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ABSTRACT

In this paper, we develop a Bayesian framework for the empirical estimation of the parameters of one of the best known nonlinear models of the business cycle: The Marx-inspired model of a growth cycle introduced by R. M. Goodwin. The model predicts a series of closed cycles representing the dynamics of labor's share and the employment rate in the capitalist economy. The Bayesian framework is used to empirically estimate a modified Goodwin model. The original model is extended in two ways. First, we allow for exogenous periodic variations of the otherwise steady growth rates of the labor force and productivity per worker. Second, we allow for stochastic variations of those parameters. The resultant modified Goodwin model is a stochastic predator-prey model with periodic forcing. The model is then estimated using a newly developed Bayesian estimation method on data sets representing growth cycles in France and Italy during the years 1960-2005. Results show that inference of the parameters of the stochastic Goodwin model can be achieved. The comparison of the dynamics of the Goodwin model with the inferred values of parameters demonstrates quantitative agreement with the growth cycle empirical data.

Keywords: economic fluctuations; business cycles, stochastic models, Bayesian estimation.

1. INTRODUCTION

The empirical estimation of business cycle and economic fluctuations models is one of the main features of the research program aimed at explaining the origins and causes of instabilities in the economy. Various theoretical business cycle models have been introduced in the literature. The models fall into two general types: exogenous and endogenous type models. Exogenous-shock models, such as the real business cycle model (RBC), hypothesize that the causes of economic fluctuations are from outside the economic system such as weather, war or technology shocks. Endogenous cycle models, on the other hand, are all based on internal mechanisms that cause the economy to experience fluctuations. The endogenous models are inherently nonlinear models. Nonlinear models generate fluctuations that are persistent. The complex dynamics generated from these models range from limit cycles (representing periodic economic cycles) to chaotic motion (representing irregular economic fluctuations). The nonlinear formulation eliminates the necessity for outside shocks to explain the persistence of fluctuations in a capitalist economy. However, one of the main shortcomings of nonlinear business cycle models is the difficulty of empirical estimations of such models. This lack of empirical work in nonlinear modeling in economics has caused a wide gap between the development of business cycle models and the necessary empirical estimation of such models. In this paper, we apply a newly developed method for the empirical estimation of nonlinear stochastic differential equations to one of the well-known nonlinear models of the business cycle: the Goodwin model.[1] The Marx-inspired model of a growth cycle introduced by R. M. Goodwin has attracted much attention recently.[2-5]

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The model predicts a series of closed cycles which show the relationship between labor's national income share (u) and employment rate (v). These variables are said to be analogous to the predator and prey respectively in the Lotka-Volterra model[6-7] which according to Solow, the capitalist being the prey and the workers the predators.[8] The model governs the following dynamic: when both workers' share and employment rate are above the mean value, workers can bargain for higher wages which will squeeze profit margins and reduce investment. Given assumptions of constant growth of productivity and labor force, lower investment will decrease the employment rate which in turn strengthen the capitalists' bargaining position and increase their profit-share.

In this paper, the Goodwin model is extended to allow for exogenous periodic variations of the otherwise steady growth rates of the labor force and productivity per worker. Second, we allow for stochastic variations of those parameters. The resultant modified Goodwin model is a two-dimensional stochastic predator-prey model with periodic forcing. The resultant SDE is then estimated using a novel Bayesian estimation method on data sets representing growth cycles in France and Italy during the years 1960-2005. Results show that a straightforward inference of the parameters of Goodwin's model can be achieved. The comparison of the dynamics of the Goodwin's model with the inferred values of parameters demonstrates qualitative agreement with the collected empirical data. The paper is divided as follows. Section 2 introduces the Goodwin model and its extension. Section 3 presents model inference with synthetic data and data from France and Italy growth cycles. Section 4 concludes.

2. THE PREDATOR-PREY MODEL

In this section, we present a skeletal of the original Goodwin model. The model governs the dynamics of employment and the share of labor in a capitalist economy. The resultant nonlinear dynamical model is a predator-prey model. The model is then rendered stochastic with periodic forcing through the noise-excitation of its parameters.

2.1 The Goodwin model

The economy is divided into two classes: capitalists and workers. Let u be the labor's share of national output, w is the wage and a the productivity of labor then u can be written as $u = \frac{w}{a}$. The capitalists share becomes $(1-\frac{w}{a})$. Assuming that the capitalists save and invest all the profits, we can write investment $\dot{k}$ as

$$\dot{k} = \left(1 - \frac{w}{a}\right) q . \tag{1}$$

where q = output. The rate of change of output is then given by

$$\frac{\dot{k}}{k} = \frac{\dot{q}}{q} = \frac{(1 - w/a) q}{k} = \frac{(1 - w/a)}{\sigma} \tag{2}$$

where σ is the constant capital–output ratio. If we let l = employment, n =labor force, $n = n_0 e^{\beta t}$, $a = a_0 e^{\beta t}$ then the employment rate and the labor share are governed by

$$\frac{\dot{v}}{v} = \left[\frac{1-u}{\sigma} - (\alpha + \beta) \right] \tag{3}$$

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \alpha \tag{4}$$

Finally, given the well-known Phillips curve relationship

$$\frac{\dot{w}}{w} = -\gamma + \rho v \quad (5)$$

then the equations of Goodwin model of economic growth cycle are given by

$$\dot{v} = \left[\frac{1-u}{\sigma} - (\alpha+\beta)\right] v \quad (6)$$

$$\dot{u} = \left[-(\alpha+\gamma) + \rho v\right] u \quad (7)$$

2.2 Extension of the Goodwin model

We further relax the restrictions of the original Goodwin's model in two important respects. First, we allow for the exogenous periodic variations of the otherwise steady growth rates of the labor force and productivity per worker through the perturbation of parameters α and β in equations (6) and (7). Second, we allow for stochastic variations of the parameters. With these modifications, the Goodwin model becomes a stochastic-predator prey equation

$$\frac{\dot{v}}{v} = \frac{1-u}{\sigma} - (\alpha+\beta) + r_1 \sin \omega t + s_{11}\xi_1(t) \quad (8)$$

$$\frac{\dot{u}}{u} = -(\alpha+\gamma) + \rho v + r_2 \sin \omega t + s_{22}\xi_2(t) \quad (9)$$

Here $\xi_1(t)$ and $\xi_2(t)$ are N(0,1), s is the noise matrix, and $D = ss^T$ is the diffusion matrix. In what follows we will infer parameters of this model directly from the experimental data for economic growth in two OECD countries, compare the outcome of the model dynamics with the experimental data, and try to make predictions for the economic growth in those countries in the near future.

3. TESTING MODEL INFERENCE

3.1 Dynamical inference method

To infer parameters of the stochastic nonlinear model a newly developed Bayesian estimation method is applied.[8] It can be briefly summarized as follows. Let us write the general form of stochastic differential equations as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \varepsilon(t) = \mathbf{f}(\mathbf{x}) + \sigma\xi(t) \quad (10)$$

where $\xi(t)$ is an additive stationary white Gaussian vector noise process characterized by diffusion matrix ***D***. It is assumed that the trajectory $x(t)$ of this system is observed at sequential time instants $\{t_k; k = 0,1,...,K\}$ and a series $S = \{s_k \equiv s(t_k)\}$ thus obtained is related to the (unknown) "true" system states $X = \{x_k \equiv x(t_k)\}$ through some conditional PDF $p_o(S \mid X)$. A *prior* PDF of the model parameters $p_{pr}(M)$ is related to the improved PDF by measurements (so-called the *posterior* conditional PDF) $p_{post}(M \mid S)$ via Bayes' theorem:

$$p_{post}(M \mid S) = \frac{\ell(S \mid M)\, p_{pr}(M)}{\int \ell(S \mid M)\, p_{pr}(M)\, \mathrm{d}M} \quad (11)$$

where $\ell(S \mid M)$ is the *likelihood* that relates measurements S to the dynamical model. The posterior computed from block S serves as the prior for the next block S', etc. For a sufficiently large number of observations, $p_{\text{post}}(M \mid S, S', \ldots)$ is sharply peaked at a certain most probable model $M = M^*$. The problem is to construct the likelihood function and to optimize it with respect to the model parameters. A number of methods can be applied to infer $\{\mathbf{x}(t_k)\}$ from a set of measurements $\{s(t_k)\}$, including e.g. global nonlinear optimization[9] developed in our earlier work. In what follows, however, we will assume for the sake of simplicity that the measurement noise is small enough and that all the dynamical variables can be measured directly, so that $\mathbf{x}(t_k) = s(t_k)$.

This problem is solved analytically by using parameterization of the vector field of (10) in the form

$$\mathbf{f}(\mathbf{x}) = \hat{\mathbf{U}}(\mathbf{x})\,\mathbf{c} \equiv \mathbf{f}(\mathbf{x};\mathbf{c}), \mathrm{k} \tag{12}$$

and applying the results of the path-integral theory of fluctuations, we write the probability density function for the system to have a given trajectory (the likelihood function) in the form

$$-\frac{2}{K}\log \ell(X \mid M) = \ln\left(\left|\hat{\mathbf{D}}\right|\right) + N\ln(2\pi h) + \frac{h}{K}\sum_{k=0}^{K-1}\left[\frac{\mathbf{v}(\mathbf{x}_k)\mathbf{c}}{2} + (\dot{\mathbf{x}}_k - \mathbf{f}(\mathbf{x}_k;\mathbf{c}))^T\,\hat{\mathbf{D}}^{-1}(\dot{\mathbf{x}}_k - \mathbf{f}(\mathbf{x}_k;\mathbf{c}))\right]. \tag{13}$$

which relates the dynamical variables $\mathbf{x}(t)$ of the system (10) to the observations $\mathbf{s}(t)$. Here we introduce the following notations $\hat{\mathbf{U}}_k \equiv \hat{\mathbf{U}}(\mathbf{y}_k)$, $\dot{\mathbf{y}}_k \equiv h^{-1}(\mathbf{y}_{k+1} - \mathbf{y}_k)$ and vector $\mathbf{v}(\mathbf{x})$ with components

$$\mathrm{v}_m(\mathbf{x}) = \sum_{n=1}^{N}\frac{\partial U_{nm}(\mathbf{x})}{\partial x_n}, \quad m = 1:M. \tag{14}$$

The vector elements $\{c_m\}$ and the matrix elements $\{D_{nn'}\}$ together constitute a set $M = \{\mathbf{c}, \hat{\mathbf{D}}\}$ of unknown parameters to be inferred from the measurements S. Choosing the prior PDF in the form of Gaussian distribution

$$p_{\text{pr}}(M) = \sqrt{\frac{\det(\hat{\Sigma}_{\text{pr}}^{-1})}{(2\pi)^M}}\exp\left(-\frac{1}{2}(\mathbf{c} - \mathbf{c}_{\text{pr}})^T\,\hat{\Sigma}_{\text{pr}}^{-1}(\mathbf{c} - \mathbf{c}_{\text{pr}})\right) \tag{15}$$

and substituting $p_{\text{pr}}(M)$ and the likelihood $\ell(S \mid M)$ into (11) yields in the posterior $p_{\text{post}}(M \mid lX) = \text{const} \times \exp[-S(M \mid X)]$, where

$$S(M \mid X) = \frac{R}{2} - \sum_{m=1}^{M} c_m w_m + \frac{1}{2}\sum_{m,l=1}^{M} c_m A_{ml}^{(1)} c_l \tag{16}$$

$$R = \int_0^T dt \sum_{m,l=1}^{M} \dot{x}_m(t_k) Q_{ml}\, \dot{x}_l(t_k) + K\ln(\det\hat{\mathbf{D}}) \tag{17}$$

$$w_m = \sum_{l=1}^{M} A_{ml}^{(0)} c_l + \int_0^T dt\left[\sum_{n,n'=1}^{L} U'_{mn}(t_k) Q_{nn'} t x_{n'}(t_k) - \frac{v_m(\mathbf{x}(t_k))}{2}\right] \tag{18}$$

Here $Q = D^{-1}$ is an inverse diffusion matrix. The analytical form of the inverse covariance matrix of the model parameters $A^{(1)}$ found by optimization of S has the form

$$A^{(1)}_{ml} = A^{(0)}_{ml} + \int_0^T dt U'_{mn}(t_k) Q_{nn'} U_{n'l}(t_k). \tag{19}$$

For the values of updated model parameters we have

$$c'_l = \left(A^{(1)}\right)^{-1}_{ml} w_m \tag{20}$$

The optimal values of the elements of diffusion matrix have the form

$$\tilde{D}_{nn'} \equiv \frac{1}{T}\int_0^T dt\left[\dot{\mathbf{x}}(t_k) - \hat{\mathbf{U}}(\mathbf{x}(t_k))\mathbf{c}\right]_n \left[\dot{\mathbf{x}}(t_k) - \hat{\mathbf{U}}(\mathbf{x}(t_k))\mathbf{c}\right]'_{n'} \tag{21}$$

Equations (19)-(21) provide an analytical solution of the problem and allow one to find mean values of the model parameters $\mathbf{c}$, the covariant matrix of their distribution $\left(A^{(1)}\right)^{-1}$ and also values of the diffusion matrix $D_{nn'}$.

3.2 Inferring parameters with synthetic data

First, we validate the novel inference technique by applying to the analysis of synthetic data generated by the stochastic growth cycle model of economy set of equations in the form

$$\dot{x} = \left(a + r_1 \sin \omega t + s_{11}\xi_1(t)\right)x + bxy \tag{22}$$

$$\dot{y} = \left(d + r_2 \sin \omega t + s_{22}\xi_2(t)\right)y + cxy \tag{23}$$

where $a = 1/\sigma - (\alpha + \beta)$, $b = -1/\sigma$, $d = -(\alpha + \gamma)$, and $c = \rho$.

Table 1 The results of the parameter estimation of the model based on Eqs. (22-23) using the Bayesian method and the time-series shown in Fig. 1. The total time of the measurements in this test was T=56 years, the sampling time was Δt=0.03 year, and the number of measured points was N=2000.

Parameter	Actual values	Inferred values
a	0.2	0.213
d	-0.1	-0.1029
c^*x_0	0.1	0.1051
b^*y_0	-0.2	-0.2155
r_1	0.05	0.0487
r_2	0.05	0.0366
D_{11}	0.000625	0.0006152
D_{22}	0.000625	0.0006429

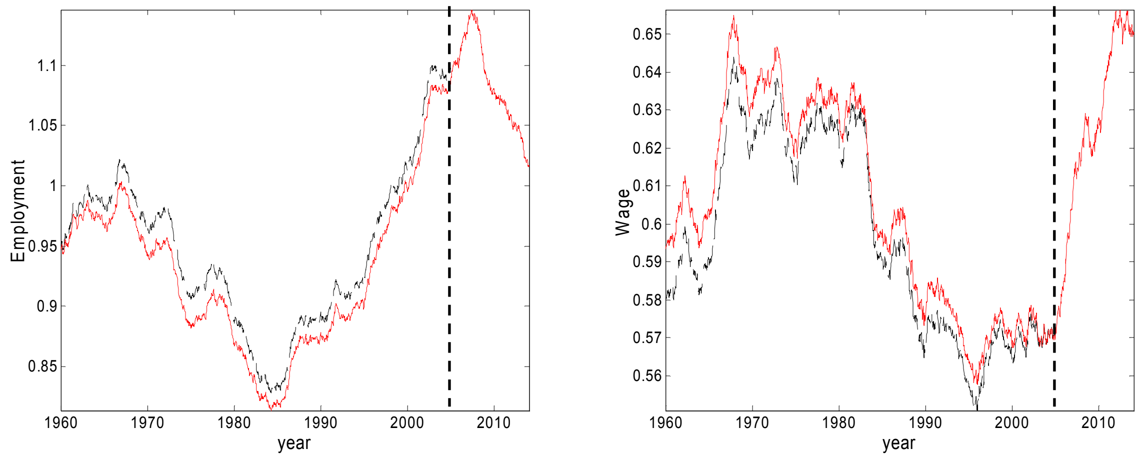

Fig. 1 Black solid lines show employment (left) and wages (right) generated by the set of equations (22-23) with the following parameters: a = 0.2; b = -0.005/1.35; c = 0.001*1.065; d = -0.1; e = -0.0; r1 = .05; r2 = .05. The number of points per block N=2000 and sampling time Δt =0.03. The historical simulation is given by the grey lines. The ex-ante forecasting results are shown beyond vertical dashed line.

The simulations presented in figure 1 are generated using Eqs. (22-23) with the model parameters inferred from the given time-series data. The initial conditions are chosen in such a way that the difference between the time-series data and the predictions is minimized in the least square sense. The initial prediction time is indicated by the vertical dashed lines. The results of the inference of the model parameters are summarize in table 1. It can be seen that the method can infer parameters of nonlinear stochastic predator-prey Lotka-Volterra model with reasonable accuracy. The stable fixed point of the system is given by coordinates $x_0 = -\frac{d}{c}$ and $y_0 = -\frac{a}{b}$. The period of the oscillations is given by $T = \frac{2\pi}{\sqrt{ad}}$.

3.3 Analysis of the time-series for France in 1960-2005

We now compare synthetic data with the time-series representing French economic cycles in 1960-2005. It is clear from the comparison of the synthetic data to the measurements of the French employment rate (v) and worker's share of the national income (u) presented in figure 2 that the model (22)-(23) can qualitatively reproduce the observed dynamics of the growth cycles of the French economy. The results of the inference are summarized in the Table 2. It is clear from the comparison that the results of inference are also in qualitative agreement with the model parameters selected to generate synthetic data. We now compare measured time-series data for the French economy with data generated by the model (4) using parameters inferred directly from the measured data. The results of the comparison are shown in the Fig. 2

Table 2 The results of the inference of the parameters of French economy for the stochastic model

Parameter	Inferred values	Model parameters
a	0.053	0.23
d	-0.127	-0.095
$c*x_0$	0.126	0.095
$b*y_0$	-0.0554	-0.23
r_1	-0.00125	0.015
r_2	0.00152	0.015
D_{11}	$8.5\cdot10^{-7}$	$6.25\cdot10^{-6}$
D_{22}	$5\cdot10^{-6}$	$6.25\cdot10^{-6}$

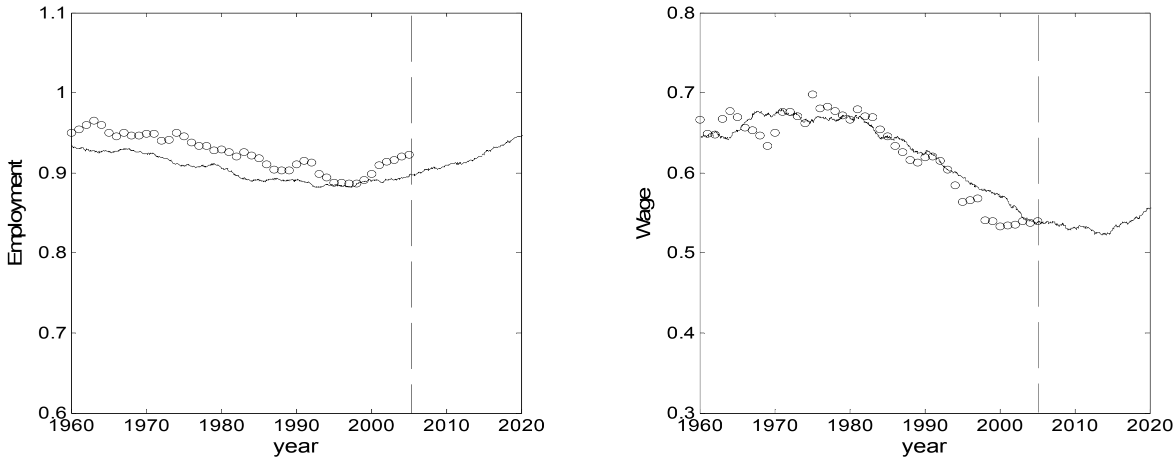

Fig. 2 Employment (left, circles) and wages (right, circles) for the French economy in 1960-2005 as compared to the synthetic data generated by the set of equations (22)-(23) (lines) using parameters inferred directly from the measured time-series data. Vertical dashed line indicates the initial time for ex-ante forecasting. Data Source: AMECO Database, European Commission.

3.4 Analysis of the time-series for Italy in 1960-2005

We now compare synthetic data with the time-series representing Italian economic cycles in 1960-2005. It is clear from the comparison of the synthetic data to the measurements of the Italian employment rate (v) and worker's share of the national income (u) that the model (22)-(23) can again qualitatively reproduce observed dynamics of the growth cycles of the Italian economy.

The results of the inference are summarized in table 3. It is clear from the comparison that the results of inference are also in qualitative agreement with the model parameters selected to generate synthetic data.

Table 3 The results of the inference of the parameters of Italian economy for the stochastic model

Parameter	Inferred values	Model parameters
a	0.025	0.27
d	-0.23	-0.09
c^*x_0	0.229	0.09
b^*y_0	-0.025	-0.27
r_1	-0.0003	0.015
r_2	0.0014	0.015
D_{11}	$8.7\cdot10^{-7}$	$6.25\cdot10^{-6}$
D_{22}	$8.8\cdot10^{-6}$	$6.25\cdot10^{-6}$

We now compare measured time-series data for the Italian economy with data generated by the model (22)-(23) using parameters inferred directly from the measured data. The results of the comparison are shown in figure 3.

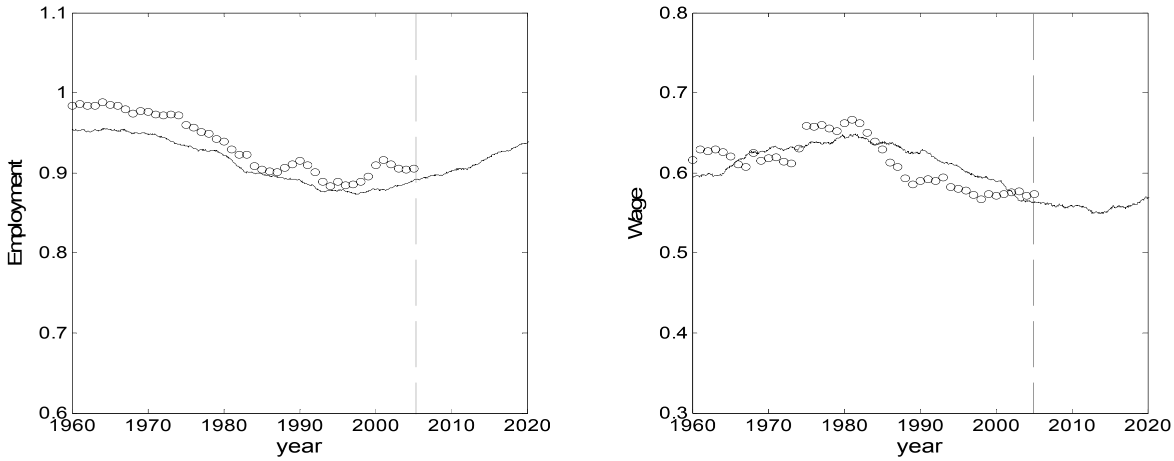

Fig. 3 Employment (left, circles) and wages (right, circles) for the Italian economy in 1960-2005 as compared to the synthetic data generated by the set of equations (4) (lines) using parameters inferred directly from the measured time-series data. Vertical dashed line indicates the initial time for ex-ante forecasting. Data Source: AMECO Database, European Commission.

4. CONCLUSION

The obtained results demonstrate that a straightforward inference of the parameters of a modified stochastic Goodwin model of the French and Italian economies leads to the quantitative agreement between the measured data and the dynamics predicted by the model. This result agrees and improves on earlier empirical estimations of the Goodwin model that used econometric-based tools. Further research will attempt to improve the model by applying results obtained recently in the analysis of limit cycles in predator-prey communities in the context of resources-consumers dynamics.

We note also that the proposed approach can be used as an effective tool for prediction of economic growth. It can also be extended to encompass inference of model parameters from empirical observations for a number of other stochastic models in economics and finance such as option pricing models and models of financial crashes.

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