1, 2, and 3. Figures 5(b), 5(c) and 5(d) show—for Cases 1, 2, and 3, respectively—the vertical displacements at different radial positions of the point load, with the relative error between the analytical and numerical results visually represented at intervals of $\pi/4$. Figures 6(b), 6(c) and 6(d) show—for Cases 1, 2, and 3, respectively—the angular twist at different radial positions of the point load, with the relative error between the analytical and numerical results shown at intervals of $\pi/4$.

As can be observed from Figure 5(a), for all 3 Cases of ring footings, the maximum vertical displacement of the footing takes place at a radial position of $\theta = 0$. The vertical displacement decreases rapidly with increasing value of θ, and reaches zero (almost zero for Case 1) at $\theta = \pi/4$. The effect of the concentrated load on the foundation settlement beyond a radial position of $\theta = \pi/2$ becomes negligible. This behaviour is validated by the FEA modelling, which provides slightly, but consistently, lower values for the displacement when compared with the analytical solution. This behaviour was anticipated and consistent with basic structural and mechanics theories, given the assumption of elastic soil. As the radius of the ring footing increases and the foundation stiffness thus decreases, the effect of the point load on the displacement vectors reaches shorter radial distances from the point of application. This explains why for the smaller and stiffer Case 1 footing, the effect of the point load on the displacements continued to be noticeable until a radial position of $\theta = \pi/2$, while for the larger and more flexible Cases 2 and 3 footings, this effect was negligible beyond a radial position of $\theta = \pi/4$ from the point of application of the load. It is expected that had the soil been assumed not to be elastic, the effect of the point load on the displacement vectors would reach farther radial positions than $\theta = \pi/2$.

[Figure 5]

[Figure 6]

6. Conclusions and Recommendations

An analytical solution for the problem of a ring foundation resting on an elastic sub-grade was developed using the Lagrangian multiplier method. The analytical expressions for the translational and rotational displacements were derived and expressed in a simple form which is especially convenient for use and application in modern computer codes. The results of the analytical solution were compared to and verified against a numerical finite element model for several foundation sizes and varying load positions. The analytical solution results are shown to closely match those of numerical values with maximum and average relative errors of 4% and 2%, respectively.

While several issues were addressed in this paper, there are many open questions and issues that need to be researched. Future research should include a demonstration of the validity of the analytical model for a series of concentrated loads. Future research should also include experimental validation of the analytical model using laboratory scaled ring foundation specimens. Similar analytical expressions can be derived using a similar approach for the case of uniform distributed load such as that applied by a water tank on grade.

Notations

E	=	Modulus of elasticity for ring
F	=	Vertical point load
G	=	Shear modulus for ring
I	=	Cross sectional moment of inertia for footing about its centroidal horizontal axis
J	=	Polar moment of inertia of the cross section of the ring
k	=	Modulus of subgrade reaction for vertical displacement
k'	=	Modulus of subgrade reaction for rotational displacement
M	=	Moment at s
n	=	Integer
P	=	Equivalent distributed load
q	=	Vertical distributed load acting along s
R	=	Radius of curvature of the curved beam
S_R	=	Soil resistance per unit length of the ring foundation
s	=	Distance along centroidal axis of the ring
T	=	Torsion at s
t	=	Distributed twisting moment along s in the plane normal to the centroidal axis of the ring
V	=	Shear at s
w	=	Vertical displacement at s
$\overline{w}$	=	Prescribed value of w
θ	=	Radial position of the calculated displacement as determined from the point of application of the point load
φ	=	Rotational displacement at s in the plane normal to the centroidal axis of the ring
$\overline{\varphi}$	=	A prescribed value of φ

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Table 1. Ring Footing radius, width and depth

Case Number	**Radius cm (inch)**	**Ring Width (*b*) cm (inch)**	**Ring Depth (*h*) cm (inch)**
1	305 (120)	61 (24)	30.5 (12)
2	610 (240)	91.5 (36)	45.75 (18)
3	915 (360)	122 (48)	61 (24)

Figure 1. General Representation of the Curved Beam of Constant Cross Section

Figure 2. Ring Footing Geometry

Figure 3. FEA Model of Ring Footing

Figure 4. Case 1 Deflection of Ring Footing in mm at $\theta = 0$

Figure 5(a). Vertical Displacement Versus Radial Position of Load

Figure 5(b). Vertical Displacement Versus Radial Position of Load (Case 1)

Figure 5(c). Vertical Displacement Versus Radial Position of Load (Case 2)

Figure 5(d). Vertical Displacement Versus Radial Position of Load (Case 3)

Figure 6(a). Angular Twist Versus Radial Position of Load

Figure 6(b). Angular Twist Versus Radial Position of Load (Case 1)

Figure 6(c). Angular Twist Versus Radial Position of Load (Case 2)

Figure 6(d). Angular Twist Versus Radial Position of Load (Case 3)