A New Family of Space-Time Codes for Pulse Amplitude and Position Modulated UWB Systems

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***Abstract*— This paper presents the construction of new totally-real space-time coding schemes suited for carrier-less ultra-wideband transmissions. These schemes are associated with pulse position modulation and with hybrid pulse amplitude and position modulation where the input data is modulated onto both the pulse amplitudes and positions. The new schemes have a uniform average transmitted energy per antenna and achieve full rate and full diversity with hybrid M-PPM-M'-PAM for all values of M' and for $M \geq 3$, $M \geq 5$ and $\{M = 5, M \geq 7\}$ with $n = 2$, 3 and 4 transmit antennas respectively.**

I. Introduction

There is a growing interest in applying space-time (ST) coding techniques on impulse radio ultra-wideband systems (IR-UWB) [1], [2]. IR-UWB is used in conjunction with pulse amplitude modulation (PAM) and (or) pulse position modulation (PPM). In what follows, we use hybrid M-PPM-M'-PAM constellations where the amplitude of the pulse transmitted during each position can take M' possible values. This modulation scheme is appealing since it takes advantage of the high temporal resolution to deliver higher data rates with lower complexity [3]. This constellation is given by:

$$\mathcal{C} = \{(2m'-1-M')e_{m+1}; m' = 1, \dots, M'; m = 0, \dots, M-1\} \tag{1}$$

where e_m is the m-th column of the $M \times M$ identity matrix I_M. $\mathcal{C}$ entails PPM and PAM as special cases.

Among the different classes of ST codes, those constructed from cyclic division algebras (CDAs) are particularly simple [4], [5]. Moreover, CDAs result in a systematic code design for any number of antennas [6], [7]. Designate by $\mathbb{K}/\mathbb{F}$ the cyclic field extension of degree n whose Galois group is given by $Gal(\mathbb{K}/\mathbb{F}) = <\sigma>$ with $\sigma^n = 1$. Elements of the cyclic algebra $A(\mathbb{K}/\mathbb{F}, \sigma, \gamma)$ have the following matrix representation:

$$C(x_1, ..., x_n) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \gamma\sigma(x_n) & \sigma(x_1) & \cdots & \sigma(x_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma\sigma^{n-1}(x_2) & \gamma\sigma^{n-1}(x_3) & \cdots & \sigma^{n-1}(x_1) \end{bmatrix} \tag{2}$$

where $x_1, \dots, x_n \in \mathbb{K}$. γ is chosen such that there is no element in $\mathbb{K}$ whose norm is equal to γ^t for $t = 1, \dots, n-1$ [4]. γ is taken to be transcendental in [5] with $|\gamma| = 1$ resulting in information lossless codes [8]. On the other hand, if γ is chosen to be algebraic, then C has a non-vanishing determinant [6], [7] and achieves the diversity-multiplexing gain (D-MG) tradeoff over the Rayleigh fading channel [7].

On the other hand, in carrier-less IR-UWB systems, information is conveyed through real-valued pulses. Therefore, the complex-valued codes in [6], [9] are not suitable for such systems. This can be simply remedied by choosing $\mathbb{K}$, $\mathbb{F}$ and γ to be real. For example $\gamma = 2$ results in a non-vanishing determinant when $\mathbb{K}$ is the maximal real subfield of the cyclotomic field [2]. Moreover, eq. (2) can be readily applied with PPM-PAM. In this case, the scalars $x_1, \dots, x_n$ are replaced by M-dimensional vectors corresponding to the information symbols taken from $\mathcal{C}$ [2]. However, for totally-real constructions, energy-balanced codes ($|\gamma| = 1$) can be obtained uniquely by $\gamma = \pm 1$. This shows the non-existence of totally-real energy balanced-codes based on cyclic division algebras for $n \geq 3$ since $\gamma^2 = 1$ is always a norm. For $n = 2$, $\gamma = -1$ results in codes that are not information lossless [2].

In this paper, instead of adopting the classical approach of constructing ST codes over infinite fields ($\mathbb{Z}$ in the totally-real case), we profit from the particular structure of the PPM-PAM constellations in eq. (1) to construct new coding schemes suitable for these modulations. These schemes are energy-balanced and introduce no shaping losses. The advantage over the codes in [5], [6] and [9] is that the proposed codes are totally-real (there are no phase rotations in the codewords).

Note that we insist on the shaping constraint rather than the non-vanishing constraint for two reasons. First, the D-MG tradeoff (achieved by codes having non-vanishing determinants) is considered over Rayleigh fading channels while the propagation of UWB signals is subject to lognormal fading [10]. For these channels, and even for single-antenna situations, it is easy to find that the slope of the outage probability curves tends to infinity for large SNRs implying an ambiguity on the D-MG tradeoff with lognormal fading. On the other hand, at high SNRs, the spectral efficiency of M-PPM-M'-PAM can be increased by extending the constellation in the time domain (rather than the amplitude domain). This extension does not introduce a decrease in the coding gain. Denote by $\delta_{M,M'}$ the minimum determinant of eq. (2) with this constellation, we have $\delta_{N,M'} \geq \delta_{M,M'}$ for $N \geq M$ since N-PPM-M'-PAM is obtained by adding new dimensions to the initial signal set while $\delta_{M,N'} \leq \delta_{M,M'}$ for $N' \geq M'$ since M'-PAM is a subset of N'-PAM for $N' \geq M'$.

II. Code Construction

Consider a multi-antenna UWB system with $P = n$ transmit antennas, Q receive antennas and L fingers Rake. When using M-PPM-M'-PAM constellations with ST codes whose block length is T, the received signal can be expressed as:

$$Y = HX + N \tag{3}$$

where Y and N are $QLM \times T$ matrices corresponding to the decision variables and the noise terms respectively. X is the $PM \times T$ codeword whose $((p-1)M + m, t)$-th entry corresponds to the amplitude of the pulse (if any) transmitted at the m-th position of the p-th antenna during the t-th symbol duration for $p = 1, \dots, P$, $m = 1, \dots, M$ and $t = 1, \dots, T$. H is the $QLM \times PM$ channel matrix. $H = [H_{q,p}]$ for $q = 1, \dots, Q$ and $p = 1, \dots, P$ where $H_{q,p} = [H_{q,p,l}]$ is a $LM \times M$ matrix. $H_{q,p,l}$ is a $M \times M$ matrix for $l = 0, \dots, L-1$. The (m, m')-th element of $H_{q,p,l}$ corresponds to the impact of the signal transmitted during the m'-th position of the p-th antenna on the m-th correlator placed after the l-th Rake finger of the q-th receive antenna. For example, with the time hopping multiple access scheme [2]:

$$H_{q,p,l}(m, m') = r_{q,p}((m - m')\delta + \Delta_l) \tag{4}$$

where T_w and δ are the pulse-width and the modulation delay respectively. Δ_l is the l-th finger delay. $r_{q,p}$ corresponds to the frequency selective channel between antennas p and q.

We propose the following structure for the codewords:

$$X = DC \tag{5}$$

where C is a $PM \times P$ matrix that has the same form as eq. (2) but now the the "non-norm" scalar γ will be replaced by the $M \times M$ permutation matrix Ω given by:

$$\Omega = \begin{bmatrix} \mathrm{O}_{M-1}^T & 1 \\ I_{M-1} & \mathrm{O}_{M-1} \end{bmatrix} \tag{6}$$

where O_M is the M-dimensional vector whose components are all equal to 0. In eq. (2), the M-dimensional vectors x_i take the form $x_i = \sum_{j=0}^{n-1} a_{(i-1)n+j+1}\theta^j$ for $i = 1, \dots, n$ where $a_i \in \mathcal{C}$ are the information vectors given in eq. (1) for $i = 1, \dots, n^2$. $\{1, \theta, \dots, \theta^{n-1}\}$ is an integral basis of $\mathbb{K}$. D is a $PM \times PM$ diagonal matrix introduced for normalization purposes. Moreover, the correct choice of D will result in information lossless codes as discussed later.

A. 2×2 *code*

Consider the quadratic field extension given by $\mathbb{K} = \mathbb{Q}(\sqrt{5})$. For 2 transmit antennas and M-dimensional constellations, the coding scheme is constructed over the ring of integers of $\mathbb{K}$ given by $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}(\theta)$ with $\theta = \frac{1+\sqrt{5}}{2}$.

We must show that the rank of $C(X_1, X_2)$ is equal to 2 for all values of $(X_1, X_2) \neq (\mathrm{O}_M, \mathrm{O}_M)$ where $X_i = x_i - x'_i$ for $i = 1, 2$. $X_1, X_2 \in \mathcal{A}$:

$$\mathcal{A} = \{(a - a') + (b - b')\theta \mid a, a', b, b' \in \mathcal{C}\} \subset \mathcal{O}_{\mathbb{K}}^M \tag{7}$$

where $\mathcal{C}$ is given in eq. (1). When there is no ambiguity, $C(X_1, X_2)$ will be referred to as C. Denote by $X_{i,m}$ the m-th component of the vector X_i for $i = 1, 2$ and $m = 1, \dots, M$.

Proposition 1: if $\exists\, i, m \,|\, X_{i,m} = 0$ then rank$(C(X_1, X_2)) = 2$ unless $X_1 = X_2 = \mathrm{O}_M$.

Proof: Designate by π the cyclic permutation given by: $\pi(i) = i \mod (M) + 1$. π defines a bijection over the set $\{1, \dots, M\}$. $C(X_1, X_2)$ can be written as:

$$\begin{bmatrix} X_{1,1} & \cdots & X_{1,M} & \sigma(X_{2,\pi^{-1}(1)}) & \cdots & \sigma(X_{2,\pi^{-1}(M)}) \\ X_{2,1} & \cdots & X_{2,M} & \sigma(X_{1,1}) & \cdots & \sigma(X_{1,M}) \end{bmatrix}^T \tag{8}$$

Suppose that $X_{1,m} = 0$ for a given value of $m \in \{1, \dots, M\}$. When rank$(C) < 2$, its two columns have the same direction. Therefore, considering the first M rows of C, $X_{1,m} = 0 \Rightarrow X_{2,m} = 0$. Now we have $\sigma(X_{2,m}) = 0$ (since $X_{2,m} = 0$ and $\{1, \theta\}$ is an integral basis of $\mathcal{O}_{\mathbb{K}}$). Considering the last M rows of C, $\sigma(X_{2,m}) = 0 \Rightarrow \sigma(X_{1,\pi(m)}) = 0 \Rightarrow X_{1,\pi(m)} = 0$. Starting the same procedure again with $\pi(m)$ rather than m, we can conclude by iteration that $X_{1,m} = \cdots = X_{1,\pi^{M-1}(m)} = 0$ and $X_{2,m} = \cdots = X_{2,\pi^{M-1}(m)} = 0 \Leftrightarrow X_1 = X_2 = \mathrm{O}_M$ since π is a bijection over $\{1, \dots, M\}$. The same proof holds if $\exists m \,|\, X_{2,m} = 0$.

Lemma 1: The code achieves full diversity for $M > 4$.

Proof: From the definition of $\mathcal{A}$ in eq. (7), X_1 and X_2 are linear combinations of any 4 columns of I_M. Therefore for $M > 4$, X_1 and X_2 each has at least one zero component resulting in full rank as shown in proposition 1.

We must now verify that C has a full rank when all of its components are nonzero. In this case, rank$(C) < 2$ implies that:

$$\frac{X_{2,1}}{X_{1,1}} = \ldots = \frac{X_{2,M}}{X_{1,M}} = \ldots = \frac{\sigma(X_{1,M})}{\sigma(X_{2,\pi^{-1}(M)})} = k \tag{9}$$

where $k \in \mathbb{K}$. After some manipulations, eq. (9) becomes:

$$X_{1,1} = (\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(k))^{M+1-m} X_{1,m} \;\; ; \;\; m = M, \dots, 2 \tag{10}$$

where $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(k)$ corresponds to the norm of k. On the other hand, $X_{1,m}, X_{2,m} \in \mathcal{O}_{\mathbb{K}}^* = \mathbb{Z}^* \oplus \theta\mathbb{Z}^* \oplus \mathcal{O}'_{\mathbb{K}}$ for all values of m where $\mathcal{O}'_{\mathbb{K}} = \{a + b\theta \mid a, b \in \mathbb{Z}^* = \mathbb{Z} \backslash \{0\}\}$. Since $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(k) \in \mathbb{Q}$ for $k \in \mathbb{K}$, eq. (10) implies that $X_{1,1}, \dots, X_{1,M}$ (and $X_{2,1}, \dots, X_{2,M}$ in an equivalent manner) must belong to one of the following sets $\mathbb{Z}^*$, $\theta\mathbb{Z}^*$ or $\mathcal{O}'_{\mathbb{K}}$ simultaneously.

Following from the structure of $\mathcal{A}$, a maximum number of 2 components of X_1 (or X_2) can contain an integer or an integral multiple of θ. Therefore the code is not fully diverse with $M = 2$. From eq. (7), both entries of X_1 can belong to $\mathbb{Z}^*$ (resp. $\theta\mathbb{Z}^*$) when $b = b'$ (resp. $a = a'$) and $(a, a') = (x_1 e_i, x_2 e_j)$ (resp. $(b, b') = (x_1 e_i, x_2 e_j)$) for $i, j = 1, 2$ and $i \neq j$. In the same way, $X_{1,1}$ and $X_{1,2}$ can both be in $\mathcal{O}'_{\mathbb{K}}$. In this case, the vector X_1 takes the form $X_1 = (x_1 + \theta x_2)e_i + (x_3 + \theta x_4)e_j$ where $x_1, \dots, x_4$ are M'-ary PAM symbols and $i \neq j$.

For $M = 3$, when $X_{1,m} \neq 0$ for $m = 1, \dots, M$, X_1 belongs to the set of all possible permutations of:

$$\mathcal{A}' = \{[x_1, x_2, x_3\theta]^T, [x_1, x_2\theta, x_3\theta]^T, [x_1, x_2\theta, x_3 + x_4\theta]^T\}$$

where $x_1, \dots, x_4 \in \mathbb{Z}^*$. Therefore, a maximum number of two components of X_1 can be in $\mathbb{Z}^*$ (or $\theta\mathbb{Z}^*$) at the same time while only one component can belong to $\mathcal{O}'_{\mathbb{K}}$. This is in contradiction with eq. (10) which proves that the proposed code is fully diverse.

For $M=4$, eq. (10) is in contradiction with the structure of $\mathcal{A}$. When $X_{1,m} \neq 0$ for $m=1,\ldots,M$, and in order to occupy 4 positions, X_1 must be a permutation of the vector $(x_1e_1+\theta x_2e_2+x_3e_3+\theta x_4e_4)$ where $x_1,\ldots,x_4$ are M'-PAM symbols. This implies that there are two values $X_{1,i}, X_{1,j} \in \mathbb{Z}^*$ while the other two values $X_{1,k}, X_{1,l} \in \theta\mathbb{Z}^*$. Therefore, the components of X_1 can not belong simultaneously to $\mathbb{Z}^*$, $\theta\mathbb{Z}^*$ or $\mathcal{O}'_{\mathbb{K}}$. Considering lemma 1 and the cases $M=3$ and $M=4$, we conclude that C achieves full diversity for $M \geq 3$.

B. 3×3 code

For 3 transmit antennas, the code is constructed over the field extension $\mathbb{K}=\mathbb{Q}(\theta)$ whose Galois group is $Gal(\mathbb{K}/\mathbb{Q}) = \langle\sigma\rangle$ with $\sigma^3=1$ and $\theta = 2\cos(\frac{2\pi}{7})$.

We must show that $\text{rank}(C(X_1,X_2,X_3)) = 3$ for $(X_1,X_2,X_3) \in \mathcal{A}^3\backslash\{(\text{O}_M,\text{O}_M,\text{O}_M)\}$ where $X_i = x_i - x'_i$ for $i=1,2,3$. $\mathcal{A}$ is given by:

$$\mathcal{A} = \{(a-a')+(b-b')\theta+(c-c')\theta^2 \mid a,a',b,b',c,c' \in \mathcal{C}\} \tag{11}$$

For an element $k \in \mathbb{K} \mid k = \sum_{j=0}^{n-1} k_j\theta^j$, $k_j \in \mathbb{Q}$ will be referred to as the j-th coordinate of k. In eq. (11), since a,a',b,b',c,c' are multiples of the columns of the $M \times M$ identity matrix, elements of $\mathcal{A}$ have the property that at most two of their components can have their j-th coordinates different from zero for $j=0,\ldots,n-1$.

$C(X_1,X_2,X_3)$ will be referred to as C for simplicity. For $M=2$, the code is not fully diverse. In fact, when $X_1 = X_2 = X_3 = [1\ 1]^T$, all the columns of C are equal.

In what follows, we will limit ourselves to the case $M \geq 3$. $\text{rank}(C) < 3$ if $\exists k_1,k_2$ such that $C_3 = k_1C_1+k_2C_2$ where C_i is the i-th column of C. Moreover, $k_1,k_2 \in \mathbb{K}$ since all of the elements of C are in $\mathbb{K}$. Solving the equations:

$$\begin{aligned} X_3 &= k_1X_1+k_2X_2 \\ X_2 &= \Omega\sigma^2(k_1)X_3+\sigma^2(k_2)X_1 \\ X_1 &= \Omega\sigma(k_1)X_2+\Omega\sigma(k_2)X_3 \end{aligned}$$

We conclude that the vectors X_i for $i=1,\ldots,3$ must verify:

$$\left(I_M+\lambda_1\Omega+\lambda_2\Omega^2\right)X_i = \text{O}_M \tag{12}$$

$$\lambda_1 = -\left(\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\sigma^2(k_1)k_2)+\text{N}_{\mathbb{K}/\mathbb{Q}}(k_2)\right)\ ;\ \lambda_2 = -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_1) \tag{13}$$

Let $\mathcal{M} = \left(I_M+\lambda_1\Omega+\lambda_2\Omega^2\right)$. $\mathcal{M} \in \mathbb{Q}^{M\times M}$ since $\lambda_1,\lambda_2 \in \mathbb{Q}$ because they are linear combinations of norms and traces that are always rational numbers.

Proposition 2: All non-zero vectors X_i that verify $\mathcal{M}X_i = \text{O}_M$ are not in $\mathcal{A}$ given in eq. (11) for $M \geq 5$.

Proof: See Appendix I. This proves that the proposed scheme achieves full diversity with $M \geq 5$.

C. 4×4 code

For 4 transmit antennas, the code is constructed over the field extension $\mathbb{K}=\mathbb{Q}(\theta)$ where $\theta = 2\cos(\frac{2\pi}{15})$.

We must show that $\text{rank}(C(X_1,X_2,X_3,X_4)) = 4$ for $(X_1,X_2,X_3,X_4) \in \mathcal{A}^4\backslash\{(\text{O}_M,\text{O}_M,\text{O}_M,\text{O}_M)\}$.

$$\mathcal{A} = \{\sum_{j=0}^{3}(a_j-a'_j)\theta^j \mid a_0,a'_0,\ldots,a_3,a'_3 \in \mathcal{C}\} \tag{14}$$

$\text{rank}(C) < 4$ if $\exists k_1,k_2,k_3$ such that $C_4 = k_1C_1+k_2C_2+k_3C_3$ where C_i is the i-th column of C. Moreover, $k_1,k_2,k_3 \in \mathbb{K}$. Solving the equations:

$$\begin{aligned} X_4 &= k_1X_1+k_2X_2+k_3X_3 \\ X_3 &= \sigma^3(k_2)X_1+\sigma^3(k_3)X_2+\sigma^3(k_1)\Omega X_4 \\ X_2 &= \sigma^2(k_3)X_1+\sigma^2(k_1)\Omega X_3+\sigma^2(k_2)\Omega X_4 \\ X_1 &= \sigma(k_1)\Omega X_2+\sigma(k_2)\Omega X_3+\sigma(k_3)\Omega X_4 \end{aligned}$$

We conclude that the vectors X_i for $i=1,...,4$ must verify:

$$\left(I_M+\lambda_1\Omega+\lambda_2\Omega^2+\lambda_3\Omega^3\right)X_i = \text{O}_M \tag{15}$$

$$\begin{aligned} \lambda_1 &= -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_3)-\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_1\sigma(k_3)) \\ &\quad -\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma(k_3)\sigma^2(k_3))-\frac{1}{2}\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma^2(k_2)) \\ \lambda_2 &= \text{N}_{\mathbb{K}/\mathbb{Q}}(k_2)-\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma^2(k_1)\sigma^3(k_1))- \\ &\text{Tr}_{\mathbb{K}/\mathbb{Q}}(k_2\sigma(k_2)\sigma^3(k_1)\sigma^2(k_3))+\frac{1}{2}\text{Tr}_{\mathbb{K}/\mathbb{Q}}(\sigma(k_1)\sigma^3(k_1)k_3\sigma^2(k_3)) \\ \lambda_3 &= -\text{N}_{\mathbb{K}/\mathbb{Q}}(k_1) \end{aligned}$$

Let $\mathcal{M} = (I_M+\lambda_1\Omega+\lambda_2\Omega^2+\lambda_3\Omega^3)$. $\mathcal{M} \in \mathbb{Q}^{M\times M}$ since $\lambda_1,\lambda_2\,\lambda_3 \in \mathbb{Q}$.

Proposition 2: All non-zero vectors X_i that verify $\mathcal{M}X_i = \text{O}_M$ are not in $\mathcal{A}$ for $M=5$ and $M \geq 7$.

Proof: See Appendix II. This proves that the proposed scheme achieves full diversity with $M=5$ and $M \geq 7$.

D. Choice of D

The diagonal matrix D in eq. (5) takes the form ($n=P$):

$$D = \text{diag}\left([\alpha \quad \sigma(\alpha) \quad \cdots \quad \sigma^{n-1}(\alpha)]\otimes 1_M^T\right) \tag{16}$$

where $\otimes$ corresponds to the Kronecker product and 1_M is the M-dimensional vector whose elements are equal to 1. Instead of transmitting $a_1,\ldots,a_{n^2}$, the coding scheme given in eq. (5) transmits permutations of the conjugates of $x_1,\ldots,x_n$:

$$\begin{bmatrix} x_{i,m} \\ \vdots \\ \sigma^{n-1}(x_{i,m}) \end{bmatrix} = D_0(\alpha)\Gamma(\theta)\begin{bmatrix} a_{n(i-1)+1,m} \\ \vdots \\ a_{ni,m} \end{bmatrix} \tag{17}$$

where $x_{i,m}$ is the m-th component of x_i for $i=1,\ldots,n$ and $m=1,\ldots,M$. $D_0(\alpha) = \text{diag}\left([\alpha \cdots \sigma^{n-1}(\alpha)]\right)$ and $\Gamma(\theta)$ is the $n\times n$ matrix whose (i,j)-th element is equal to $\sigma^{i-1}(\theta^{j-1})$. Normalizing the transmitted energy implies that $\det(D_0(\alpha)\Gamma(\theta)) = 1$ resulting in:

$$\det(D_0) = \text{N}_{\mathbb{K}/\mathbb{Q}}(\alpha) = \frac{1}{\det(\Gamma(\theta))} \triangleq \frac{1}{\sqrt{d_{\mathbb{K}}}} \tag{18}$$

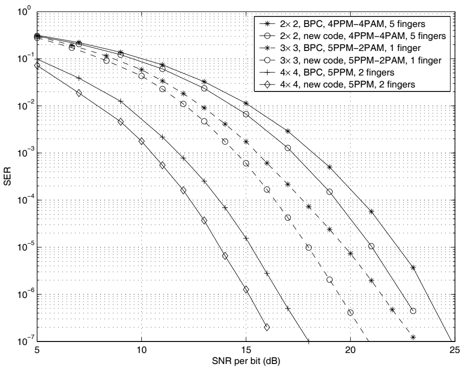

Fig. 1. The proposed codes vs. the best previously known totally-real codes (BPC) [2].

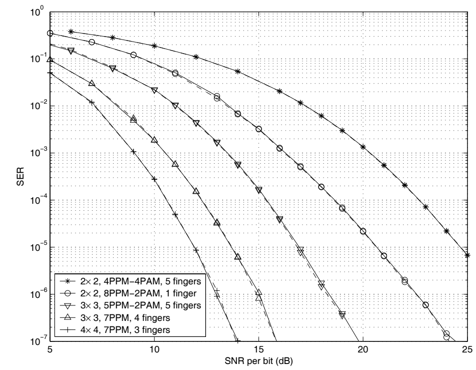

Fig. 2. Solid lines correspond to the new codes. Dashed lines correspond to [6] and [9].

where $d_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$. In order to satisfy eq. (18), we fix $\alpha = \beta$ when n is odd and $\alpha = \sqrt{\beta}$ when n is even. $\beta \in \mathbb{K}$ and it must be totally-positive when n is even.

Using KANT software [11], we find that we can choose $5\beta = 3 - \theta$, $7\beta = 1 - \theta + 2\theta^2$ and $15\beta = 5 + 6\theta - \theta^2 - 2\theta^3$ for $n = 2$, 3 and 4 respectively. This choice of D corresponds to limiting the construction in an ideal $I = \alpha\mathcal{O}_{\mathbb{K}}$ such that the volume of the lattice generated by I is equal to 1.

From eq. (17), if we choose $D_0(\alpha)\Gamma(\theta)$ to be unitary, we obtain a transmitted constellation that is a rotation of the initial PPM-PAM signal set. Therefore, we must construct an orthonormal basis $\{v_i\}_{i=1}^n$ that is a rotation of the original basis $\{u_i\}_{i=1}^n = \alpha\{1, \dots, \theta^{n-1}\}$. For example, we can use the transformation matrices given in [2]. For $n = 2$, the original basis is orthonormal. For $n = 3$, the new basis is given by $7\{v_i\}_{i=1}^3 = \{\mu, \sigma(\mu), \sigma^2(\mu)\}$ with $\mu = -2 + 2\theta + 3\theta^2$. For $n = 4$, $\{v_i\}_{i=1}^4 = \sqrt{\beta}\{1, -1 - 3\theta + \theta^2 + \theta^3, -1 - 2\theta + \theta^2 + \theta^3, -1 + 3\theta - \theta^3\}$.

Consider the matrix Φ obtained by writing eq. (5) as $\text{vec}(X) = \Phi[a_1^T, \dots, a_{n^2}^T]^T$. It is easy to show that choosing the basis $\{v_i\}_{i=1}^n$ to be orthonormal results in a unitary matrix Φ since Ω is unitary. Therefore, the proposed codes introduce no shaping losses according to the definition given in [6]. In an equivalent way, we can say that the codes are information lossless [8]. Finally, the modified version of eq. (5) is obtained by setting $D = I_{nM}$ and $x_i = \sum_{j=1}^n a_{n(i-1)+j} v_j$ for $i = 1, \dots, n$ and $a_1, \dots, a_{n^2} \in \mathcal{C}$ given in eq. (1).

III. Simulations and results

The second derivative of the Gaussian pulse is used with $T_w = \delta = 0.5$ns. The sub-channels are generated independently using the IEEE 802.15.3a channel model CM2 [10].

Fig. 1 compares the proposed codes with the best previously known totally-real codes [2]. The latter can be obtained from eq. (2) by setting $\gamma = 2$. This choice results in a non-vanishing determinant for PPM-PAM constellations. Moreover, the coding gain is maximized for $n = 2, 3$ [2]. The receiver has the same number of antennas as the transmitter. The superiority of the proposed schemes is obvious in all situations. This shows the importance of using energy-balanced codes.

In Fig. 2, the proposed codes are compared with [6], [9]. For comparison reasons, and even though it may seem practically infeasible, the UWB receivers are supposed to be equipped with IQ front ends. The results show the high performance levels of the proposed schemes. Even though they are real-valued, they show exactly the same performance as the best known codes. In fact, we can find numerically that the coding gain of the proposed 2×2 code is the same as that of the Golden code for M-PPM-M'-PAM with $M = 3$ (resp. $M \geq 4$) and $M' = 1,\ 2,\ 4,\ 8$ (resp. $M' = 2,\ 4,\ 8$). For $n = 3$, 4 there is no expression of the coding gain even though the simulations show practically the same performance as [6], [9].

IV. Conclusion

We investigated the problem of constructing ST coding schemes suitable for UWB systems using M-PPM-M'-PAM constellations. We presented new totally real constructions that are suitable to carrier-less 2×2, 3×3 and 4×4 antennas UWB systems. These constructions solve the problem of the non-existence of energy-balanced, information lossless and totally-real constructions. They outperform the best known totally-real ST codes based on cyclic division algebras.

Appendix I

When $M \geq 3$, the rank of the matrix $\mathcal{M}$ verifies the relation $r = \text{rank}(\mathcal{M}) \geq M-2$. In fact, the matrix composed from the first $M-2$ rows and $M-2$ columns of $\mathcal{M}$ is a lower triangular matrix whose diagonal elements are all equal to 1. Therefore, the determinant of this matrix is equal to 1 and consequently it has a rank of $M-2$ and so $r \geq M-2$. Since eq. (12) is valid for all values of i, we fix $Y = X_i$. Denote by Y_m the m-th component of Y for $m = 1, \dots, M$. The j-th coordinate of Y_m is denoted by $Y_{m,j}$ for $j = 0, 1, 2$.

When $r = M$, $\mathcal{M}Y = \text{O}_M \Rightarrow Y = \text{O}_M$ and therefore the only matrix C that is rank-deficient is the all-zero matrix.

When $r = M - 1$, the M components of Y can be determined from a single parameter. Without loss of generality, this parameter is taken to be equal to Y_M. In this case, $Y_m = \beta_m Y_M$ with $\beta_m \in \mathbb{Q}^*$ for $m = 1, \dots, M-1$ since $\mathcal{M} \in \mathbb{Q}^{M\times M}$. $Y_M = 0 \Leftrightarrow Y = \text{O}_M$. For non-zero vectors, denote by $Y_{M,j}$ the first non-zero coordinate of Y_M for a given value of $j \in \{0,1,2\}$. Since $\beta_m \in \mathbb{Q}^*$, all the components of Y will have non-zero j-th coordinates. Therefore, $Y \notin \mathcal{A}$ when $M \geq 3$ since, from eq. (11), any element of $\mathcal{A}$ can have a maximum number of two components whose j-th coordinates are different from zero $\forall j$. Therefore, the only vector of $\mathcal{A}$ that verifies $\mathcal{M}Y = \mathrm{O}_M$ when $r = M-1$ is $Y = \mathrm{O}_M$.

When $r = M-2$, the components of Y can be written as: $Y_m = \beta_m^{(1)} Y_{M-1} + \beta_m^{(2)} Y_M$ with $(\beta_m^{(1)}, \beta_m^{(2)}) \in \mathbb{Q}^2 \backslash \{(0,0)\}$ for $m = 1, \dots, M-2$.

Suppose that $\nexists k \mid Y_{M-1,k} \neq 0$ and $Y_{M,k} \neq 0$. Denote by j and l the indexes of the first non-zero coordinates of Y_{M-1} and Y_M respectively ($Y_{M,j} = Y_{M-1,l} = 0$). Among the first r components of Y, designate by r_1 and r_2 the number of components having their j-th and l-th coordinates different from zero respectively. $0 \leq r_i \leq r$ for $i = 1,2$. Along with Y_{M-1} and Y_M, r_1+1 and r_2+1 components of Y can have non-zero j-th and l-th coordinates respectively. Y_M can not be a rational multiple of Y_{M-1}. Moreover, since $(\beta_m^{(1)}, \beta_m^{(2)}) \neq (0,0)$ for $m = 1, \dots, r$, there are at least r non-zero coefficients among $\beta_1^{(1)}, \dots, \beta_r^{(1)}, \beta_1^{(2)}, \dots, \beta_r^{(2)}$. Therefore, $r_1 + r_2 \geq r$. Y will fall outside of $\mathcal{A}$ whenever $\exists i | r_i + 1 > 2$. The maximum value of M for which the code is not fully diverse is $M = 4$. In this case, $r_1 = r_2 = 1$ is a feasible solution ($r = 2$). For example, when $X_1 = X_3 = [1\ \theta\ 1\ \theta]^T$ and $X_2 = \Omega X_1$ the first and third columns of C are equal. When $M \geq 5$, $r_1 + r_2 \geq r \geq 3 \Rightarrow r_1 > 1$ or $r_2 > 1$.

The second case is when $\exists j \mid Y_{M-1,j} \neq 0$ and $Y_{M,j} \neq 0$. We limit ourselves to the case $M \geq 5$. In this case, it is possible to have a maximum number of one zero-component among $Y_1, \dots, Y_r$. When there are more than one zero-components, and since $r \geq 2$, this will result in $Y = \mathrm{O}_M$. Therefore, there is at least $r' = r-1 \geq 1$ components among $Y_1, \dots, Y_r$ having their j-th coordinates different from zero. Therefore, along with Y_{M-1} and Y_M, there are $r'+2$ components having their j-th coordinates different from zero. Since $r'+2 > 2$ for $M \geq 5$, this implies that $Y \notin \mathcal{A}$. From all that preceded, we conclude that for $M \geq 5$, C has full rank for $(X_1, X_2, X_3) \in \mathcal{A}^3 \backslash \{(\mathrm{O}_M, \mathrm{O}_M, \mathrm{O}_M)\}$.

Appendix II

For $M \geq 4$, $r = \text{rank}(\mathcal{M}) \geq M-3$ since the matrix constructed from the first $M-3$ rows and $M-3$ columns of $\mathcal{M}$ has a full rank. For $r \in \{M, M-1, M-2\}$, the proof is the same as in the preceding appendix. When $r = M-3$, the components of Y can be parameterized by $(t^{(1)}, t^{(2)}, t^{(3)}) = (Y_{M-2}, Y_{M-1}, Y_M)$ such that $Y_m = \sum_{k=1}^{3} \beta_m^{(k)} t^{(k)}$ with $(\beta_m^{(1)}, \beta_m^{(2)}, \beta_m^{(3)}) \neq (0,0,0)$ for $m = 1, ..., r$.

Case 1: Suppose that $\nexists\ j \in \{0, \dots, 3\} \mid t_j^{(k)} \neq 0$ for $k = 1,2,3$. Denote by j_k the first non-zero coordinate of $t^{(k)}$ for $k = 1,2,3$. Among the first r components of Y, designate by r_k the number of components whose j_k-th coordinates are different from zero. $0 \leq r_k \leq r$ for $k = 1,2,3$. Along with $t^{(1)}$, $t^{(2)}$ and $t^{(3)}$, r_k+1 components of Y can have non-zero j_k-th coordinates. Moreover, since $(\beta_m^{(1)}, \beta_m^{(2)}, \beta_m^{(3)}) \neq (0,0,0)$ for $m = 1, \dots, r$, there are at least r non-zero coefficients among $\beta_m^{(1)}, \beta_m^{(2)}, \beta_m^{(3)}$ for $m = 1, \dots, r$. Therefore, $r_1 + r_2 + r_3 \geq r$. Y will fall outside of $\mathcal{A}$ whenever $\exists k \mid r_k + 1 > 2$ implying that $M \geq 7$. Therefore, $\mathcal{M}Y = \mathrm{O}_M \Rightarrow Y \notin \mathcal{A}$ for $M \geq 7$.

Case 2: consider the complementary of case 1 where $\exists k, k'$ such that $t^{(k)}$ and $t^{(k')}$ have nonzero j-th coordinates. Consider the $r \times 3$ matrix $\mathcal{M}'$ whose (i,j)-th element is equal to $\beta_i^{(j)}$. Since the j-th coordinates are present only in $t^{(k)}$ and $t^{(k')}$, if $Y_1, \dots, Y_r$ all have zero j-th coordinates, then the k-th and k'-th columns of $\mathcal{M}'$ are linearly dependent which is impossible given that $r \geq 4$ (for $M \geq 7$). Therefore, along with $t^{(k)}$ and $t^{(k')}$, more than two components will have non-zero j-th coordinates showing that Y will fall outside of $\mathcal{A}$ for $M \geq 7$.

$\mathcal{M}$ is a circulant matrix that can be expressed as: $\mathcal{M} = \sum_{i=1}^{M} \lambda_i \Omega^i$ with $\lambda_M = 1$ since $\Omega^M = I_M$. From [12], the eigenvalues of $\mathcal{M}$ can be expressed as:

$$\mu_k = \sum_{n=0}^{M-1} \omega^{kn} \lambda_{M-n} \quad ; \quad k = 0, \dots, M-1 \tag{19}$$

where $\omega = \exp\left(\frac{2\pi i}{M}\right)$ is the M-th root of unity. $\{1, \omega, \dots, \omega^{M-1}\}$ forms a basis of degree $\mathcal{E}(M)$ over $\mathbb{Q}$ where $\mathcal{E}$ stands for Euler's function. For $\mathcal{E}(M) \geq n$ (n is the number of transmit antennas) and since $\lambda_M \neq 0$, only μ_0 can be equal to zero and therefore $r = \text{rank}(\mathcal{M}) \geq M-1$. From what preceded, $\mathcal{M}Y = \mathrm{O}_M$; $Y \in \mathcal{A}$; $r \in \{M, M-1\} \Rightarrow Y = \mathrm{O}_M$ for $M \geq 3$. Therefore, the code is fully diverse for the values of M verifying $\mathcal{E}(M) \geq n$ and $M \geq 3$. This implies that the proposed code is also fully diverse with $n = 4$ and $M = 5$.

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