This paper investigates coupling strategies for finite element modeling (FEM) of thermal elastohydrodynamic lubrication (TEHL) problems. The TEHL problem involves a strong coupling between several physics: solid mechanics, fluid mechanics, and heat transfer. Customarily, this problem is split into two parts (elastohydrodynamic (EHD) and thermal) and the two problems are solved separately while an iterative procedure is established between their respective solutions. This weak coupling strategy involves a loss of information, as each problem is not made intimately aware of the evolution of the other problem’s solution during the resolution procedure. This typically leads to slow convergence rates. The current work offers a full coupling strategy for the TEHL problem, i.e., both the EHD and thermal parts are solved simultaneously in a monolithic system. The system of equations is generated from a finite element discretization of the governing field variables: hydrodynamic pressure, solids elastic deformation, and temperature. The full coupling strategy prevents any loss of information during the resolution procedure leading to very fast convergence rates (solution is attained within a few iterations only). The performance of the full coupling strategy is compared to that of different weak coupling strategies. Out of simplicity, only steady-state line contacts are considered in this work. Nevertheless, the proposed methodology, results, and findings are of a general nature and may be extrapolated to circular or elliptical contacts under steady-state or transient conditions. [DOI: 10.1115/1.4034956]

Keywords: thermal elastohydrodynamic lubrication, numerical modeling, coupling strategies, finite elements

1 Introduction

Elastohydrodynamic lubrication (EHL) has grown to become a full-fledged scientific field owing to its importance for a proper and safe operation of many moving machine elements such as gears, bearings, and cam-followers. EHD contacts are lubricated contacts in which the contacting solids are fully separated by a lubricant film inside which the generated hydrodynamic pressures are high enough to induce an elastic deformation of the solid components. Pressures may be as high as several GPa, film thicknesses as low as a few nm, and shear stresses within the lubricant film may reach hundreds of MPa.

Under such conditions, the Newtonian limit of most lubricants is exceeded and significant heat generation by shear may occur within EHL conjunctions leading to a temperature rise that may exceed 100°C in extreme cases. As such, it becomes inevitable for any numerical model for simulating EHL contacts to account for the non-Newtonian response of the lubricant as well as the dissipation of heat within the lubricant film and contacting solids. This is essential for an accurate prediction of lubricant film thickness and friction generation within these contacts.

Over the years, many numerical models of thermal non-Newtonian EHL lubrication have emerged. One of the pioneering works on the topic is that of Cheng and his co-workers [1,2] who were the first to incorporate thermal effects into the numerical solution of the EHL line contact problem. The first full numerical solution of the point contact problem under TEH conditions was obtained by Zhu and Wen [3]. Many other researchers developed different numerical tools for the simulation of the TEHL problem such as Kim and Sadeghi [4], Guo et al. [5], Liu et al. [6], or also Sharif et al. [7] who solved the full energy equation in order to obtain the temperature distribution throughout the lubricant film. The contacting solids though were not included in the thermal analysis. Instead, temperature at the solid-fluid interfaces is imposed as an essential boundary condition and is obtained based on the full expression for a moving heat source given by Carslaw and Jaeger [8]. The latter is obtained from a direct analogy with Blok’s [9] flash temperature concept derived for solid-to-solid contacts. Another type of TEHL models that employed this simplification is that of Hartinger et al. [10] who solved the problem using a computational fluid dynamics (CFD) approach. The latter consists in solving the full Navier–Stokes equations for the hydrodynamic part instead of the simplified Reynolds equation using a finite volume discretization. However, the heavy computational overhead associated with CFD models remains prohibitive and most often limits this kind of approach to the line contact configuration. A further simplification has been adopted in many works consisting in assuming a parabolic temperature profile across the lubricant film thickness. This reduces the dimension of the thermal problem by one and avoids the heavy computational cost associated with the discretization across the film required by the energy equation. Examples of such works are that of Salehizadeh and Saka [11], Wolff and Kubo [12], and Kazama et al. [13] for the line contact problem or also Jiang et al. [14], Lee et al. [15], and Kim et al. [16,17] for the point contact problem. However, the parabolic temperature profile simplification leads to inaccurate predictions in the contact inlet as shown in Ref. [13]. This is due to the occurrence of complex thermal convective effects that are associated with significant reverse flows appearing in this region of the contact. Other works such as that of Kaneta et al. [18] or...
also Wang et al. [19] incorporated the solid components into their thermal analysis which is based on the solution of the full energy equation applied to the solids and lubricant film. Most of the previously-mentioned TEHL models are based on a finite difference discretization of the governing partial differential equations (Reynolds and energy equation). More recently, Habchi et al. [20,21] introduced a FEM for thermal elasto-hydrodynamic lubrication incorporating the solid components into both the EHL and thermal parts of the model. This allowed a later incorporation of complex effects such as surface coatings [22,23] into the model. They also solved the full energy equation in both the solid and lubricant domains. In addition, the use of finite elements allowed the adoption of nonregular nonstructured meshing of the geometric domains of the solids and lubricant film leading to significant reductions in computational overheads.

A common aspect of all the above cited TEHL models is the partition of the problem into two parts: EHL and thermal. These two problems are solved separately and an iterative procedure is established between their respective solutions until convergence is attained. This weak coupling strategy involves a loss of information, as each problem is not made intimately aware of the evolution of the other problem’s solution during the resolution procedure. This typically leads to slow convergence rates. Up to the author’s knowledge, the only work to have offered a full coupling of the TEHL problem is that of Bruyere et al. [24]. However, it is based on a CFD approach which, as stated earlier, is associated with heavy computational costs and this is probably why only line contacts were considered. Such models are based on the solution of the full Navier–Stokes equations for the hydrodynamic part of the problem instead of the simplified Reynolds equation. This allows, under certain operating conditions, a better capture of the complex reverse flows occurring in the inlet region of the contact as highlighted by Hartinger et al. [10]. However, under most conditions, the simplified Reynolds equation, which is directly derived from the Navier–Stokes equations by applying thin film approximations, succeeds in predicting these reverse flows with reasonable accuracy. This being said, the extra computational cost associated with CFD approaches remains prohibitive and does not justify their use as a replacement to the much less computationally demanding Reynolds-based ones especially that both allow perfectly comparable pressure, film thickness, and friction predictions.

The merits of full coupling have been praised by Habchi et al. [21,25] for the isothermal Newtonian EHD problem. In fact, the latter is based on the solution of the Reynolds, linear elasticity, and load balance equations. Most isothermal Newtonian EHD models in the literature solve these equations separately (weak coupling) and an iterative procedure is established between their respective solutions. The very few models that adopted a full coupling strategy [26–29] suffered from two major limitations. First, the simultaneous solution of all pressure updates meant a tedious implementation of the cavitation boundary condition at the exit of the contact and second, the use of the half-space approach to compute the elastic deformation of the contacting solids leads to a full Jacobian matrix arising in the solution of the corresponding algebraic nonlinear system of equations. The computational overhead associated with the inversion of this full Jacobian matrix turned out to be prohibitive. An alternative, the "differential deflection method," was offered by Evans and Hughes [30] and consisted in deriving a differential equation from the half-space approach with a more localized character and a less dense Jacobian matrix. However, it remained relatively dense for the point contact case [31] and the arising Jacobian matrix had a relatively large bandwidth. This required a special iterative technique to be used in order to solve the fully coupled equations efficiently. Habchi et al. [21,25] offered solutions to the limitations associated with fully coupled isothermal Newtonian EHD models. The problem of the tedious implementation of the cavitation boundary condition was overcome by adopting a penalty method as proposed by Wu [32], while the problem of the full Jacobian matrix was solved by using a classical linear elasticity approach to compute the elastic deformation of the contacting solids leading to a sparse Jacobian matrix. This allowed the authors to establish an efficient full coupling strategy for the isothermal Newtonian EHD problem. It was shown that a full coupling strategy enables much faster convergence rates allowing the solution to be attained within a few iterations instead of hundreds. The authors extended their proposed model to include thermal and non-Newtonian effects [20,21]. However, the EHL part was solved in a fully coupled way, while full coupling between the EHL and thermal parts was yet to be established. The current work attempts to fill this gap by providing a fully coupled finite element TEHD model using a Reynolds-based approach for the hydrodynamic part of the problem. The fast convergence features associated with full coupling are revealed. A thorough comparison between full coupling and different weak coupling strategies, in terms of computational cost and memory requirements, is also carried out.

2 Governing Equations

This part describes the governing equations of the TEHL problem. For the sake of simplicity, only steady-state line contacts are considered in this work. However, the proposed methodology and results are of a general nature and can be extended to the more general case of circular or elliptical contacts under steady-state or transient conditions. The geometry of a line contact can be reduced to that of a contact between a cylinder of radius R and an infinite length with a flat plane (half-space) as shown in Fig. 1. From this point on, the subscripts c and p are used to represent the cylinder and plane, respectively, and the subscript f is used for the fluid (lubricant).

The solid surfaces are assumed to be smooth and to be moving at constant unidirectional surface velocities $u_x$ and $u_y$ in the $x$-direction. A full lubricant film is assumed to separate the two solids that are pressed against each other by a constant external applied force $F$. The lubricant is assumed to have a generalized-Newtonian behavior. The model description is split into two parts: the EHL part describing the lubricant flow and elastic deformation of the solids and the thermal part describing the heat dissipation within the EHL conjunction. All equations provided in this section are written in dimensionless form using Hertzian contact parameters.

2.1 Elasto-hydrodynamic Lubrication Part. The EHL part describes the flow of lubricant through the deformed EHL conjunction. Since the length of the contact is infinite in the $y$-direction, the variations of all governing field variables in this direction are nil. Hence, the geometrical domain $\Omega$ for this part is two-dimensional (2D) in the $x$-plane. It is represented by a square of sufficient size to ensure a half-space configuration as shown in Fig. 2. Habchi et al. [25] showed that a dimensionless side length of 60 is sufficient to ensure zero displacement in the regions far away from the one-dimensional (1D) contact zone $\partial \Omega$. The latter is located on the upper side of the square and has a dimensionless length of six.

![Fig. 1 Geometry of a line contact](image-url)
The dimensionless hydrodynamic pressure $P$ generated within the EHL conjunction is governed by the generalized Reynolds equation proposed by Yang and Wen [33]

$$-\frac{\partial}{\partial X} \left( \bar{p} \frac{\partial p}{\partial X} \right) + \frac{\partial (\bar{p} \bar{H})}{\partial X} + \frac{\bar{q}}{\gamma} P = 0$$

(3)

where $P^*$ corresponds to the negative part of the pressure distribution which can be defined as $P^* = P \theta(-P)$ (where $\theta$ is the Heaviside function which is nil for a negative argument and equal to unity for a positive one). Note that the penalty term has no effect in positive pressure regions and the consistency of the generalized Reynolds equation is preserved. However, in the outlet region of the contact where negative pressures arise, the penalty term dominates Eq. (3) and forces the negative pressures toward zero provided that the arbitrary constant $\zeta$ has a sufficiently large value. The dependence of dimensionless generalized-Newtonian viscosity $\eta$ on dimensionless shear stress $\bar{\tau} = \tau/\tau_0$ could be described by any of the known non-Newtonian models. In this work, the double-Newtonian modified Carreau model [34] is used

$$\eta(P, T, \tau) = \eta_z(P, T) + \eta_1(P, T)\frac{\eta_z(P, T)}{\eta_z(P, T) + \eta_1(P, T)} \left[ 1 + \left( \frac{\bar{\tau} \tau_0}{G_c} \right)^{1/\alpha} \right]$$

(4)

where $\alpha_c$, $n_c$, and $G_c$ are constants and the choice of $\tau_0 = p_b$ is adopted. The shear stress $\tau$ is determined from the solution of the following nonlinear equation applied at every point of the contact zone $\partial \Omega_n$:

$$H a \frac{\tau_0}{\mu_1 R} \int_{\Gamma(P, T)}^{\bar{\tau} \tau_0 \eta(P, T, \tau)} dZ = u_c - u_p$$

with: $\tau = \tau_p + H a p_b \frac{\partial P}{\partial X}$

(5)

where $\tau_p$ corresponds to the dimensionless lubricant shear stress on the plane’s surface ($Z = 0$). The pressure–temperature dependence of the dimensionless Newtonian viscosities $\bar{\eta}_1$ and $\bar{\eta}_2$ in Eq. (4) is described by the Roelands [35] equation for simplicity

$$\bar{\eta}_1(P, T) = \frac{\mu_1 R}{\mu_1 R} \exp \left\{ \ln(\eta_c) + 9.67 \left[ -1 + \left( 1 + 5.1 \times 10^{-9} P p_a \right)^{2/3} \right] \right\}$$

$$\frac{\tau_p}{\tau_p + \left( \frac{T R - 138}{T R - 138} \right)^{-S_0}}$$

With: $i = 1$ or $2$, $Z_0 = \frac{\tau}{5.1 \times 10^{-9} (\ln(\eta_c) + 9.67)}$ and

$$S_0 = \frac{\beta(T R - 138)}{\ln(\eta_c) + 9.67}$$

(6)

where $\alpha$ is the lubricant’s pressure–viscosity coefficient, $\beta$ its temperature–viscosity coefficient, and $\mu_1 R$ and $\mu_2 R$ its first and second Newtonian viscosity limits at the reference state under zero and infinite shear rates, respectively. The reference state throughout this work is chosen to be $p_b = 0$ and $T_R = T_0 = 300K$ ($T_R = 1$), where $T_0$ is the ambient temperature. The dependence of the lubricant’s dimensionless density on pressure and temperature is described by the Dowson and Higginson [36] relationship for simplicity

$$\bar{\rho}(P, T) = \frac{0.59 \times 10^9 + 1.34 P p_b}{0.59 \times 10^9 + P p_b} - \gamma T_R (T - 1)$$

(7)

where $\gamma$ is the lubricant’s temperature-density coefficient. The dimensionless film thickness $H$ appearing in Eq. (1) is expressed as a function of the rigid body separation term $H_b$ and $W$ (the
Z-component of the total dimensionless elastic deformation of the solids in the contact zone $\partial \Omega_s$ as follows:

$$H = H_0 + \frac{X^2}{2} - W(X)$$  \hspace{1cm} (8)

where the middle term on the right-hand side corresponds to the nondeformed shape of the EHL conjunction. The total elastic deflection of the two solids is obtained by assuming that one of them is rigid, while the other is elastic and it accommodates the total deflection of both. Habchi et al. [21] showed that the equivalent modulus of elasticity $E$ and Poisson coefficient $\nu$ of the latter are expressed as a function of those of the cylinder and plane as follows:

$$E = \frac{E_p^2 v_c (1 + v_c)^2 + E_p^2 (1 + v_p)^2}{[E_p (1 + v_c) + E_c (1 + v_p)]^2}$$  \hspace{1cm} \text{and}  \hspace{1cm} (9)

$$\nu = \frac{E_p v_c (1 + v_c) + E_c v_p (1 + v_p)}{E_p (1 + v_c) + E_c (1 + v_p)}$$

A classical linear elasticity approach is applied to the solid domain $\Omega$ to get the total elastic deflection of the equivalent solid under the applied hydrodynamic pressure $P$ in the contact zone. A plane strain approximation is adopted since the solids have an infinite length in the $Y$-direction. The governing partial differential equations can be written as a function of the total dimensionless elastic deflection components $U$ and $W$ of the elastic deformation field $\mathbf{U}$ in the $X$- and $Z$-directions, respectively.

$$
\begin{aligned}
- \frac{\partial}{\partial X} \left[ C_1 \frac{\partial U}{\partial X} + C_2 \frac{\partial W}{\partial Z} \right] &= \frac{\partial}{\partial Z} \left[ C_1 \left( \frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} \right) \right] = 0 \\
- \frac{\partial}{\partial Z} \left[ C_3 \left( \frac{\partial U}{\partial Z} + \frac{\partial W}{\partial X} \right) \right] &= \frac{\partial}{\partial X} \left[ C_2 \frac{\partial U}{\partial X} + C_1 \frac{\partial W}{\partial Z} \right] = 0
\end{aligned}
\tag{10}

\text{With: } C_1 = \frac{E (1 - \nu) a}{(1 + \nu) (1 - 2 \nu) R p_h} \text{ and } C_2 = \frac{\nu E a}{(1 + \nu) (1 - 2 \nu) R p_h} \text{ and } C_3 = \frac{E}{2(1 + \nu) R p_h}
$$

This equation can be re-arranged in a more compact form as a function of the dimensionless stress tensor $\sigma$ and dimensionless strain tensor $\varepsilon$

$$\nabla \cdot \sigma = 0$$

where:

$$\sigma = \begin{bmatrix} \sigma_{XX} & \sigma_{XZ} \\ \sigma_{XZ} & \sigma_{ZZ} \end{bmatrix}$$

and

$$\varepsilon = D \mathbf{U} \text{ where } D = \begin{bmatrix} \frac{\partial}{\partial X} & 0 \\ 0 & \frac{\partial}{\partial Z} \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} U \\ W \end{bmatrix} \tag{11}$$

A zero displacement boundary condition is applied to $\partial \Omega_s$, while the dimensionless hydrodynamic pressure $P$ is applied as a dimensionless normal stress $\bar{\sigma}_n$ over the contact domain $\partial \Omega_s$.

\begin{equation}
\bar{\sigma}_n = \sigma_{ZZ} = C_2 \frac{\partial U}{\partial X} + C_1 \frac{\partial W}{\partial Z} = -P \tag{12}
\end{equation}

A free displacement boundary condition ($\bar{\sigma}_n = \bar{\tau}_t = 0$) is applied to the remainder of the boundaries of $\Omega$. Finally, to complete the EHL part, the load balance equation in dimensionless form reads

$$\int_{\partial \Omega_s} P \; dX = \frac{\pi}{2} \tag{13}$$

This equality ensures equilibrium of forces between the external applied load and the hydrodynamic pressure generated within the lubricant film. This is done by monitoring the value of the rigid body separation term $H_0$ as explained later.

### 2.2 Thermal Part.

The thermal part describes the heat generation and dissipation through the lubricant film and bounding solids. Since the length of the contact is infinite in the $Y$-direction, temperature variations are nil in this direction. Hence, the geometrical domain for this part is 2D in the $XZ$-plane. It is represented by three adjacent rectangular domains as shown in Fig. 3: $\Omega_s$ for the fluid domain sandwiched between $\Omega_c$ for the plane and $\Omega_q$ for the cylinder. For the solid domains, Kaneta et al. [18] and Wang et al. [19] showed that a dimensionless depth of 3.15 is enough in most cases to ensure a zero temperature gradient in regions that are far from the fluid-solid interfaces. In this work, a dimensionless depth of 3.5 is adopted. As for the fluid domain, it has a unit height since $Z = z/h$. 

Fig. 3 Computational domain of the thermal part
The heat flow within the lubricant film and solids is governed by the energy equation applied to their respective domains. In dimensionless form, after a few manipulations, the corresponding equations read

\[
\begin{align*}
\frac{\partial}{\partial x} \left( k_p \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial z} \left( k_c \frac{\partial T}{\partial z} \right) + \rho_c \frac{c_p}{\rho_c} u_t \frac{\partial T}{\partial x} &= 0 \quad \text{(Solid p)} \\
\frac{\partial}{\partial x} \left( k_s \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial z} \left( k_s \frac{\partial T}{\partial z} \right) + \rho_c c_s u_t \frac{\partial T}{\partial x} &= 0 \quad \text{(Solid s)} \\
- \frac{\partial}{\partial x} \left( k_{He} \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial z} \left( k_{He} \frac{\partial T}{\partial z} \right) + \rho_{He} c_l u_t H_i \frac{\partial T}{\partial x} &= Q_{\text{comp}} + Q_{\text{shear}} \quad \text{(Lubricant film)}
\end{align*}
\]

with:

\[Q_{\text{comp}} = - \frac{H a_p b}{R T_0} \frac{\partial \Pi}{\partial T} \frac{\partial T}{\partial x} u_t \frac{\partial P}{\partial x}
\]

\[Q_{\text{shear}} = \frac{\eta_{He} H a^2}{R T_0} \gamma_{zt}^2
\]

\[\left( \mathbf{k} \cdot \nabla T \right) \cdot \mathbf{n}, \text{ where } \mathbf{n} \text{ is a normal outward unit vector is assumed to be nil.}
\]

### 3 Finite Element Model

The TEHL problem for line contacts is governed by the system of equations formed by the generalized Reynolds equation (5), the linear elasticity equations (11), the load balance equation (13), the energy equations (17), and the shear stress equation (5). Equation (3) governs the hydrodynamic pressure build-up within the EHL conjunction. The associated unknown field variable is the 1D dimensionless pressure distribution \(P\) in the contact zone \(\partial \Omega\), and the subscript \(h\) (for “hydrodynamic”) will be associated to it. The system of Eq. (11) governs the elastic deformation of the solid components under the applied contact pressure. The associated unknown field variable is the scalar rigid body separation term \(H_0\). In fact, this equation is added to the system of TEHL equations as an ordinary integral equation while introducing \(H_0\) as an additional unknown. The subscript \(l\) (for “load balance”) will be associated to it. The system of Eq. (17) describes the heat flow within the solids and lubricant film. The associated unknown field variable is the 2D dimensionless total elastic deformation field \(T\) over the solid domain \(\Omega\) and the subscript \(e\) (for “elastic”) will be associated to it. Equation (13) defines the equilibrium of forces over the contact and is written as a function of \(P\). However, the associated unknown field variable is the scalar rigid body separation term \(H_0\). For the plane domain \(k_1 = k_2 = k_p/\alpha, \) \(\varphi = \rho_c \frac{c_r u_t}{\rho_c} \) and

\[Q = 0
\]

as for the cylinder domain \(k_1 = k_2 = k_p/\alpha, \) \(\varphi = \rho_c \frac{c_u u_t}{\rho_c} \) and

\[Q = 0
\]

For the fluid domain, \(k_1 = k_2 = k_{He}/\alpha, \) \(\varphi = \rho_{He} \frac{c_l u_t}{\rho_{He}} \) and

\[Q = Q_{\text{comp}} + Q_{\text{shear}}
\]

The boundary conditions associated with the thermal problem are shown in Fig. 3. Heat flux continuity boundary conditions are imposed on the two fluid-solid interfaces as follows:

\[
\left. \frac{k_p}{a} \frac{\partial T}{\partial z} \right|_{z=0} = \left. k_{He} \frac{\partial T}{\partial z} \right|_{z=0} \quad \text{and} \quad \left. k_s \frac{\partial T}{\partial z} \right|_{z=1} = \left. k_{He} \frac{\partial T}{\partial z} \right|_{z=1}
\]

An ambient temperature \(T_0\) is imposed (\(T = 1\)) on all inlets (left side of the computational domain) since the energy equation is hyperbolic (owing to the convective terms). Note that for the solids, \(u_t\) and \(u_p\) are assumed to be positive throughout this work and the left boundaries are actual inlet boundaries. However, for the lubricant film, because of the reverse flows occurring in the inlet region of the contact, \(u_t\) may be negative on the left side boundary and therefore the ambient temperature boundary condition only needs to be imposed on inlet sections (where \(u_t < 0\)). An ambient temperature \(T_0\) is also imposed in the depth of the solids (top and bottom boundaries). As for the outlet boundaries (right side) of the solids and lubricant film, a convective heat flux boundary condition is assumed. That is, the convective heat flux
Find \((P, \overline{U}, H_0, T, \tau_p)\) such that \(\forall(N_p, N_T, N_T')\), one has:

\[
\begin{align*}
\int_{\Omega} \varepsilon(N_T')^T C \varepsilon(\overline{U}) d\Omega + \int_{\partial\Omega} P N_w n d\Omega &= 0 \\
\int_{\Omega} \left( \frac{\partial P}{\partial x} \frac{\partial N_p}{\partial x} - \tau_p' \frac{\partial N_p}{\partial x} + \varepsilon P' N_p \right) d\Omega &= 0 \\
\int_{\partial\Omega} P d\Omega - \frac{\pi}{2} &= 0 \\
\int_{\Omega/\partial\Omega} \left( \nabla N_T^T \cdot \mathbf{k} \cdot \nabla T + \varphi \frac{\partial T}{\partial x} N_T - Q N_T \right) d\Omega &= 0 \\
\int_{\partial\Omega/\partial\Omega} \left( \tau_k \left( \frac{\partial P}{\partial x} \frac{\partial N_p}{\partial x} \right) R_h d\Omega &= 0 \\
H a^2 \int_{\Omega/\partial\Omega} \tau_p' d\Omega + \frac{H a^2 P}{\overline{P}} \frac{\partial P}{\partial x} d\Omega &= - \frac{\mu T}{\overline{T}} (u_c - u_p) \quad \text{(at every point of } \partial\Omega) \\
\end{align*}
\]

(19)

The penalty term coefficient \(\varepsilon\) is defined over a given element \(\Omega\), of \(\partial\Omega\), as \(\varepsilon = 10^6 \times h_e\), where \(h_e\) is the length of \(\Omega\). [21] Note that for highly loaded contacts, the generalized Reynolds equation gives rise to numerical oscillations in the pressure distribution when a standard Galerkin formulation is employed. Hachchi et al. [21] explained that these oscillations appear because the generalized Reynolds equation can be written as a classical convection-diffusion equation in which convection becomes dominant for highly loaded contacts. The instability of the standard Galerkin formulation to resolve fine error scales under convection-dominant diffusion equation in which convection becomes dominant for over conduction. Adding the SUPG stabilizing terms to the Galerkin formulation to resolve fine error scales under convection-dominant diffusion equation in which convection becomes dominant. The system of Eq. (20) is nonlinear and thus its resolution requires a special treatment. It is solved using a Newtonlike procedure. The latter gives rise to a linearized system of equations in which the unknowns are the increments of the field variables \(\delta \overline{U}, \delta P, \delta H_0, \delta T, \text{ and } \delta \tau_p\). These are discretized in the usual finite element sense and the arising overall linearized matrix system to be solved at every Newton iteration \(i\) as a function of the nodal values of the unknown increments of the field variables is of the form:

\[
\begin{align*}
K_{\delta \overline{U}} &\begin{bmatrix} \delta \overline{U} \\ \delta P \\ \delta H_0 \\ \delta T \\ \delta \tau_p \end{bmatrix} = R_i \\
&= \begin{bmatrix} K_{\delta \overline{U}} & K_{\delta P} & K_{\delta H_0} & K_{\delta T} & K_{\delta \tau_p} \\ K_{\delta P} & P_{\delta P} & 0 & 0 & 0 \\ K_{\delta H_0} & 0 & H_{\delta H_0} & 0 & 0 \\ K_{\delta T} & 0 & 0 & H_{\delta T} & 0 \\ K_{\delta \tau_p} & 0 & 0 & 0 & H_{\delta \tau_p} \end{bmatrix} \begin{bmatrix} \delta \overline{U} \\ \delta P \\ \delta H_0 \\ \delta T \\ \delta \tau_p \end{bmatrix} \\
&= \begin{bmatrix} R_i \\ R_i \\ R_i \\ R_i \\ R_i \end{bmatrix} \\
\end{align*}
\]

(22)

The matrix on the left-hand side is the Jacobian matrix to be evaluated at every Newton iteration \(i\) as a function of the field variables obtained at the previous iteration \(i - 1\). The right-hand side is formed by the residual vectors of the hydrodynamic (\(R_0\)), load balance (\(R_0\)), thermal (\(R_0\)), and shear stress (\(R_0\)). These are also evaluated at every Newton iteration \(i\) as a function of the field variables obtained at the previous iteration \(i - 1\). Note that the residual vector for the elastic problem (\(R_0\)) is nil since the linear elasticity equations are linear and the corresponding initial guess is chosen to satisfy them. The starting point for the overall numerical procedure is the selection of appropriate guesses for all the field variables. For \(P\), a dimensionless Hertzian pressure distribution is adopted and its corresponding elastic deformation is used to initialize \(\overline{T}\). A homogeneous ambient temperature distribution (\(T = 1\)) is used to initialize the temperature field and for \(\tau_p\), an initial guess of zero could be used or a fraction of the Hertzian pressure distribution (typically 5%). Starting from the initial guess, the system of Eq. (22) is solved for the increments of the field variables using a sparse direct linear system solver (UMFPACK [40]). The equations being highly nonlinear, a damped-Newton [41] procedure is employed to ensure convergence. The latter consists in adding at every Newton iteration \(i\), a carefully selected fraction \(\delta' \in [0,1]\) of the solution.
integral terms arising in Eq. (1). A six-point Gauss quadrature rule where a structured rectangular meshing is employed. This is to be used for all domains except for the fluid domain in the thermal part and thermal domains. Note that nonstructured triangular meshing is the latter. Figure 4 shows the extra coarse mesh case for the EHL domain and these variations become smaller and smaller with distance from the contact region. This is because all field variables exhibit significant variations in the vicinity of the contact region and these variations become smaller and smaller with distance from the latter. Figure 4 shows the extra coarse mesh case for the EHL part. This is to avoid any unnecessary interpolation operations for the field variables of each part over a nonmapped mesh in the other. Also note that the mesh of the thermal part is symmetric with respect to an axis passing through the midlayer of the lubricant film in the x-direction. Second-order Lagrange interpolation functions are used for all elements in both the EHL and thermal domains.

Table 2 provides the mesh specifications for all five considered mesh cases in terms of number of elements in the 1D contact domain \( \partial \Omega \) \((n)\), number of elements in the 2D elastic domain \( \Omega \) \((n)\), number of elements for the 2D plane domain \( \Omega \), in the thermal part \((n)\) which is the same as the number of elements for the cylinder domain (given the mesh symmetry discussed above), and the number of elements in the film thickness direction for the fluid domain of the thermal part \((n)\). The numbers of degrees-of-freedom (unknown field variables) for the hydrodynamic problem \((N\text{def})\) which is equal to that of the shear stress problem \((N\text{def})\), the elastic problem \((N\text{def})\), and the thermal problem \((N\text{def})\) are also indicated along with the total number of degrees-of-freedom \((N\text{def})\) for each mesh case.

Table 2 clearly shows that the size (or number of degrees-of-freedom) of the hydrodynamic or shear stress problems is relatively small compared to the elastic or thermal problems. This is because the former are 1D, while the latter are 2D.

Table 1 Lubricant properties, solid material properties, and operating conditions

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<th>Lubricant properties</th>
<th>Solid material properties</th>
<th>Operating conditions</th>
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<tr>
<td>( \mu ) = 0.1 Pa·s</td>
<td>( c = 1500 J/kg \cdot K )</td>
<td>( T_R = T_0 = 300 \text{ K} )</td>
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<tr>
<td>( \mu / \mu_R )</td>
<td>( k = 0.1 \text{ W/m} \cdot \text{K} )</td>
<td>( R = 15 \text{ mm} )</td>
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<tr>
<td>( \mu / \mu_R = 0.5 ) or 0.01</td>
<td>( \rho_e = \rho_r = 7850 \text{ kg/m}^3 )</td>
<td>( u_{\text{av}} = 0.1 ) or ( 1 ) ( \text{ m/s} )</td>
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<tr>
<td>( \alpha = 15 ) or 25 ( \text{ GPa}^{-1} )</td>
<td>( G_c = 0.01 \text{ MPa} )</td>
<td>( SRR = 0.0 ) or ( 0.5 ) or ( 1 )</td>
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<tr>
<td>( \beta = 0.05 \text{ K}^{-1} )</td>
<td>( a_e = 2.2 )</td>
<td>( F = 0 ) or ( 2 ) ( \text{ MN/m} )</td>
</tr>
<tr>
<td>( \gamma = 0.00075 \text{ K}^{-1} )</td>
<td>( n_c = 0.8 )</td>
<td>( \rho_b = 0.7 ) or ( 2.2 ) ( \text{ GPa} )</td>
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<tr>
<td>( \rho_b = 750 \text{ kg/m}^3 )</td>
<td>( E_p = E_r = 210 \text{ GPa} )</td>
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4 Results

This section presents typical TEHL results obtained using the full coupling strategy introduced previously. The computational performance of the corresponding finite element model is also examined and compared to that of different weak coupling strategies. Throughout this section, steel-onsteel contacts are considered. The properties of the considered steel as well as the lubricant’s properties and the operating conditions are summarized in Table 1.

Note that two loading conditions are considered: a moderate load with \( F = 0.2 \text{ MN/m} \) corresponding to a Hertzian contact pressure \( p_h = 0.7 \text{ GPa} \) and a high load with \( F = 2 \text{ MN/m} \) corresponding to \( p_h = 2.2 \text{ GPa} \). All numerical tests are run on a personal laptop using a single Intel Core i7 - 2.7 GHz processor.

4.1 Mesh Selection and Specifications. Five different mesh cases are considered in this work. These are named “Extra Coarse,” “Coarse,” “Normal,” “Fine,” and “Extra Fine” from the coarsest to the finest. All mesh cases are specifically tailored toward the EHL problem. That is, a fine mesh is used in the contact region and its surrounding and the mesh size is progressively increased with distance from the contact region. This is because all field variables exhibit significant variations in the vicinity of the contact region and these variations become smaller and smaller with distance from the latter. Figure 4 shows the extra coarse mesh case for the EHL and thermal domains. Note that nonstructured triangular meshing is used for all domains except for the fluid domain in the thermal part where a structured rectangular meshing is employed. This is to allow an easier handling/computation of the density and viscosity integral terms arising in Eq. (1). A six-point Gauss quadrature rule was found to be sufficiently accurate for the evaluation of these integrals. It is noteworthy to mention that the distancing between points in the x-direction for the fluid domain in the thermal part is chosen to exactly map the mesh over the contact region \( \partial \Omega \) in the EHL part. This is to avoid any unnecessary interpolation operations for the field variables of each part over a nonmapped mesh in the other. Also note that the mesh of the thermal part is symmetric with respect to an axis passing through the midlayer of the lubricant film in the x-direction. Second-order Lagrange interpolation functions are used for all elements in both the EHL and thermal domains.

Table 2 Mesh specifications in terms of numbers of elements and degrees-of-freedom

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<th>Degrees-of-freedom</th>
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</table>

Fig. 4 “Extra coarse” mesh case for the EHL domain (a) and the thermal domain (b)
thickness ($H_m$), dimensionless minimum film thickness ($H_{m0}$), and maximum temperature rise ($\Delta T_{\text{max}}$) with mesh size for the moderately (left) and highly (right) loaded test cases. Film thicknesses are chosen to represent the convergence of the EHL part, while the maximum temperature rise represents that of the thermal part. It is clear from Fig. 5 that convergence is attained with the “Normal” mesh case and that any increase in the mesh density (or any further decrease in mesh size) does not lead to any meaningful improvement in the solution accuracy. So, from this point on, unless stated otherwise, the “Normal” mesh case is used for all calculations as it ensures grid-independent solutions with minimal memory requirements and computational overhead.

4.2 Typical Test Case Results. In this section, a typical test case with $F = 0.2\, \text{MN/m}$ ($p_H = 0.7\, \text{GPa}$), $u_m = 0.1\, \text{m/s}$, SRR = 0.5, $\mu_{2R}/\mu_1 = 0.5$, and $\alpha = 25\, \text{GPa}^{-1}$ is considered. The corresponding dimensionless pressure and film thickness distributions over the contact domain are reported in Fig. 6 (left). Figure 6 (middle) shows the corresponding temperature rise through the contact over the solid components’ surfaces and the midlayer of the lubricant film, while Fig. 6 (right) shows the distribution of dimensionless lubricant shear stress over the plane’s surface.

The results of Fig. 6 show typical features of TEHL contacts such as the usual pressure spike and film thickness constriction, and the temperature rise within the lubricant film which exceeds that of the solid surfaces. This validates the employed approach (at least qualitatively).

4.3 Full Versus Weak Coupling. Contrary to full coupling (FC), weak coupling (WC) consists in a segregated resolution of the problem by splitting it into several parts. These are solved individually while setting the value of the field variables of the remainder parts to that obtained at the previous iteration or step. Thus, the solution of every individual part is not made intimately aware of the simultaneous evolution of the solutions of the remainder parts but rather of an “older” nonsynchronized version of those. Obviously, this leads to a loss of information during the resolution process which translates into slower convergence rates. But, on the other hand, the sizes of the individual parts and their arising matrix systems become smaller requiring a reduced computational overhead for their inversion. In addition, the computation of the extra-diagonal submatrices required for a full coupling scheme is no longer required. But it remains to be identified whether or not these advantages outweigh the increased or accelerated convergence rates attained with a full coupling strategy. Four different weak coupling strategies are considered. These are detailed in the flowcharts of Fig. 7. The iteration numbers for different loops within these strategies are denoted $n_1$, $n_2$, and $n_3$ as shown in Fig. 7. Note that the “Solve EHD Problem” block in Fig. 7 implies a damped-Newton resolution of the generalized Reynolds, linear elasticity, and load balance equations in a fully coupled way for fixed temperature and shear stress distributions. On the other hand,
the "Solve Thermal Problem" block implies a damped-Newton resolution of the energy equations for fixed pressure, film thickness, and shear stress distributions, while the "Solve Shear Stress Problem" implies a damped-Newton resolution of the shear stress equation (5) at every discretization point of the contact domain $\partial \Omega$, for fixed pressure, film thickness, and temperature distributions. For all weak coupling strategies, the stopping criterion for the overall procedure is that the $L_2$-norm (normalized with respect to the problem size/total number of unknowns) of the overall solution increment vector between two consecutive global iterations falls below $10^{-6}$. As for internal blocks, the stopping criteria are those of the damped-Newton procedure [41].

Note that pressure, temperature, film thickness, and shear stress results are in perfect agreement among all considered coupling strategies. Given that weak coupling finite element modeling of the TEHL problem has been validated against experiments by the author and his coworkers on several occasions (e.g., see Refs. [20,22,42–43]), this validates the current proposed methodologies.

The performance of the full coupling strategy is compared to that of all considered weak coupling strategies in terms of numbers of iterations required for convergence and computational (CPU) times. The pressure-viscosity coefficient for these test cases is taken to be $\mu = 15$ GPa$^{-1}$, while $F$, $u_\alpha$, SRR, and $\mu_{1, R}/\mu_{2, R}$ are varied. The results are reported in Tables 3 and 4, respectively. Table 3 provides the numbers of iterations in outer loops and the summations of the numbers of iterations in internal loops over all outer loops for all coupling strategies, while Table 4 provides the corresponding overall CPU times.

First, comparing the different weak coupling strategies, the results of Tables 3 and 4 suggest that WC1, WC2, and WC4 are more robust and efficient than WC3. These generally require an overall smaller number of iterations to attain convergence resulting in faster CPU times. In fact, for WC3, some internal loops (which numbers of iterations are marked with an "a" in Table 3) did not reach convergence and had to be broken when a preset maximum allowable number of iterations was exceeded. The final result though is a converged one. This is a clear indication that stagnation points are reached within these loops, whereby the solution oscillates back and forth around a given stagnation point. This is a typical indicator of the lack of robustness of a given algorithm. This lack of robustness also appears for the other weak coupling strategies. In fact, in some cases, an increased overall number of iterations and CPU times are observed for no apparent reason, e.g., rows 2, 8, and 10 from the bottom in Tables 3 and 4. Though sometimes no internal loops are broken as a result of stagnation, this suggests that stagnation points or slow convergence rates may be encountered within some internal loops, but these end up converging. Stagnation points are characteristic of weak coupling strategies. These are a direct result of the lack of information associated with the omission of coupling terms which are not necessarily negligible in these cases. This feature vanishes with full coupling as indicated by the results of Tables 3 and 4 which reveal more consistent convergence rates and CPU times for FC strategy.

It is clear from Table 3 that a full coupling strategy guarantees a smaller (and in most cases much smaller) overall number of iterations, compared to weak coupling strategies to attain convergence under all considered operating conditions. However, this does not always reflect into smaller CPU times. In fact, Table 4 reveals that for the moderately loaded case, CPU times are generally smaller for weak coupling strategies even though the number of iterations is higher. The same applies for pure-rolling cases under any loading or mean entrainment speed conditions. This is because in both cases, the difference in the overall number of iterations between full and weak coupling strategies is not sufficiently large to compensate for the increased computational overheads associated with full coupling. These result from the introduction of additional coupling terms to be evaluated and more importantly, the inversion of a larger arising matrix system (22) compared to weak coupling strategies. This observation is not surprising as for these cases, shear-thinning as well as heat generation are relatively mild, leading to a weak dependence of the TEHL problem’s solution on the thermal and shear-thinning parts. As such, the additional coupling terms are relatively negligible and have very little effect on the overall solution. Therefore, the
additional computational overhead associated with their evaluation and the inversion of a larger matrix system is not justified. On the other hand, under high loading and rolling–sliding conditions, the overall number of iterations as well as computational times becomes relatively smaller for the full coupling strategy compared to weak coupling strategies. This is because under such conditions, shear-thinning as well as heat generation becomes significant and has a more pronounced effect on the overall solution. Thus, it becomes essential to incorporate these effects in a fully coupled way. The associated additional computational overhead in this case is offset by the significant reduction in the overall number of iterations required for convergence.

Table 3 Comparison of numbers of iterations required for convergence of full and weak coupling strategies

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<th>$u_m$ (m/s)</th>
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*Internal loop convergence not attained. Iterative process of corresponding internal loop stopped when the maximum number of iterations is attained.

Table 4 Comparison of CPU times of full and weak coupling strategies

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<tr>
<th>$\rho_1 \rho_2$</th>
<th>$F$ (MN/m)</th>
<th>$u_m$ (m/s)</th>
<th>SRR</th>
<th>CPU (s)</th>
<th>FC</th>
<th>WC1</th>
<th>WC2</th>
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5 Conclusion

This paper presents a finite element fully coupled approach to the resolution of TEHL problems. The approach consists in solving the governing equations: generalized Reynolds, linear elasticity, load balance, energy, and shear stress equations in a monolithic system using a damped-Newton resolution procedure. The generalized Reynolds, linear elasticity, and energy equations being of a partial differential nature are discretized using a finite element procedure, while the load balance and shear stress equations being simple integral equations are directly added to the arising algebraic system of equations. Special stabilized finite element formulations are employed for the solution of the generalized Reynolds equation under highly loaded regime and for the solution of the energy equations under convection-dominated regime. Results clearly reveal the fast convergence characteristics associated with full coupling. In fact, solutions are attained within a few iterations only with a total CPU time of 1 min or less.

The performance of the proposed fully coupled scheme was compared to different weak coupling strategies. It turned out that, in some cases, weak coupling leads to a loss of robustness, resulting in a significant increase in the overall number of iterations for no apparent reason and increased CPU times. Full coupling on the other hand turned out to be very robust, with consistent convergence rates and CPU times. It guarantees a smaller (and in most cases much smaller) overall number of iterations to attain convergence, regardless of operating conditions. However, this does not always reflect into smaller CPU times. In fact, for moderately loaded cases, or also for pure-rolling cases under any loading or mean entrainment speed conditions, CPU times are smaller for weak coupling strategies. This is because the difference in the overall number of iterations between full and weak coupling strategies is not sufficiently large to compensate for the increased computational overheads of a full coupling strategy. Actually, for these cases, shear-thinning as well as heat generation is relatively mild, leading to a weak dependence of the TEHL problem’s solution on the thermal and shear-thinning parts. As such, the additional coupling terms are relatively negligible and have very little effect on the overall solution. This being said, the additional computational overhead associated with their evaluation and the inversion of a larger matrix system is not justified. On the other hand, under high loading and rolling–sliding conditions, the overall number of iterations as well as CPU times becomes relatively smaller for the full coupling strategy. This is because under these conditions, shear-thinning as well as heat generation becomes significant and has a more pronounced effect on the overall solution.

Nomenclature

- \( a \) = Hertzian contact radius (m)
- \( c \) = lubricant’s heat capacity (J/kg K)
- \( c_c \) = cylinder’s heat capacity (J/kg K)
- \( \bar{E} \) = equivalent Young’s modulus (Pa)
- \( E_c \) = cylinder’s Young’s modulus of elasticity (Pa)
- \( E_p \) = plane’s Young’s modulus of elasticity (Pa)
- \( F \) = contact external applied load per unit length (N/m)
- \( h \) = lubricant film thickness (m)
- \( H \) = dimensionless lubricant film thickness
- \( H_0 \) = film thickness constant parameter
- \( k \) = lubricant’s thermal conductivity (W/m K)
- \( k_c \) = cylinder’s thermal conductivity (W/m K)
- \( k_p \) = plane’s thermal conductivity (W/m K)
- \( N_P, N_T, N_T \) = finite element weight functions for field variables
- \( P, U \) and \( T \) = pressure (Pa)
- \( p_{h} \) = Hertzian contact pressure (Pa)
- \( p_{h} \) = dimensionless pressure
- \( R \) = ball’s radius (m)
- \( SRR = \frac{u_c - u_p}{u_m} \)
- \( T \) = temperature (K)
- \( T_{0} \) = dimensionless temperature
- \( T_0 \) = ambient temperature (K)
- \( T_R \) = reference temperature (K)
- \( T_R \) = dimensionless temperature at reference state
- \( u, w \) = x and z-components of the solid’s elastic deformation field (m)
- \( U, W \) = dimensionless x and z-components of the solid’s elastic deformation field
- \( u_m \) = mean entrainment speed = \( \frac{(u_c + u_p)}{2} \) (m/s)
- \( u_c \) = cylinder’s surface velocity (m/s)
- \( u_p \) = plane’s surface velocity (m/s)
- \( u_f \) = lubricant’s velocity field x component (m/s)
- \( x, y, z \) = space coordinates (m)
- \( X, Y, Z \) = dimensionless space coordinates
- \( \gamma \) = lubricant’s density-temperature coefficient (K\(^{-1}\))
- \( \eta \) = lubricant’s generalized Newtonian viscosity (Pa s)
- \( \eta \) = dimensionless lubricant’s generalized Newtonian viscosity
- \( \mu_1, \mu_2 \) = lubricant’s first Newtonian viscosity at reference state (Pa s)
- \( \mu_2 \) = lubricant’s second Newtonian viscosity at reference state (Pa s)
- \( \mu_1 \) = dimensionless lubricant’s first Newtonian viscosity
- \( \mu_2 \) = dimensionless lubricant’s second Newtonian viscosity
- \( \rho \) = lubricant’s density (kg/m\(^3\))
- \( \bar{\rho} \) = dimensionless lubricant’s density
- \( \rho_c \) = cylinder’s density (kg/m\(^3\))
- \( \rho_p \) = plane’s density (kg/m\(^3\))
- \( \rho_f \) = lubricant’s density at reference state (kg/m\(^3\))
- \( \tau \) = lubricant shear stress (Pa)
- \( \tau_f \) = lubricant’s dimensionless shear stress
- \( \tau_p \) = lubricant’s dimensionless shear stress on plane’s surface
- \( v \) = equivalent Poisson coefficient
- \( \nu \) = cylinder’s Poisson coefficient
- \( \nu_p \) = plane’s Poisson coefficient

Subscripts

- \( c \) = cylinder
- \( e \) = elastic
- \( f \) = fluid/lubricant
- \( h \) = hydrodynamic
- \( l \) = load balance
- \( p \) = plane
- \( s \) = shear stress
- \( t \) = thermal

Dimensionless Parameters

\[
X = \frac{x}{a}, \quad H = \frac{h R}{a^2}, \quad U = \frac{u R}{a^2}, \quad V = \frac{v R}{a^2},
\]

\[
P = \frac{p}{p_h}, \quad \bar{T} = \frac{T}{T_0}, \quad \bar{\rho} = \frac{\rho}{\rho_p}, \quad \bar{\eta} = \frac{\eta}{\mu_1}, \quad \tau = \frac{\tau}{\tau_p}
\]

\[
Z = \left\{ \begin{array}{l}
\frac{z}{a} \quad \text{Solids} \\
\frac{z}{h} \quad \text{Lubricant film}
\end{array} \right.
\]

References
