Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations

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by

Sirine Abdallah Labban

Under the Direction of

Dr. Iman Osta

LEBANESE AMERICAN UNIVERSITY

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LEBANESE AMERICAN UNIVERSITY

We hereby approve the project of

Student's Full Name: Sirine Abdallah Labban

Full Title of the Project: "Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations"

Date Submitted: February 23, 2005

Division: Education

Dr. Iman Osta
Supervisor

Dr. May Hamdan
Committee Member

A copy of the project report is available at the University Library.

Student Signature

Date: 23/02/05
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AN ABSTRACT OF THE RESEARCH PROJECT OF

Sirine Abdallah Labban for Masters of Arts

Major: Education/ Math

Title: Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations

In this study we investigated the thinking strategies of grade 7 students in solving a problem situation involving first-degree equations, prior to formal instruction in algebra. Twelve students participated in individual, talk-aloud problem solving sessions and were interviewed about their attempts to solve the given problem. The sessions were videotaped for further analysis. Students generated different solution plans, using arithmetical rather than algebraic methods. The majority of participants used trial-and-error strategies for solving the given problem. Very few were those who constructed an algebraic equation but then shifted to an arithmetic approach while solving it, representing the unknown by a “?” (a question mark) or “_” (a blank space). The study revealed that it is yet unfamiliar to students who represented the problem’s mathematical content with a mathematical equation form to continue and proceed algebraically. Thus, before being introduced to the concept of equation and unknown, and before being instructed on the algorithms for solving equations, students do not operate algebraically on equations, which are not yet symbolically formalized.
CHAPTER 1

Introduction

Nowadays, mathematics curricula, all over the world, call for greater understanding of the fundamentals of algebra and algebraic reasoning by all members of the society.

Algebra is more than memorizing rules for manipulating symbols and solving prescribed types of problems. It is part of the reasoning process, a problem solving strategy, and a key to think and to communicate with mathematics (The National Council of Teachers of Mathematics, 1989).

Accordingly, the National Council of Teachers of Mathematics recommends that algebra be studied by all students of all grade levels, including those who are low achieving and underserved (NCTM, 1989).

When do children start to learn algebra? When do we begin to teach students algebra? Realistically, children learn the basics of algebra at an early stage, at the elementary level. When children first learn addition and subtraction, they begin to learn algebra. They may not recognize it because an “x” or a “y” is not present. When learning their addition and subtraction facts, children work with problems such as $3 + 4 = \square$ or $3 + \square = 4$. The box represents the number the students are trying to find. It represents then a mark that covers a hidden number, the unknown. Conceptually, there is no difference between $3 + 4 = \square$ and $3 + 4 = x$ or $3 + \square = 4$ and $3 + x = 4$. Either way, the student is being asked to find an unknown. Moreover, sometimes students see blanks such as $3 + 4 = \ldots$ or $3 + \ldots = 4$ rather than boxes.
Thus, when students are asked to fill in the blanks with the correct response, they are solving an algebraic equation (Stacey & MacGregor, 1997).

Thus, the concept of equation is informally introduced in the primary grades, when the equation is first defined as an “arithmetic identity with a hidden number”. However, it is only until grade 7 that the concept of equation is formally taught as an independent mathematical algebraic concept. The hiding that was done first by a finger then by a box is finally represented by a letter, which appears most of the time on both sides of the “equal” sign. Nevertheless, this use of letters to represent the unknowns is considered a major and significant shift from arithmetic to algebra.

Rationale

Experiences in the field of teaching show that 7th grade students encounter major problems in shifting from arithmetic to algebra. The most frequent difficulty remains in writing and solving first-degree equations representing real-life problem situations. In fact, in arithmetic, students are used to deal with known information to get the unknown quantities without the use of any symbols or equations to express relationships. However, in algebra, starting to use a symbol to represent the unknown is problematic for pupils (Stacey & MacGregor, 1997).

In order to minimize those problems and to effectively introduce the concept of equations, proactive measures should be considered. Teachers need to investigate which didactical means enable students to make a smooth transition from arithmetic to algebra. A way for approaching this should start from students’ informal strategies to build more formal methods out of these, and provide more meanings to the symbols and procedures being used.

Unfortunately, although extensive research on algebra learning has been conducted, educators do not have a complete picture of what students can do in algebra prior to formal
instruction. They do not know exactly which aspects of informal knowledge create useful foundation upon which instruction can be built (Swafford & Langrall, 2000). Moreover, no research has been conducted on students’ reasoning and knowledge construction while solving problem situations involving first-degree equations, before the introduction of the concept of equation and of unknown. In fact, the majority of the research projects focused on the difficulties that students encounter while learning algebra, mainly while solving equations, constructing equations from word problems, as well as interpreting, rewriting, and simplifying algebraic expressions (Herscovics & Kieran, 1980; Ishida, 2002; Stacey & MacGregor, 1999b; Swafford & Langrall, 2000; Van Amerom, 2003; Warren, 2003).

Therefore, conducting a study to discover the difficulties that students face while shifting from arithmetic to algebra is essential in order to avoid the problems that those students might face later in solving first-degree equations.

_Purpose, Statement of the Topic, and Operational Definitions_

The purpose of this study is to explore the _thinking strategies_ of grade 7 students in solving problem situations involving _first-degree equations_, prior to formal instruction in algebra, thus before being introduced to the notions of first-degree equations and of unknowns.

According to Greeno (1997), _thinking strategies_ refer to the processes by which individuals try to find solutions to problems through reflection. These processes involve thoughtful and effective use of cognitive skills and strategies for a particular context or type of thinking task where individuals engage in activating prior schemata and in integrating new subject matters into meaningful knowledge structures. Thus, thinking strategies include the abilities to identify a problem and its associated assumptions, to analyze, understand, and make use of inferences, inductive and deductive logic, as well as to judge the validity and reliability of
assumptions, source of data or information available (Greeno, 1997; Pithers & Soden, 2000; Pugalee, 2001).

An equation is a symbolic representation of a situation involving the equality of two terms. In the Lebanese 7th grade mathematics textbook, a first-degree equation with one unknown is defined by:

"... an equality of the form: \( ax + b = cx + d \); where \( a, b, c, \) and \( d \) are numbers and \( x \) denotes an unknown number" (Naji et al., 1999, p. 134).

**Hypothesis**

In this research study, it was assumed that even if students are not taught the formal approaches of solving algebraic equations, they might bring some aspects of those procedures along from their elementary school experience. Therefore, taking into consideration that students are active constructors and builders of knowledge, it was presumed that when students solve problems such as \( 3 + 4 = \square \) or \( 3 + \square = 4 \) they must have some understanding of the unknown or the missing number, of an equation, thus can manipulate algebraic expressions or even solve problem situations involving first-degree equations by using their own previous knowledge, their own informal techniques and algebraic representations, before being introduced to the concept of equation.
CHAPTER II

Literature Review

Mathematics is considered to be a set of defined concepts, symbols, and systems of representations, proved results, and valid computational procedures (Enright, 1998). It is as much about axioms and theorems as it is about methods, highly structured processes, and less structured strategies (Stylianou, 2002).

Algebra, as defined by Stacey and MacGregor (1999b), is “a language for expressing mathematical information” (p. 10). It is “signified by the use of letters to denote unknown or variable quantities” (Stacey & MacGregor, 1999a, p. 1). Moreover, algebra involves “reasoning about unknown quantities and generalized relations; modeling situations and abstract relations with symbols and solution methods that are imbued with meaning; and working with a variety of representational forms, including equations, tables, diagrams, and verbal relations” (Koedinger & Nathan, 2000, p. 214).

Research showed that students encounter major difficulties in learning algebra, especially when shifting from arithmetic to algebra. Such problems are first witnessed at the 7th grade, when students start learning their first purely algebraic concept of first-degree equations, thus when students start solving problem situations involving first-degree equations.

The importance of algebraic problem solving has been widely documented. However, all studies done on the use of algebra for solving problems have been conducted on students already instructed about the concept of equations. Such research proved that most students resist the use of algebra and apply their own strategies, and very few utilize the algebraic methods taught in class (Stacey & MacGregor, 1999b).
A Critical Shift from Arithmetic to Algebra

Algebra is sometimes defined as a "generalized arithmetic". However, in learning algebra, students need number knowledge that goes far beyond arithmetic calculations and basic skills (Stacey & MacGregor, 1997). According to Stylianou (2002), students need training in order to acquire the knowledge and the concepts presented implicitly in everyday reasoning, and the strategies enabling them to solve problems, which are not explicitly discussed in textbooks.

Research on students' abilities to model and solve problems using algebra focused on difficulties that students encounter while interpreting symbols, formulating and solving equations, word problems, constructing and interpreting graphic representations of functions (Izsák, 2003; Sutherland, 1989). It was found that students struggle with the idea of the unknown and its representation, and encounter problems while solving equations. Most research done on the subject found that students have difficulties in understanding the meaning of the "unknown" (Cai, 1998; Ellis & Labato, 2002; Koedinger & Nathan, 2000; Stacey & MacGregor, 1997, 1999a, 1999b; Sutherland, 1989; Yackel, 1997) and the equality sign (Falkner, Levi & Carpenter, 1999; Herscovics & Kieran, 1980; Lubinski & Otto, 2002; VanDyke & Craine, 1997).

In arithmetic, students deal with known information to get the unknown quantities without the use of any symbols or equations to express the relationships (Cai, 1998; Van Amerom, 2003). A study done by Stacey & MacGregor (1999b) showed that starting to use a symbol to represent the unknown is problematic for students who fail to distinguish the different notions of unknown quantities in arithmetic and in algebra. Consequently, this has led to the fact that students build three different perceptions of an equation: (a) a formula for working out the answer, (b) a narrative describing operations yielding a result, and (c) a description of essential relationships.
Researchers assert that students need to know that letters of the alphabet are used to stand for numbers and that answers can be accepted or rejected (Stacey & MacGregor, 1997; Sutherland, 1989).

In addition, students’ beliefs that any procedure can be translated arithmetically cause a problem. This resistance to the use of algebra generates difficulties in writing formulas from numbers, patterns, and tables.

Conceptual difficulties for learning algebra are more widespread than commonly believed (Herscovics & Kieran, 1980). Some researchers believe that real algebra is when the unknown is on both sides of the equal sign, and this is much more difficult to solve arithmetically without the use of algebra (Stacey & MacGregor, 1999a, 1999b; Van Amerom, 2003). Stacey and MacGregor (1999b) mentioned several reasons that explain the cognitive discontinuities involved in the shift from arithmetic reasoning to algebraic reasoning: (a) the change from calculating with numbers to operating with unknowns, (b) the interpretation of algebraic expressions as being procedural or operational rather than being structural or conceptual, and (c) the obligation to calculate preventing students from attempting an algebraic approach. Thus, students tend to have problems in writing algebraic expressions due to their lack of experience and to their wrong expectations of what they should be doing. Further, Stacey and MacGregor (1999b) as well as Ellis and Labato (2002) showed that some of the students assign different meanings to the unknown such as: (a) “x” refers to different quantities in one equation, (b) “x” refers to different quantities at different stages, (c) “x” is a general label for any unknown quantity (Stacey & MacGregor, 1999b), (d) “x” is a label meaning “goes by” or “every time”, (e) “x” is a part of a memorized equation that students write but whose meaning is not known to them, and (f) “x” represents a set of given values (Ellis & Labato, 2002).
One of the key conceptual changes in the transition from arithmetic to algebraic problem solving methods is the recognition that the unknowns can be used as if they were knowns (Herscovics & Linchevski, 1994, see in Stacey & MacGregor, 1999a; Sutherland, 1989).

In addition to difficulties with the unknown, students seem to have problems related to the equality sign, which can be traced back to kindergarten (Falkner et al., 1999; Lubinski & Otto, 2002). Reading the equal sign as “makes” or “gives” and using it to link parts of a calculation gives students assumptions that they carry into the formal language of algebra. Thus, they have difficulties understanding the presence of the unknown on both sides of the equal sign since they assume that what follows the equal sign should strictly be a specific numerical answer (Falkner et al., 1999; Herscovics & Kieran, 1980; Stacey & MacGregor, 1997; Van Amerom, 2003). VanDyke & Craine (1997) emphasized the fact that students must recognize and understand the equivalence of algebraic expressions that describe the particular expression with which they are working. This understanding can be achieved through the following steps: (a) recognize equivalence, (b) analyze why expressions are equivalent, (c) determine why they would want to use one form or another, and (d) develop skills in going from one form to another.

Thus, the shift from arithmetic to algebra is considered to be a difficult but an essential step for mathematical progress (Stacey & MacGregor, 1999a). According to Warren (2003), this shift involves a move from knowledge required to solve “arithmetic equations” operating on or with numbers to knowledge required to solve “algebraic equations” operating on or with the unknown or variable, and entails a mapping of standard mathematical symbols onto pre-existing mental models of arithmetic.
Equations and Word Problems

Equations are mathematically pure by nature, thus they are devoid of context (Femiano, 2003). In teaching algebra, teachers must keep in mind that students tend to lose sight of equivalence. However, with practice, it becomes possible for students to perform procedures perfectly but without necessarily grasping the underlying meaning (VanDyke & Craine, 1997). It is when equations are transformed into concrete problems that children start understanding the logic behind them (Femiano, 2003).

Equations can then be translated into word problems essential to make relevance for algebra. The analysis of students’ problem solving processes performed by Koedinger & Nathan (2000) showed that verbal problems are easier for students to solve than symbolic problems because verbal problems elicit informal strategies.

Problem Solving

Problem solving helps young children uncover essential mathematical relationships and concepts by building on their own knowledge basis. Since problems can often be solved in more than one way, students can begin with the knowledge they already have then start exploring (Femiano, 2003; Moyer, 2000; Pugalee, 1999).

One of the most famous references in the literature concerning problem solving is George Polya (1945) who compiled a list of heuristic suggestions for successful problem solving based on his own experience with math (see in Stylianou, 2002).

The NCTM Curriculum and Evaluation Standards for School Mathematics (1989) emphasized the importance and use of problem solving. It stated that problem solving should be one of the major concerns of school mathematics and that instructional programs from K-12 should enable all students to: (a) build new mathematical knowledge through problem solving,
(b) solve problems arising in mathematics and in other contexts, (c) apply and adapt a variety of appropriate strategies to solve problems, and (d) monitor/reflect on the process of mathematical problem solving (Bay, 2000).

The vision for problem solving in mathematics, teaching, and learning is still expanding. Schroeder & Lester (1989) (see in Bay, 2000) describe three ways in which problem solving is interpreted in the classroom: (a) teaching “for” problem solving where the mathematical concept is taught to students for future application in solving problems, (b) teaching “about” problem solving consisting of teaching strategies to solve problems, for example, teaching the four-step processes developed by Polya (1971): understand the problem, devise a plan, carry out the plan, and look back, and (c) teaching “via” problem solving which is teaching mathematics content in a concrete problem solving environment and eventually moving to abstraction (Bay, 2000).

Erickson (1993) (see in Bay, 2000) found that teaching “for” problem solving involves much direct instruction, while teaching “via” problem solving facilitates students’ exploration and improves not only their problem solving abilities but also their conceptual understanding and skills. This is based on the rationale that mathematical concepts can be best developed in the context of modeling real world problems (Bay, 2000).

Research proves that modeling tasks provides a rich platform for students’ independent development of powerful mathematical ideas. When given a sequence of model development tasks, students engage themselves in multiple cycles of interpretation and re-interpretation of the context, of the quantities, of the relationships between and among quantities, and of the representations of these relationships in such a way as to provide a mathematical basis for decisions. Such processes shift the focus from guiding students toward particular ways of
thinking about a problem to engaging them in revising, refining, and extending their own ways of thinking about the problem (Izsák, 2003).

*Mathematical Communication*

In his study about visualization and analysis in problem solving, Stylianou (2002) affirmed that teachers are concerned with the ways students apply strategies while solving problems, and are exploring the various aspects of students' problem solving behavior to support their progress in math. In fact, this exploration provides valuable information regarding the students' mathematical thinking and reasoning including the way in which they process a problem, represent and communicate their mathematical ideas and thoughts (Cai & Hwang, 2002; Stylianou, 2002).

Mathematical communication is considered to be the process of exchanging and conveying mathematical information. It requires students to reach agreement about the meaning of words and to recognize the crucial importance of commonly shared definitions, concepts, principles, etc... (NCTM, 1989).

In this context, mathematical communication is best depicted involving the transmission of thoughts mediated by language thus involving the ability to speak, read, and write mathematics and to interpret meanings and ideas (Pugalsky, 2001).

In order to achieve this, students must encode and decode messages, formulate and express information orally, in writing and/or with the help of technological and mathematical tools.

According to the curriculum standards, communication plays an essential role in clarifying, developing and enhancing students' mathematical knowledge, understanding, and reasoning (Pugalsky, 2001). It helps them construct links between their informal, intuitive notions,
the abstract language and symbolism of mathematics. Thus, it plays a major role in helping children make important connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas (Fortescue, 1994; Lewis et al., 1993; NCTM, 1989).

In fact, communication serves as an act of knowledge production and understanding. Thus, emphasizing communication in a math class helps shift the classroom from an environment in which students are totally dependent on the teacher to one in which students assume more responsibility for validating their own thinking (NCTM, 1989).

**Mathematical Representations and Problem Solving Abilities**

An important educational goal is for students to learn how to use multiple forms of representations in communicating mathematically (Fennell & Rowan, 2001; Greenco & Hall, 1997; Lubinski & Otto, 2002; Pugalee, 1999).

Representation is a process, an essential component of both teaching and learning, a way to model mathematics, and a way for students to show their thinking about mathematics by translating a mathematical idea into a form that can be mentally or physically manipulated (Fennell & Rowan, 2001; Pugalee, 1999; Watanabe, 2002). It involves the skills that enable individuals to construct and alternate various mathematical models, such as equations, matrices, and other symbolic or graphical forms (Pugalee, 1999).

According to the NCTM, the term “representation” refers both to process and to product. A process for being an act of capturing a mathematical concept or relationship in some form and a product when the form is an aim in itself. Moreover, the term applies to processes and products that are observable externally as well as those that occur internally or mentally (Alagic, 2003; Fennell & Rowan, 2001; Greenco & Hall, 1997; NCTM, 1989; Stylianou, 2002; Watanabe, 2002).
Representations have proven to be essential elements in building students' understanding of mathematical concepts and relationships, in organizing their thinking, and in communicating their mathematical arguments and understanding to one's self and to others. Thus, representations are essential for developing students' problem solving abilities (Alagie, 2003; Fennell & Rowan, 2001; Greeno & Hall, 1997; Lubinski & Otto, 2002; Stylianou, 2002).

**Critical Thinking and Learning to Think**

Research shows that the more information teachers can obtain about students' knowledge and critical thinking in problem solving, the more opportunities they can offer for students' success (Cai & Hwang, 2002). Critical thinking refers to identifying, clarifying and focusing on a problem; analyzing, understanding, and making use of inferences, inductive and deductive logic (Pithers & Soden, 2000).

One might approach "learning to think" from different perspectives:

- **According to the Associationist/Behaviorist perspective**, learning to think is an acquisition of higher order skills while thinking could be spontaneous or purposive. In spontaneous thinking, the learner's ideas flow at random aiming at no desired objective; while in purposive or controlled thinking, the usefulness of the learner's thoughts is judged in the light of the general system of ideas and purposes of the topic at hand.

- **According to the Domain-Structural/Cognitive perspective**, learning to think is the acquisition of schemata and strategies for understanding, reasoning, and problem solving. Successful thinking depends on whether the individual's cognitive structures are consistent with correct concepts or whether the individual has acquired
appropriate procedures and strategies in solving problems and reasoning (Greeno, 1997).

- According to the *Situative perspective*, learning to think is the effective participation in social practices where individuals develop their identities as learners and knowers (Greeno, 1997); whereas thinking should be aimed to persistent participation in communities (Greeno, 1997; Greeno & Hall, 1997).

*Visualization / Analysis Model in Problem Solving*

Students' thinking about problem solving can be described in terms of explicit steps of visualization and analysis as specified by the *Visualization / Analysis model* (the V/A model), which was developed based on Piaget's work on perception. This model assumes that both visualization and analysis work together in the problem solving process (Greeno & Hall, 1997; Pithers & Soden, 2000; Stylianou, 2002).

Thinking, as it is described by the V/A model, starts with a visualization step which can be the drawing of a picture, diagram, etc. followed by an analysis and an attempt to extract additional information from the representational system that students were not able to deduce from the statement of the problem (Greeno & Hall, 1997; Pithers & Soden, 2000; Stylianou, 2002). The new information is then used during a few moments of mathematical elaboration where students combine all new and old information and manipulate them as needed to set new goals such as how to proceed in solving the problem (Pithers & Soden, 2000; Stylianou, 2002).

Zazkis, Dubinsky & Dautermann (1996) emphasize that the act of visualization is a translation from external to mental or vice versa. On the other hand, an act of analysis is any mental manipulation of objects or processes with or without the aid of symbols (see in Stylianou,
2002). This process goes on until the student reaches the final solution of the problem (Stylianou, 2002).

Metacognition plays an important role in monitoring and reflecting the analysis process. It involves students' awareness and self-regulation of mental processes and a variety of decisions and strategies including behaviors such as predicting, planning, revising, selecting, checking, guessing, and classifying (Pugalee, 2001). Thus, students' problem solving behavior is influenced by metacognitive knowledge, beliefs and skills (Greeno & Hall, 1997; Pithers & Soden, 2000; VanLehn, 1996). Many research studies showed aspects of the metacognitive understanding of the students, which appeared to function as a vital element contributing to successful problem solving by allowing students to identify and work strategically (Ishida, 2002; Pugalee, 2001).

Strategies Used for Solving Problems Involving Equations

In the research done by Stacey and MacGregor (1999b) on strategies used by students in solving problem situations involving equations, interviews and written responses showed that when students choose a good mathematical strategy, they either value its efficiency and apply it to find a solution, or they do not appreciate its value and disregard it. However, students who choose poorer mathematical methods consider them easier to use and to understand. In the same research study, students were found to apply the following different routes while solving algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known numbers at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing the equation, and (e) algebraic route:
solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning (Stacey & MacGregor, 1999b).

**Acquiring and Developing Algebraic Reasoning**

Research used “progressive formalization” to model how students move from informal reasoning (arithmetic) to formal conventional symbol-system (algebra). This is summarized in the work done by Brown & Herbert (1997), Ellis & Labato (2002), and Greeno & Hall (1997) who described the acquisition of algebraic reasoning as follows: (a) pattern seeking: extracting information from a given problem, (b) pattern recognition: mathematical analysis and mathematical representation of information in the form of diagrams, graphs, tables, equations, etc., and (c) generalization: interpreting and applying mathematical findings such as finding unknowns, identifying relationships... As a result, students can understand the power of algebraic thinking (Brown & Herbert, 1997; Ellis & Labato, 2002).

Another way for extending students’ thinking is to ask them to solve similar but more challenging problems to the previously solved ones (Brown & Herbert, 1997).

Teachers play a crucial role in assisting students in developing ways to construct and record their thinking (Yackel, 1997). Teachers must seek a balance between strengthening students’ arithmetic thinking and developing the powerful new patterns of reasoning provided by algebra. As such, they can promote algebraic problem solving by: (a) allowing students more time for thinking, (b) believing that students can solve problems, (c) listening carefully to students’ explanations, and (d) structuring an environment that values the work done by students (Buschman, 2003).

Algebraic problem solving has proven to be an invaluable tool in helping children develop mathematical and logical thinking skills. Not only it strengthens conceptual
understanding, but it also provides many other benefits, from reducing mathematics anxiety to increasing participation levels (Femiano, 2003).

For students, using algebra in solving problems represents an extra difficulty imposed by teachers for no obvious purpose. So to promote mathematical literacy in students, teachers should limit the use of algebra to problems that cannot be solved easily without algebra (Cai, 1998; Stacey & MacGregor, 1999b; Sutherland, 1989). And this measure is considered to be reactive.

The above literature emphasizes the power of algebraic thinking as well as the importance of algebraic problem solving in developing students’ mathematical skills and thinking strategies. Despite this emphasis, research showed that most students resist the use of algebra and apply their own informal strategies rather than use the “difficult” formal algebraic methods taught in class (Stacey & MacGregor, 1999b). As mentioned by Sutherland (1989), students perceive a need to use algebraic thinking only when pre-algebraic thinking is very inefficient or no longer adequate to solve the problem at hand. Furthermore, one can notice that all studies done on the use of algebra for solving problems have been conducted on students who have been already introduced to the concept of equations. Thus, investigating students’ thinking strategies before the introduction of the concept of equations can lead to the development of proactive measures.
CHAPTER III
Methodology

This paper is an action research study. As defined by Burns (2000), an action research deals with an everyday problem situation with a future view of improving the circumstances due to such problem.

The study uses a descriptive analytical methodology. It utilizes "clinical interview" technique to analyze students' preferred problem solving methods. Moreover, it describes and compares the strategies that students employ successfully or unsuccessfully while solving a problem situation involving first-degree equations.

Sample

Twelve 7th grade students were selected to participate in this study: (a) two high achiever girls and two high achiever boys, referred to by: H1, H2, H3, H4, (b) two average girls and two average boys, referred to by: V1, V2, V3, V4, and (c) two low achiever girls and two low achiever boys, referred to by: L1, L2, L3, L4.

Participants are students of International College (IC), in Beirut. IC is a private, reputable, multicultural, liberal arts institution based on the concepts and precepts of American education. It offers three different instruction programs: the Lebanese program, the English College Preparatory Program, and the French program. The Lebanese program is taught in two languages, in different sections: the English and the French sections. The student population comes from middle and high socioeconomic class families, many of whom have lived abroad and/or acquired a second citizenship in addition to Lebanese.

All twelve subjects follow the Lebanese Curriculum in the French language, are distributed over two different sections, and are taught by the same teacher.
were taken into consideration in the statement of the problem, and a new version was added.

Two problems were then administered to students: problem A and problem B.

Each version of the given problem was solved by six students of different achievement levels (a high achiever boy, a high achiever girl, an average boy, an average girl, a low achiever boy, and a low achiever girl).

The purpose of this research, its procedures, and its educational implications were described to the students as well as to their parents.

Subjects were ensured that a high degree of confidentiality will be maintained so that any reader of the research would be unable to infer their identity.

A sheet stating the problem and a pencil were available to each student. Subjects were instructed not to use any other scratch paper and to write all their solving attempts and thinking on the given problem sheet.

Directions emphasized that the aim of the study is not to evaluate or grade students' work but rather to explore their thinking strategies while solving the given problem.

Each session was audio taped and video taped with a camera focusing on the student in order to record his/her facial expressions, his/her body movements, as well as his/her written work.

Students were informed that it is up to them to decide when the given task is accomplished, when their work on the problem is achieved, thus when the solution paper should be submitted.

Students were asked to think out loud while working. They were required to reflect on and develop written records of what strategies they used in order to solve the problem. Furthermore, students were encouraged to describe any difficulty they might have faced.
All along the interviews, the researcher-interviewer tried to create a relaxing and motivating atmosphere. However, care was taken in order not to interfere in the solution process or suggest any solution path. Thus, the researcher-interviewer was careful not to give hints about possible responses and refrained from expressing approval, surprise or shock at any of the respondent’s answers.

The researcher-interviewer was asking questions in a contingent manner and was requesting reflection on the part of the subject. He/she asked students “how” and “why” they approached the problem in a way or another, including questions such as:

- What difficulty (ies) are you facing?
- Why did you choose this particular method to solve the problem?
- Do you think the solution that you found is the right one? How do you know that?
- Do you think that your method is the best way for solving the problem? Why do you think that?

The researcher-interviewer was interested in the knowledge that students used and developed as they were working and answering questions. He/she was also interested in the participants’ strength of belief, whether they believe that their solutions and results are correct or not.

In addition, the researcher-interviewer was reminding each participant to “keep talking” while working on the problem in order to clarify his/her intentions, strategy (ies), and explanations. Thus, he/she was asking the student to verbalize his/her thoughts, to give reasons for his/her actions, and to reflect on what he/she has done.
In fact, the major purpose of those clinical interviews was to give insights into students’
reasoning and to explain much of the thinking underlying the written solutions that were
collected from the sample study.

*Data collection and analysis*

Qualitative data were first collected through the transcriptions of problem solving
sessions and interviews.

Then, students’ problem-solving process actions and approaches from the
video/audiotapes, written solutions, and transcriptions were individually summarized and
gathered in a rubric. This rubric mentions the major categories and sub-categories of the actions
and approaches that emerged from students’ written work and answers to clinical interview
questions (refer to Appendix).

Next, all data collected from the rubric, transcriptions, and students’ written work were
analyzed and summarized under the following titles:

- **Major difficulties faced by students while solving the problem**
- **Different strategies used by students for solving the problem, such as:**
  1. Arithmetic logic reasoning: (a) performing arithmetic calculations (b) using
     backward or inverse operations by calculating from known numbers at every
     stage, and (c) using a diagram, flow chart or any other mathematical
     representation.
  2. Estimation / assumption: estimating the measures of the needed information.
  3. Guess-and-check: trying a single number as a solution and verifying that the
given relationships are satisfied.
4. Trial-and-error: repeating process using forward operations inherent in the problem situation, testing different numbers in the statement of the problem. There are two types of trial-and-error method: (a) random trial-and-error and (b) sequential trial-and-error.

5. Inductive reasoning: trying to generalize a mathematical rule.

6. Writing a formula

7. Writing an algebraic equation: using the equality of two terms involving operating with unknown numbers.

8. Writing an arithmetic equation: writing the equality of two terms and representing the unknowns by their original names in the figure (such as BI or [BI]) or by their role (such as: perimeter).

At this stage, it was taken into consideration whether students have used one solution plan or have tried many in solving the problem, and whether they have adopted certain representational forms related to first-degree equations, such as the use of letters to designate unknowns.

- Different steps used by students in executing and verifying the solution plan
- Students’ motivation and desire to solve the problem
- Students’ strength of belief about the used method and obtained result

Finally, students’ actions and responses across all data were compared and contrasted. Triangulation of data from the observed actions, interviews, and students’ written solution plans contributed to the validation of this qualitative analysis.
Problems

The problems were given in French to students. Additionally, an English version will be provided in this section.

French versions administered to students

Problem A

Dans la figure ci-dessous, trouve la mesure du segment [BI] sachant que le périmètre du triangle ABC est le triple de la mesure du segment [BI].

Problem B

Dans la figure ci-dessous, trouve la mesure du segment [BI] sachant que le périmètre du triangle ABC est le triple de la mesure du segment [BI].
English versions of the given problems

Problem A
In the following figure, find the measure of the segment [BI] knowing that the perimeter of the triangle ABC is triple the measure of the segment [BI].

Problem B
In the following figure, find the measure of the segment [BI] knowing that the perimeter of the triangle ABC is triple the measure of the segment [BI].
Analysis of the problems

The above problems were carefully constructed. They are clear, free of grammatical and contextual features. Special attention was given to the choice of easy /straightforward wording of the problems.

The two problems are similar. They both represent the same given data and can be solved using the same strategy or procedures. However, the given measure of the segment [BC] differs in problem A and problem B.

In problem A, the measure of the segment [BC] was chosen in a way so that the obtained measure of the segment [BI] is a whole number.

In problem B, the measure of the segment [BC] was chosen in a way so that the obtained measure of the segment [BI] is a decimal number. In fact, this version of the problem was constructed in order to reduce the probability of using trial-and-error method since it is usually difficult for students to deal with decimal numbers mentally while testing different numbers in the statement of the problem. This way, it was hoped to direct students toward using other strategies for solving the problem and the probability of writing an equation that represents the problem situation might be increased.

The problems are not routine problems and can be solved using multiple strategies. To test their validity, they were independently reviewed by six experienced mathematics teachers who certified that the problems were mathematically and developmentally appropriate for 7th grade students.

To solve each of these problems, students are required to apply familiar geometric and arithmetic concepts, skills, or relationships that they have already acquired at the elementary level (such as operating with numbers, calculating the perimeter of a triangle, calculating the sum
and difference of segments, applying problem solving skills and strategies). An equation, which represents an algebraic relationship and requires the unknown to occur on both sides of the equal sign, is inherent in the structure of the problems.
CHAPTER IV

Results And Analysis

Difficulties that Students Faced while Solving Each of the Given Problems

Clinical interviews showed that 4 out of the 12 participants claimed not facing any difficulties in solving the given problem. Those students did not consider the given problem as representing a problem situation but rather viewed it as a simple exercise to which it was easy finding a solution. However, 2 out of those 4 students did not solve the problem correctly and two did reach a correct solution.

The study also revealed that other two subjects described the problem as easy, at first sight, because they understood the statement of the problem and were able to identify the given and needed information, but once involved in finding the measure of the segment [BI] or the perimeter of the triangle ABC, they changed their mind as they found themselves incapable of solving the problem.

On the other hand, data showed that the remaining six students faced major difficulties in finding the measure of the segment [BI] as well as the measure of the perimeter of the triangle ABC, in both versions of the problem A and B. They could not find the measure of the segment [BI] because the measure of the perimeter of the triangle ABC is not given in the statement of the problem, or find the measure of the perimeter of the triangle ABC because the measure of the segment [BI] is not provided. Thus, the fact that the two unknown measures ([BI] and perimeter) are related to each other (the perimeter is triple the measure of the segment [BI]) constituted an obstacle that prevented the students from solving the problem.

H1: “If we don’t know the measure of the perimeter, how to find the measure of the segment [BI]?"
V3: “If the measure of the perimeter is the triple of the measure of the segment [BI] and we don't know the measure of [BI], how to find then the measure of the perimeter?”

A student also claimed that the difficulty of the problem remains in finding the measure of the segment [BA] whose measure could help in calculating the measure of the segment [BI] (since BI = BA − AI and [AI] being known to measure 2 cm).

Some students judged the given problem difficult since it was the first time they were faced with such kind of problems especially in the absence of geometrical instruments.

Finally, students who used estimation, trial-and-error, or guess-and-check methods for finding the measure of the segment [BI] and the perimeter of the triangle ABC, faced difficulties in finding a formal mathematical method which justifies their results or their conjectures (V1: “why half the sum of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI]?”). Yet they believed that their strategies are informal, not mathematically acceptable.

**Strategies Used by Students to Solve the Given Problem**

In this study, the twelve 7th grade students who tried to solve the given problem showed remarkable ability to describe qualitatively the relationship between the measure of the segment [BI] and the perimeter of the triangle ABC rather than representing it symbolically. Even though, one student was able to write an arithmetic equation and two students were able to write an algebraic equation, none of them was able to solve it algebraically. In fact, they used the equation as a list of operations to be performed and when they were able to solve the problem, it was through trial-and-error method.

A summary of the different strategies used by students in trying to solve the given problem is represented in Table 1. However it is important to mention that in addition to the
stated strategies all students have used arithmetic logic reasoning strategy by performing arithmetic operations and calculations. Moreover, the table shows whether students have reached a correct or incorrect result, whether they could justify their conjectures or not, and whether they were able to solve the written arithmetic or algebraic equation (if applicable).

Table 1.

<table>
<thead>
<tr>
<th>Student</th>
<th>Strategy (ies) for problem A</th>
<th>Student</th>
<th>Strategy (ies) for Problem B</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>RTE</td>
<td>H3</td>
<td>RTE</td>
</tr>
<tr>
<td></td>
<td>A shift to STE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H2</td>
<td>GC</td>
<td>H4</td>
<td>RTE</td>
</tr>
<tr>
<td>V1</td>
<td>AIE with RTE, A shift to IR</td>
<td>V2</td>
<td>AO</td>
</tr>
<tr>
<td></td>
<td>ES with STE</td>
<td>V4</td>
<td>AO</td>
</tr>
<tr>
<td>L1</td>
<td>ArE, Return to ArE</td>
<td>L3</td>
<td>AO with ES, Return to AO with ES</td>
</tr>
<tr>
<td>L2</td>
<td>AO</td>
<td>L4</td>
<td>RTE, A shift to AIE with BO</td>
</tr>
</tbody>
</table>

Abbreviations: AO: arithmetic logic reasoning performing arithmetic operations; BO: arithmetic logic reasoning performing backward operations; AIE: writing an algebraic equation; ArE: writing an arithmetic equation; ES: estimation; GC: guess-and-check; IR: inductive reasoning; RTE: random trial-and-error; STE: sequential trial-and-error; CR: correct result; IR: incorrect result; JC: justified conjecture; NS: no solution.

Research findings showed the following results:

- Five out of the 12 participating students used random trial-and-error method to solve problem A or problem B. Those students considered different measures, in no particular order, for the segment [BI]. For each considered length, students calculated the perimeter of the triangle ABC. If the obtained measure of the perimeter was not
the triple of the considered measure of the segment [BI], then the number was rejected. This process continued until the given condition (the perimeter of the triangle ABC is the triple of the measure of the segment [BI]) was verified.

- Two students used sequential trial-and-error method for solving problem A. Those students considered the same procedures followed by students who used random trial-and-error method. The only difference is that the various measures considered for the segment [BI] were taken in sequence order.

- Only one student used guess-and-check method for solving problem A. This student considered a single measure of the segment [BI] (BI = 8cm) then calculated the perimeter of the triangle ABC and found that the given relationship (the perimeter of the triangle ABC is the triple of the measure of the segment [BI]) was verified.

- Two students relied on their visual perception and estimated the measure of the segment [BI] according to its length, as shown on the figure.

- One student used inductive reasoning for solving problem A. This student intuitively tried to generalize a mathematical rule justifying the following conjecture he made:

   "In a scalene triangle ABC, if a point I belongs to the side [BA] such that the perimeter of the triangle is the triple the measure of the segment [BI], then half the sum of the measures of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI]".

The student constructed a similar problem to the given one (Figure 1) but considered the following segment measures: 40cm as the measure of the segment [AI], 20cm as the measure of the segment [AC], and 30cm as the measure of the segment [BC]. For [BI] equal half the sum of the measures 40cm, 20cm, and 30cm, the student obtained
a perimeter of 135 cm. Since 135 is the triple of 45 then the student evaluated his 
conjecture as justified thus as true for all such problem cases. However, since this 
student is not used to such proving method, he considered his strategy as informal.

\[ \begin{align*} 
\text{Figure 1. Student's construction of a similar problem to the given one.} 
\end{align*} \]

- Only one student wrote two arithmetic equations for calculating the measure of the 
  segment [BI] (Figure 2(a) and Figure 2(b)). For each written equation, the student 
  wrote an equality of two terms by representing the unknowns by their status and 
  original names in the given figure. However, this student could not solve any of the 
  written equations.

\[ [BI] = \text{perimeter} - (AC + BC + AI) \]

\[ [BI] = \text{perimeter} - (5cm + 3cm + 8cm) \]

\[ [BI] = \text{perimeter} - 16 cm \]

\[ \text{Figure 2 (a). Arithmetic equation for calculating [BI] knowing the} \]
\[ \text{perimeter of the triangle.} \]
Two students wrote algebraic equations. One of the students represented the unknown (measure of the segment [BI]) by “?” (a question mark) and solved it using random trial-and-error method (Figure 3). The other student represented the unknown (measure of the segment [BI]) by “_” (a blank space), thought about using backward operations to solve the written equation but could not reach a solution (Figure 4).

\[ 15 = 2\text{ cm} + 3\text{ cm} + ? \]
\[ 2\text{ cm} + 5\text{ cm} + 5\text{ cm} + ? = 2 \times 3 \]
\[ \underbrace{2\text{ cm} + 5\text{ cm} + 5\text{ cm} + ?}_{12} = 6 \]

*Figure 3. Algebraic equation representing the unknown with a “?” (a question mark).*

\[ BI = (IA + IC + BC) ÷ 3 \]

*Figure 4. Algebraic equation representing the unknown with a “_” (a blank space).*
In addition, the study revealed that half the students did not adopt only one solution method but rather followed multiple strategies for solving the given problem. In fact, 4 out of the 12 subjects used a combination of two different strategies such as:

- **V1** who wrote an algebraic equation then solved it using random trial-and-error method (Figure 3).

- **L4** who wrote an algebraic equation (Figure 4) then tried to solve it mentally using backward operations.

  L4: “Solving this equality is similar to solving a riddle. Usually when we think about solving a riddle we consider inverse operations to the used one. The inverse operation of addition is subtraction and the inverse operation of division is multiplication, but I don’t know how to solve: $B1 = (IA + AC + BC + \_\_\_\_\_) \div 3$.”

- **V3** who estimated the length of the segment $[B1]$ (“I estimated, it seems that the length of the segment $[B1]$ is equal to 8cm”) then justified the estimated measure using sequential trial-and-error method (“If I try $B1 = 7$ cm, the sum of $[B1]$, $[A1]$, $[AC]$, and $[BC]$ will be 23 which is not divisible by 3”).

- **L3** who used arithmetic logic reasoning by performing arithmetic operations to calculate the measure of the segment $[B1]$ (“We find the sum of $[A1]$, $[AC]$, and $[BC]$ then $[B1]$ will be the triple of the obtained sum. $[B1]$ is the triple of 15 thus we have to calculate $15 \times 15 \times 15$... “We calculate $AC + BC = 13$ cm. Now, I calculate $13$ cm - $A1$. The result is $11$ cm, then $B1 = 11 \times 11 \times 11$... “I will try now to calculate $A1 + AC = 2$ cm + $5$ cm = $7$ cm, then $B1 = 7 \times 7 \times 7$”) then rejected each of the obtained results after comparing them to some estimated lengths of the segment $[B1]$ (“The result is illogic. $[B1]$ should be equal to 8cm or 9 cm according to the figure”...
“Maybe [BI] is equal to 10cm”… “It can’t be equal to the obtained result”).

However, three other students used multiple unrelated strategies while solving the given problem. Those students shifted from a certain type of strategy to other different methods. Findings showed that two of the successful students who used random trial-and-error method for solving problem A, tried to shift to another method. In fact, they were searching for a formal mathematical way for solving the given problem since they considered their trial-and-error strategy as being an informal, a non mathematical, and a non acceptable way for solving the problem, for calculating the measure of the segment [BI] and the perimeter of the triangle ABC, thus for proving their obtained results. Nevertheless, a shift to another type of strategy was necessary to a third student who used random trial-and-error method for solving problem B. In fact, the use of such strategy proved to be helpless in finding the decimal length of the segment [BI] in problem B, more so because this student was considering only whole numbers.

On the other hand, the study showed that 3 out of the 12 participants returned to previous applied methods while they were solving the given problem. For example,

- L1 wrote an arithmetic equation in order to calculate the measure of the segment [BI] knowing the measure of the perimeter of the triangle ABC (Figure 2(a)). However, when L1 realized that he cannot solve the written equation because the measure of the perimeter is not given, he returned to the same previous strategy but wrote, this time, another arithmetic equation in order to calculate the measure of the segment [BI] knowing the measure of the segment [BA] (Figure 2(b)).

- V2 started using arithmetic logic reasoning by performing arithmetic operations with angle measures. V2 wanted to prove that the triangle BCI is isosceles having two equal angles in order to deduce that the length of the segments [BI] and [BC] are
equal. When V2 realized that he couldn’t prove that the triangle BCI is isosceles having two equal angles, he thought he should change his method but when he couldn’t find any other, he returned to the same previous type of strategy (arithmetic logic reasoning by performing arithmetic operations). However, this time, V2 performed arithmetic operations using segment measures (“I will multiply each given measure by 3. Thus BC = 24, AC = 15, and AI = 6. To find the measure of [BI] we do 24-15-6”... “If I calculate 8-5-2 then multiply the result by 3, I might obtain the measure of [BI]”).

- L3 returned more than once to the same arithmetic logic reasoning and estimation strategies. L3 started first by performing arithmetic operations for calculating the measure of the segment [BI] (“we find the sum of [AI], [AC], and [BC] then [BI] will be the triple of the obtained sum. [BI] is the triple of 15 thus we have to perform \(15 \times 15 \times 15\)”). When L3 obtained results that are not equal or close to the estimated measures she considered for the segment [BI], she decided to change her strategy (“The result is illogic. [BI] should be equal to 8cm or 9cm according to the figure”... “Maybe [BI] is equal to 10cm”... “It can’t be equal to 3475cm. I will try to find another method to calculate the measure of the segment [BI]”). However, L3 unconsciously returned to the same previous method by performing exactly the same procedures and computations. When L3 realized that she’s always obtaining the same result (\(BI = 3475\) cm), she continued using the same strategies (arithmetic logic reasoning and estimation) but changed her calculation techniques and used different operations (“We calculate AC + BC = 13 cm. Now, I calculate 13 cm - AI. The result is
11 cm, then \( BI = 11 \times 11 \times 11 \)"... "No, the result is illogical"... "I will try now to calculate \( AI + AC = 2 \text{ cm} + 5 \text{ cm} = 7 \text{ cm} \), then \( BI = 7 \times 7 \times 7 \)."

**Steps Used by the Students in Executing and Verifying the Solution Plan**

All students, with no exceptions, were able to identify the given and the needed information of the given problem. Students used to re-read the problem and re-observe the figure several times throughout the orientation, organization, and/or verification steps of their plan. Such re-reading and re-observation occurred more often simultaneously. Students reconsidered their actions by re-reading the problem for finding alternative plans and strategies to reach a correct solution. Thus, if they could not find the solution quickly and if they did not have a plan for the solution, they would re-read the problem statement and try to find a clue to assist them in order to re-orient themselves, to monitor their actions, and to reconsider the processes used in solving the problem.

Few were the students who asked themselves out loudly process oriented questions such as "how" and "why", rather than just stating their thinking strategies, the procedures they followed, the calculations they performed, etc...

While solving the problem, most students relied on the figure’s shape as well as on their visual perceptions in order to come up with conclusions that might help them find the measure of the segment \([BI]\) and the perimeter of the triangle \(ABC\). Nevertheless, those students ended up attributing some non-valid properties to the elements of the given figure. For instance, certain students considered the triangle \(ABC\) as being isosceles since the lengths of the sides \([AB]\) and \([BC]\) seemed equal, others considered the triangle \(ABC\) as being right angled at \(C\) because the angle \(ACB\) seemed to be equal to 90 degrees. Furthermore, some participants considered the segment \([CI]\) as being the height relative to the side \([AB]\) in the triangle \(ABC\) or the base of the
isosceles triangle BIC, or even the bisector of the angle ACB. However, those students were totally aware of the necessity of proof. Therefore, when they realized that none of their assumptions could be proved, they reconsidered the triangle ABC as being scalene.

On the other hand, data showed that certain students thought about some irrelevant concepts or un-existing relationships / properties while they were trying to make certain conjectures in order to find the needed information. For instance, in order to calculate the measure of the segment [BI], some subjects started thinking about angle measures, about the distance between two parallel lines, about congruent triangles as well as other irrelevant mathematical properties such as:

- The height relative to a side in a triangle divides this side into two parts such that the measure of the first part is four times the measure of the second part.
- Since BI = 8 cm and AI = 2 cm then the perimeter of the triangle BIC is four times the perimeter of the triangle AIC.
- If the triangle ABC is right angled at C then the sum of the sides [BC] and [AC] is equal to the length of the side [AB].

Some students such as H1 and V1 noticed that half the sum of the measures of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI] but did not know why and were not able to prove it, hence considered such conclusion as an irrelevant idea that occurred to their mind.

In addition, clinical interviews and written solutions showed that certain students did not have a clear understanding of the meaning of “perimeter”, “length”, and “segment”. Such students used to confuse between these three notions and misused their terminologies. For example, a student wrote “the measure of the segment ABC” to describe “the perimeter of the
triangle ABC” and considered, along with another student, the sum of the measures of the segments \([AI], [AC], \text{ and } [BC]\) as representing the perimeter of the triangle ABC.

On the other hand, the study revealed that the majority of the participants performed mentally most of their calculations. Few were those who showed or wrote all their computations on the solution paper.

However, 4 out of the 12 subjects performed certain irrelevant calculations that are not related to the problem’s situation. Those students were just trying to translate, word by word, the given information of the problem into different simple arithmetic computations. Data indicated that such calculations were due to the following two reasons: (a) a search for some plausible data, where certain students were trying to find reasonable acceptable values that might help them find a reasonable acceptable measure for the segment \([BI]\), or (b) a lack of understanding of the notions of “perimeter” and “triple”, where certain students considered the perimeter of the triangle ABC equal to the sum of the three given segment measures or translated the notion of “triple” into “an exponent of 3”.

In addition, findings showed that almost all students checked their calculations mentally, in writing, or orally in order to avoid calculation mistakes.

When students solved the problem incorrectly but had confidence in their answer, they did not check if their result is correct. Such participants were so sure of their solution plan that they did not verify whether their answer is reasonable, although some of them re-checked their strategic plan. However, when students solved the problem correctly, they often checked their results, the reasonableness of their answer, as well as their solution plan but not necessarily checked whether they have answered the problem’s question or whether they have stated or written the final result on the solution paper.
Motivation and Desire to Solve the Problem

Written solutions and clinical interviews showed that most students, in general, did not give up easily the problem and did not allow their frustration to influence their persistence in seeking a solution. Nevertheless, 5 out of the 12 participating students, who tried unsuccessfully to calculate the measure of the segment [BI] and the perimeter of the triangle ABC using one or multiple solution methods, decided to stop their work as they found themselves incapable of solving the problem or unable of reaching a correct solution. Two of those students used arithmetic logic reasoning with arithmetic operations but could not continue due to the unreasonable results they obtained. Another student who used random trial-and-error method could not reach a right answer because she was considering mainly different whole measures for the segment [BI] in problem B rather than a variation of decimal numbers. As for the last two subjects, one of them found himself incapable of solving his algebraic equation using backward operations, and the other could not solve any of his arithmetic equations since, according to him, the unknowns' measures were not provided.

In addition, data showed that four of the subjects who used random or sequential trial-and-error techniques considered themselves unable of solving the problem at the beginning but when they decided to continue they managed and reached a correct result. However two of those students viewed their trial-and-error method as informal and tried to search for another formal mathematical way for calculating the measure of the segment [BI] and the perimeter of the triangle ABC but failed to achieve their objective.

On the other hand, the three remaining students considered the given problem easy to solve and claimed not facing any difficulty while calculating the measure of the segment [BI] and the perimeter of the triangle ABC. However, one of those students only was capable of
reaching the correct result through guess-and-check method and not arithmetic logic reasoning, which was used by the other subjects who obtained wrong answers.

Strength of Belief about the Obtained Result

Research findings showed that 7 out of the 12 participating students solved the given problem and believed that their final answer is correct. However, two of those students did not, in reality, obtain a correct result even though they used arithmetic logic reasoning and were confident that their solutions were reasonable. Contrary to the others, those two subjects solved the problem without checking the correctness or the reasonableness of their results.

In addition, data showed that 2 out of the 5 students who decided to stop solving the problem after several trials, did not present a final solution because they believed that the results they got were incorrect. In fact, they claimed obtaining unreasonable or wrong measures for the segment [BI].

As for the remaining three students, they could not decide whether their result is correct or not since they did not actually solve the problem or reach any final answer.

Strength of Belief about the Used Method

The study revealed that 9 out the 12 participating students believed that there must be another different method to solve the problem than the one they used. In addition, one of the subjects who wrote an algebraic equation and could not solve it declared that there must be a way for calculating the unknown measure of the segment [BI]. However, all those students claimed that they were unable to find this “other method” because they don’t know it and didn’t learn it. Some of those subjects referred to this “other method” by an easier or shorter or more precise method and others referred to it by a formal mathematical way for proving their results or calculating the length of the segment [BI] and the perimeter of the triangle ABC.
On the other hand, one student only considered her method (estimation with sequential trial-and-error) as being the best way for solving the given problem since it was the only strategy which helped her reach the final correct result.

Concerning the student who wrote two arithmetic equations, he was not sure whether there exist any methods to solve those equations. This student claimed that the problem is unsolvable and believed that the problem’s given information is incomplete since the measure of the segments [BI] and [AB] as well as the measure of the perimeter of the triangle ABC were not provided.

*Between Arithmetic and Algebra*

Written samples and clinical interviews showed that students’ mathematical thinking was diverse. Students brought a variety of ideas and interpretations and generated many different solution plans that varied in sophistication and technique.

Most students in our sample demonstrated capabilities in comprehending and solving the given problem by non-algebraic methods. Analyses of students’ solution strategies have revealed widespread use of arithmetic logic reasoning as well as other informal arithmetic solution methods such as: random or sequential trial-and-error, guess-and-check, and estimation. The most commonly used strategy was trial-and-error. Nevertheless, many students did not identify this method as a problem solving strategy. They rather viewed it more as an approach of guessing or trying to find out the result in their own non-mathematical ways.

Few were the students who formulated an equation to represent the problem situation. Only 3 out of the 12 participating subjects wrote equations. However, although these students began along the algebraic route but when they started solving their equations they switched to an arithmetic method. Whether they succeeded or not to reach a final answer, those students tried to
use a sequence of arithmetic calculations to find or to guess the value of the unknown, the measure of the segment [BI].

One student wrote two arithmetic equations (Figures 2 (a) and 2(b)) in order to calculate the measure of the segment [BI] in two different ways (knowing the measure of the perimeter of the triangle ABC or the measure of the segment [BA]) but could not proceed. This student viewed the problem as unsolvable and considered its given information incomplete since the measures of the perimeter of the triangle ABC or the segment [BA] are missing. Therefore, students are not yet used to deal, at the same time, with two related unknown numbers.

Another participant tried to apply, mentally, the working-backwards approach that students usually use in arithmetic problem solving but did not know how to continue in order to solve his first-degree algebraic equation. In fact, the presence of the unknown on both sides of the equal sign, as shown in figure 4, made it difficult for this student to proceed and find the measure of the segment [BI].

Only one student succeeded in solving his algebraic equation and finding the measure of the segment [BI] (the unknown) but through arithmetic trial-and-error method (Figure 3), which he judged being an informal non-mathematical way for solving a word problem.

In addition, it was found that none of the written equations introduced the unknown by an alphabetical letter. The measure of the segment [BI] was presented by its original name “[BI]” (Figures 2 (a) and 2 (b)), by a “?” (Figure 3) or by a “___” (Figure 4).

The study shows that even if students get to write an arithmetic or algebraic equation representing a problem situation, before being introduced to the technique of solving the equation, they usually proceed arithmetically with the solution plan. They combine their algebraic way of thinking about unknowns with their arithmetic procedural conception of
numbers and operations. In fact, it is yet unfamiliar for those students who represented the problem's mathematical content with a mathematical equation form to continue and proceed algebraically. For such students, translating a word problem into an equation is tantamount to translating it into a strange language with which they don't know yet how to deal. For example, students whose works are shown in figures 2(a), 2(b), 3, and 4, succeeded in writing an equation representing the given problem but did not and could not operate on their equations. The equation helped those students to structure the problem but was not part of the solution process. Even though the unknown number was an integral part of students' reasoning, its original name ([BI]) or the symbol (“?” or “___”) representing it was not used along the students' calculation process.

Therefore, the study shows that, prior to formal instruction in algebra, students do not function in a pure algebraic mode. They are always deflected from the algebraic route by reverting to thinking grounded in arithmetic problem solving methods. Students might transform their prior knowledge in arithmetic into building algebraic equations but they return and proceed arithmetically and informally in order to solve it.

Interestingly, although these methods such as trial-and-error, guess-and-check, or backwards strategies are arithmetic and viewed as being informal, they may be also considered proto-algebraic methods in that they contain some of the essential conceptual underpinnings of the formal symbol-manipulation methods. Such methods of reasoning and symbolizing constitute a way for students to facilitate the transition from arithmetic to an algebraic mode of problem solving. Trial-and-error or guess-and-check method uses an iterative approach by substituting the unknown with a certain numerical value until a solution, which satisfies the quantitative constraints of the problem, is reached. Such strategies instantiate the algebraic concept of
unknowns and show equations as mathematical structures. On the other hand, working backwards strategy embodies the notion of inverting operations, hence stresses the process of “transposing” known by “bringing a number from one side of the equal sign to another and changing the operation sign”.

As mentioned earlier, most students in this study solved the given problem using intuitive, non-algebraic methods. Very few were those who used algebraic symbolism and presented the problem by a first-degree equation. However, in both cases, findings demonstrated that students were unlikely to switch from an arithmetic approach. At this stage a question arises: “Would it be possible for students to construct algebraic ways of solving equations without any prior instruction or intervention from their teacher?”

Nevertheless, introducing the concept of equation through problem solving was essential to this project in that it created relevance and need for algebra. In fact, the given word problem allowed students to grasp the concept of equation intuitively and implicitly in their own ways before being formally introduced to it thus, before it becomes symbolically formalized. Moreover, this research showed that by applying their arithmetic knowledge, students constructed explicitly a certain meaning for algebraic equation. Such construction anchors the concept of equation in arithmetic thereby making the notion meaningful later on, when students learn to operate algebraically with letters and when they are taught the formal transposing strategy for solving first-degree equations.
CHAPTER V

Conclusions and Recommendations

In this study, examining students’ different responses to the same problem gave a robust qualitative description of their solutions or their pre-instructional use of equations in solving such a problem situation.

It was demonstrated that prior to formal instruction in algebra, few students might be able to construct an algebraic equation representing a problem situation but when it comes to solving it they proceed logically and arithmetically by using their own informal mathematical strategies such as trial-and-error.

This project has set out to show how students deal with a new logic of thinking based on their prior arithmetic knowledge for solving problems involving first-degree equations, prior to formal instruction in algebra. For this purpose, some equation examples in which informal notations deviated from conventional algebra syntax (such as the inconsequent use of letters and the pseudo-absence of the unknown in solving equations) were displayed and described. In reality, this fact brings new considerations for future research and raises the following question: “How can teachers bridge the gap between students’ intuitive and meaningful notations and the more formal level of conventional symbolism and use of strategies for solving first-degree equations?"

Although a limitation of the study remains in the inability to generalize the results to a wider population of students of different schools following different programs in various language sections, it is hoped that this project can feed into the development of interventions that improve instruction in the teaching of first-degree equations. One way that this work can contribute is that the presented results and analysis can help understand how students develop
their symbolic reasoning abilities, how they approach a problem involving first-degree equation in their own ways before being introduced to the concept itself. This could potentially help teachers diagnose misconceptions and adjust their existing instruction in order to avoid difficulties that students may encounter later. Thus, teachers will know some of what to expect as the class proceeds: what kind of inventions or alternative reasoning methods students might offer, which inventions offer productive avenues to follow, and how to help guide students' work in these productive directions. Therefore, if teachers are provided with a reasonably good idea of what to expect, their burden will be reduced. They will be able to address students' concerns and make better instructional decisions.

Finally, the presented analysis is expected to support the hypothesis and to help develop proactive measures to prevent or reduce the difficulties encountered by students in writing and solving first-degree equations.
REFERENCES


APPENDIX

Rubric Listing Students’ Problem-Solving Process Actions and Approaches.
### Categories / Sub-Categories

#### I- Understanding the Problem:

<table>
<thead>
<tr>
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<th>H1</th>
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<th>L1</th>
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<td>Reads the problem</td>
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<td>Re-reads the problem when into solution</td>
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<td>Observes the figure</td>
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<td>Identifies needed information</td>
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<td>Asks continuous process oriented questions</td>
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#### II- Solving the Problem:

##### A- Strategies:

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<td>Arithmetic logic reasoning: using backward operations</td>
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<td>Guess-and-check</td>
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<td>Random trial-and-error: guessing answers in no particular order</td>
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<td>Sequential trial-and-error: trying numbers in sequence</td>
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<td>Writing an arithmetic equation</td>
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<td>Writing an equation and representing the unknown by a question mark</td>
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##### B- Carrying out a Plan

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<tr>
<td>Uses and shows computations</td>
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<td>Performs computations mentally</td>
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<td>Uses operations related to the problem situation</td>
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<td>Attributes non-valid properties to elements of the figure</td>
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<td>Selects plausible data</td>
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<td>Uses irrelevant concepts, not related to the problem situation</td>
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<td>Modifies the solution plan</td>
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<td>Returns to previous method</td>
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<td>States the result</td>
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### III- Motivation:

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<td>Expresses inability to solve the problem and desire to stop</td>
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<td>Expresses inability to solve the problem but still continues</td>
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### IV- Strength of Belief:

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<td>Expresses satisfaction, belief that the answer is correct</td>
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<td>Expresses dissatisfaction, belief that the answer is incorrect</td>
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<td>Expresses belief that the used method is the best for solving the problem</td>
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<td>Expresses belief that there is a better method for solving the problem</td>
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<td>Expresses belief that the problem cannot be solved</td>
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### V- Verification

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<td>Checks calculations</td>
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<td>Checks solution plan</td>
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<td>Justifies assumptions or estimations</td>
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<td>Verifies the reasonableness of the result</td>
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<td>Verifies the result</td>
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<td>Verifies that the question was answered</td>
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Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations

A Research Project Presented to the Faculty of the Division of Education and Social Sciences

In Partial Fulfillment of the Requirements for the Degree of Master of Arts in Education Emphasis: Math Education

by

Sirine Abdallah Labban

Under the Direction of Dr. Iman Osta

LEBANESE AMERICAN UNIVERSITY

February, 2005
LEBANESE AMERICAN UNIVERSITY

We hereby approve the project of

Student’s Full Name: Sirine Abdallah Labban

Full Title of the Project:  "Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations"

Date Submitted: February 23, 2005

Division: Education

Dr. Iman Osta
Supervisor

Dr. May Hamdan
Committee Member

A copy of the project report is available at the University Library.

Student Signature

Date 23/02/05
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APPENDIX 53
Title: Exploring the Thinking Strategies of Grade 7 Students in Solving Problem Situations Involving First-Degree Equations

In this study we investigated the thinking strategies of grade 7 students in solving a problem situation involving first-degree equations, prior to formal instruction in algebra. Twelve students participated in individual, talk-aloud problem solving sessions and were interviewed about their attempts to solve the given problem. The sessions were videotaped for further analysis. Students generated different solution plans, using arithmetical rather than algebraic methods. The majority of participants used trial-and-error strategies for solving the given problem. Very few were those who constructed an algebraic equation but then shifted to an arithmetic approach while solving it, representing the unknown by a “?” (a question mark) or “_” (a blank space). The study revealed that it is yet unfamiliar to students who represented the problem’s mathematical content with a mathematical equation form to continue and proceed algebraically. Thus, before being introduced to the concept of equation and unknown, and before being instructed on the algorithms for solving equations, students do not operate algebraically on equations, which are not yet symbolically formalized.
CHAPTER I

Introduction

Nowadays, mathematics curricula, all over the world, call for greater understanding of the fundamentals of algebra and algebraic reasoning by all members of the society.

Algebra is more than memorizing rules for manipulating symbols and solving prescribed types of problems. It is part of the reasoning process, a problem solving strategy, and a key to think and to communicate with mathematics (The National Council of Teachers of Mathematics, 1989).

Accordingly, the National Council of Teachers of Mathematics recommends that algebra be studied by all students of all grade levels, including those who are low achieving and underserved (NCTM, 1989).

When do children start to learn algebra? When do we begin to teach students algebra? Realistically, children learn the basics of algebra at an early stage, at the elementary level. When children first learn addition and subtraction, they begin to learn algebra. They may not recognize it because an "x" or a "y" is not present. When learning their addition and subtraction facts, children work with problems such as \( 3 + 4 = \square \) or \( 3 + \square = 4 \). The box represents the number the students are trying to find. It represents then a mark that covers a hidden number, the unknown. Conceptually, there is no difference between \( 3 + 4 = \square \) and \( 3 + 4 = x \) or \( 3 + \square = 4 \) and \( 3 + x = 4 \). Either way, the student is being asked to find an unknown.

Moreover, sometimes students see blanks such as \( 3 + 4 = \ldots \) or \( 3 + \ldots = 4 \) rather than boxes.
Thus, when students are asked to fill in the blanks with the correct response, they are solving an algebraic equation (Stacey & MacGregor, 1997).

Thus, the concept of equation is informally introduced in the primary grades, when the equation is first defined as an “arithmetic identity with a hidden number”. However, it is only until grade 7 that the concept of equation is formally taught as an independent mathematical algebraic concept. The hiding that was done first by a finger then by a box is finally represented by a letter, which appears most of the time on both sides of the “equal” sign. Nevertheless, this use of letters to represent the unknowns is considered a major and significant shift from arithmetic to algebra.

Rationale

Experiences in the field of teaching show that 7th grade students encounter major problems in shifting from arithmetic to algebra. The most frequent difficulty remains in writing and solving first-degree equations representing real-life problem situations. In fact, in arithmetic, students are used to deal with known information to get the unknown quantities without the use of any symbols or equations to express relationships. However, in algebra, starting to use a symbol to represent the unknown is problematic for pupils (Stacey & MacGregor, 1997).

In order to minimize those problems and to effectively introduce the concept of equations, proactive measures should be considered. Teachers need to investigate which didactical means enable students to make a smooth transition from arithmetic to algebra. A way for approaching this should start from students' informal strategies to build more formal methods out of these, and provide more meanings to the symbols and procedures being used.

Unfortunately, although extensive research on algebra learning has been conducted, educators do not have a complete picture of what students can do in algebra prior to formal
instruction. They do not know exactly which aspects of informal knowledge create useful foundation upon which instruction can be built (Swafford & Langrall, 2000). Moreover, no research has been conducted on students’ reasoning and knowledge construction while solving problem situations involving first-degree equations, before the introduction of the concept of equation and of unknown. In fact, the majority of the research projects focused on the difficulties that students encounter while learning algebra, mainly while solving equations, constructing equations from word problems, as well as interpreting, rewriting, and simplifying algebraic expressions (Herscovics & Kieran, 1980; Ishida, 2002; Stacey & MacGregor, 1999b; Swafford & Langrall, 2000; Van Amerom, 2003; Warren, 2003).

Therefore, conducting a study to discover the difficulties that students face while shifting from arithmetic to algebra is essential in order to avoid the problems that those students might face later in solving first-degree equations.

Purpose, Statement of the Topic, and Operational Definitions

The purpose of this study is to explore the thinking strategies of grade 7 students in solving problem situations involving first-degree equations, prior to formal instruction in algebra, thus before being introduced to the notions of first-degree equations and of unknowns.

According to Greeno (1997), thinking strategies refer to the processes by which individuals try to find solutions to problems through reflection. These processes involve thoughtful and effective use of cognitive skills and strategies for a particular context or type of thinking task where individuals engage in activating prior schemata and in integrating new subject matters into meaningful knowledge structures. Thus, thinking strategies include the abilities to identify a problem and its associated assumptions, to analyze, understand, and make use of inferences, inductive and deductive logic, as well as to judge the validity and reliability of
assumptions, source of data or information available (Greeno, 1997; Pithers & Soden, 2000; Pugalee, 2001).

An equation is a symbolic representation of a situation involving the equality of two terms. In the Lebanese 7th grade mathematics textbook, a first-degree equation with one unknown is defined by:

"... an equality of the form: \( ax+b= cx+d \); where a, b, c, and d are numbers and \( x \) denotes an unknown number" (Naji et al., 1999, p. 134).

**Hypothesis**

In this research study, it was assumed that even if students are not taught the formal approaches of solving algebraic equations, they might bring some aspects of those procedures along from their elementary school experience. Therefore, taking into consideration that students are active constructors and builders of knowledge, it was presumed that when students solve problems such as \( 3 + 4 = \square \) or \( 3 + \square = 4 \) they must have some understanding of the unknown or the missing number, of an equation, thus can manipulate algebraic expressions or even solve problem situations involving first-degree equations by using their own previous knowledge, their own informal techniques and algebraic representations, before being introduced to the concept of equation.
CHAPTER II

Literature Review

Mathematics is considered to be a set of defined concepts, symbols, and systems of representations, proved results, and valid computational procedures (Enright, 1998). It is as much about axioms and theorems as it is about methods, highly structured processes, and less structured strategies (Stylianou, 2002).

Algebra, as defined by Stacey and MacGregor (1999b), is “a language for expressing mathematical information” (p. 10). It is “signified by the use of letters to denote unknown or variable quantities” (Stacey & MacGregor, 1999a, p. 1). Moreover, algebra involves “reasoning about unknown quantities and generalized relations; modeling situations and abstract relations with symbols and solution methods that are imbued with meaning; and working with a variety of representational forms, including equations, tables, diagrams, and verbal relations” (Koedinger & Nathan, 2000, p. 214).

Research showed that students encounter major difficulties in learning algebra, especially when shifting from arithmetic to algebra. Such problems are first witnessed at the 7th grade, when students start learning their first purely algebraic concept of first-degree equations, thus when students start solving problem situations involving first-degree equations.

The importance of algebraic problem solving has been widely documented. However, all studies done on the use of algebra for solving problems have been conducted on students already instructed about the concept of equations. Such research proved that most students resist the use of algebra and apply their own strategies, and very few utilize the algebraic methods taught in class (Stacey & MacGregor, 1999b).
A Critical Shift from Arithmetic to Algebra

Algebra is sometimes defined as a "generalized arithmetic". However, in learning algebra, students need number knowledge that goes far beyond arithmetic calculations and basic skills (Stacey & MacGregor, 1997). According to Stylianou (2002), students need training in order to acquire the knowledge and the concepts presented implicitly in everyday reasoning, and the strategies enabling them to solve problems, which are not explicitly discussed in textbooks.

Research on students' abilities to model and solve problems using algebra focused on difficulties that students encounter while interpreting symbols, formulating and solving equations, word problems, constructing and interpreting graphic representations of functions (Izsák, 2003; Sutherland, 1989). It was found that students struggle with the idea of the unknown and its representation, and encounter problems while solving equations. Most research done on the subject found that students have difficulties in understanding the meaning of the "unknown" (Cai, 1998; Ellis & Labato, 2002; Koedinger & Nathan, 2000; Stacey & MacGregor, 1997, 1999a, 1999b; Sutherland, 1989; Yackel, 1997) and the equality sign (Falkner, Levi & Carpenter, 1999; Herscovics & Kieran, 1980; Lubinski & Otto, 2002; VanDyke & Craine, 1997).

In arithmetic, students deal with known information to get the unknown quantities without the use of any symbols or equations to express the relationships (Cai, 1998; Van Amerom, 2003). A study done by Stacey & MacGregor (1999b) showed that starting to use a symbol to represent the unknown is problematic for students who fail to distinguish the different notions of unknown quantities in arithmetic and in algebra. Consequently, this has led to the fact that students build three different perceptions of an equation: (a) a formula for working out the answer, (b) a narrative describing operations yielding a result, and (c) a description of essential relationships.
Researchers assert that students need to know that letters of the alphabet are used to stand for numbers and that answers can be accepted or rejected (Stacey & MacGregor, 1997; Sutherland, 1989).

In addition, students’ beliefs that any procedure can be translated arithmetically cause a problem. This resistance to the use of algebra generates difficulties in writing formulas from numbers, patterns, and tables.

Conceptual difficulties for learning algebra are more widespread than commonly believed (Herscovics & Kieran, 1980). Some researchers believe that real algebra is when the unknown is on both sides of the equal sign, and this is much more difficult to solve arithmetically without the use of algebra (Stacey & MacGregor, 1999a, 1999b; Van Amerom, 2003). Stacey and MacGregor (1999b) mentioned several reasons that explain the cognitive discontinuities involved in the shift from arithmetic reasoning to algebraic reasoning: (a) the change from calculating with numbers to operating with unknowns, (b) the interpretation of algebraic expressions as being procedural or operational rather than being structural or conceptual, and (c) the obligation to calculate preventing students from attempting an algebraic approach. Thus, students tend to have problems in writing algebraic expressions due to their lack of experience and to their wrong expectations of what they should be doing. Further, Stacey and MacGregor (1999b) as well as Ellis and Labato (2002) showed that some of the students assign different meanings to the unknown such as: (a) “x” refers to different quantities in one equation, (b) “x” refers to different quantities at different stages, (c) “x” is a general label for any unknown quantity (Stacey & MacGregor, 1999b), (d) “x” is a label meaning “goes by” or “every time”, (e) “x” is a part of a memorized equation that students write but whose meaning is not known to them, and (f) “x” represents a set of given values (Ellis & Labato, 2002).
One of the key conceptual changes in the transition from arithmetic to algebraic problem solving methods is the recognition that the unknowns can be used as if they were knowns (Herscovics & Linchevski, 1994, see in Stacey & MacGregor, 1999a; Sutherland, 1989).

In addition to difficulties with the unknown, students seem to have problems related to the equality sign, which can be traced back to kindergarten (Falkner et al., 1999; Lubinski & Otto, 2002). Reading the equal sign as "makes" or "gives" and using it to link parts of a calculation gives students assumptions that they carry into the formal language of algebra. Thus, they have difficulties understanding the presence of the unknown on both sides of the equal sign since they assume that what follows the equal sign should strictly be a specific numerical answer (Falkner et al., 1999; Herscovics & Kieran, 1980; Stacey & MacGregor, 1997; Van Amerom, 2003). VanDyke & Craine (1997) emphasized the fact that students must recognize and understand the equivalence of algebraic expressions that describe the particular expression with which they are working. This understanding can be achieved through the following steps: (a) recognize equivalence, (b) analyze why expressions are equivalent, (c) determine why they would want to use one form or another, and (d) develop skills in going from one form to another.

Thus, the shift from arithmetic to algebra is considered to be a difficult but an essential step for mathematical progress (Stacey & MacGregor, 1999a). According to Warren (2003), this shift involves a move from knowledge required to solve "arithmetic equations" operating on or with numbers to knowledge required to solve "algebraic equations" operating on or with the unknown or variable, and entails a mapping of standard mathematical symbols onto pre-existing mental models of arithmetic.
Equations and Word Problems

Equations are mathematically pure by nature, thus they are devoid of context (Femiano, 2003). In teaching algebra, teachers must keep in mind that students tend to lose sight of equivalence. However, with practice, it becomes possible for students to perform procedures perfectly but without necessarily grasping the underlying meaning (VanDyke & Craine, 1997). It is when equations are transformed into concrete problems that children start understanding the logic behind them (Femiano, 2003).

Equations can then be translated into word problems essential to make relevance for algebra. The analysis of students’ problem solving processes performed by Koedinger & Nathan (2000) showed that verbal problems are easier for students to solve than symbolic problems because verbal problems elicit informal strategies.

Problem Solving

Problem solving helps young children uncover essential mathematical relationships and concepts by building on their own knowledge basis. Since problems can often be solved in more than one way, students can begin with the knowledge they already have then start exploring (Femiano, 2003; Moyer, 2000; Pugalee, 1999).

One of the most famous references in the literature concerning problem solving is George Polya (1945) who compiled a list of heuristic suggestions for successful problem solving based on his own experience with math (see in Stylianou, 2002).

The NCTM Curriculum and Evaluation Standards for School Mathematics (1989) emphasized the importance and use of problem solving. It stated that problem solving should be one of the major concerns of school mathematics and that instructional programs from K-12 should enable all students to: (a) build new mathematical knowledge through problem solving,
(b) solve problems arising in mathematics and in other contexts, (c) apply and adapt a variety of appropriate strategies to solve problems, and (d) monitor/reflect on the process of mathematical problem solving (Bay, 2000).

The vision for problem solving in mathematics, teaching, and learning is still expanding.

Schroeder & Lester (1989) (see in Bay, 2000) describe three ways in which problem solving is interpreted in the classroom: (a) teaching “for” problem solving where the mathematical concept is taught to students for future application in solving problems, (b) teaching “about” problem solving consisting of teaching strategies to solve problems, for example, teaching the four-step processes developed by Polya (1971): understand the problem, devise a plan, carry out the plan, and look back, and (c) teaching “via” problem solving which is teaching mathematics content in a concrete problem solving environment and eventually moving to abstraction (Bay, 2000).

Erickson (1993) (see in Bay, 2000) found that teaching “for” problem solving involves much direct instruction, while teaching “via” problem solving facilitates students’ exploration and improves not only their problem solving abilities but also their conceptual understanding and skills. This is based on the rationale that mathematical concepts can be best developed in the context of modeling real world problems (Bay, 2000).

Research proves that modeling tasks provides a rich platform for students’ independent development of powerful mathematical ideas. When given a sequence of model development tasks, students engage themselves in multiple cycles of interpretation and re-interpretation of the context, of the quantities, of the relationships between and among quantities, and of the representations of these relationships in such a way as to provide a mathematical basis for decisions. Such processes shift the focus from guiding students toward particular ways of
thinking about a problem to engaging them in revising, refining, and extending their own ways of thinking about the problem (Izsák, 2003).

*Mathematical Communication*

In his study about visualization and analysis in problem solving, Stylianou (2002) affirmed that teachers are concerned with the ways students apply strategies while solving problems, and are exploring the various aspects of students' problem solving behavior to support their progress in math. In fact, this exploration provides valuable information regarding the students' mathematical thinking and reasoning including the way in which they process a problem, represent and communicate their mathematical ideas and thoughts (Cai & Hwang, 2002; Stylianou, 2002).

Mathematical communication is considered to be the process of exchanging and conveying mathematical information. It requires students to reach agreement about the meaning of words and to recognize the crucial importance of commonly shared definitions, concepts, principles, etc… (NCTM, 1989).

In this context, mathematical communication is best depicted involving the transmission of thoughts mediated by language thus involving the ability to speak, read, and write mathematics and to interpret meanings and ideas (Pugalee, 2001).

In order to achieve this, students must encode and decode messages, formulate and express information orally, in writing and/or with the help of technological and mathematical tools.

According to the curriculum standards, communication plays an essential role in clarifying, developing and enhancing students' mathematical knowledge, understanding, and reasoning (Pugalee, 2001). It helps them construct links between their informal, intuitive notions,
the abstract language and symbolism of mathematics. Thus, it plays a major role in helping children make important connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas (Fortescue, 1994; Lewis et al., 1993; NCTM, 1989).

In fact, communication serves as an act of knowledge production and understanding. Thus, emphasizing communication in a math class helps shift the classroom from an environment in which students are totally dependent on the teacher to one in which students assume more responsibility for validating their own thinking (NCTM, 1989).

*Mathematical Representations and Problem Solving Abilities*

An important educational goal is for students to learn how to use multiple forms of representations in communicating mathematically (Fennell & Rowan, 2001; Greeno & Hall, 1997; Lubinski & Otto, 2002; Pugalee, 1999).

Representation is a process, an essential component of both teaching and learning, a way to model mathematics, and a way for students to show their thinking about mathematics by translating a mathematical idea into a form that can be mentally or physically manipulated (Fennell & Rowan, 2001; Pugalee, 1999; Watanabe, 2002). It involves the skills that enable individuals to construct and alternate various mathematical models, such as equations, matrices, and other symbolic or graphical forms (Pugalee, 1999).

According to the NCTM, the term “representation” refers both to process and to product. A process for being an act of capturing a mathematical concept or relationship in some form and a product when the form is an aim in itself. Moreover, the term applies to processes and products that are observable externally as well as those that occur internally or mentally (Alagic, 2003; Fennell & Rowan, 2001; Greeno & Hall, 1997; NCTM, 1989; Stylianou, 2002; Watanabe, 2002).
Representations have proven to be essential elements in building students' understanding of mathematical concepts and relationships, in organizing their thinking, and in communicating their mathematical arguments and understanding to one's self and to others. Thus, representations are essential for developing students' problem solving abilities (Alagie, 2003; Fennell & Rowan, 2001; Greeno & Hall, 1997; Lubinski & Otto, 2002; Stylianou, 2002).

**Critical Thinking and Learning to Think**

Research shows that the more information teachers can obtain about students' knowledge and critical thinking in problem solving, the more opportunities they can offer for students' success (Cai & Hwang, 2002). Critical thinking refers to identifying, clarifying and focusing on a problem; analyzing, understanding, and making use of inferences, inductive and deductive logic (Fithers & Soden, 2000).

One might approach “learning to think” from different perspectives:

- According to the *Associationist/Behaviorist perspective*, learning to think is an acquisition of higher order skills while thinking could be spontaneous or purposive. In spontaneous thinking, the learner’s ideas flow at random aiming at no desired objective; while in purposive or controlled thinking, the usefulness of the learner’s thoughts is judged in the light of the general system of ideas and purposes of the topic at hand.

- According to the *Domain-Structural/Cognitive perspective*, learning to think is the acquisition of schemata and strategies for understanding, reasoning, and problem solving. Successful thinking depends on whether the individual’s cognitive structures are consistent with correct concepts or whether the individual has acquired
appropriate procedures and strategies in solving problems and reasoning (Greeno, 1997).

- According to the Situative perspective, learning to think is the effective participation in social practices where individuals develop their identities as learners and knowers (Greeno, 1997); whereas thinking should be aimed to persistent participation in communities (Greeno, 1997; Greeno & Hall, 1997).

**Visualization / Analysis Model in Problem Solving**

Students' thinking about problem solving can be described in terms of explicit steps of visualization and analysis as specified by the Visualization/Analysis model (the V/A model), which was developed based on Piaget's work on perception. This model assumes that both visualization and analysis work together in the problem solving process (Greeno & Hall, 1997; Pithers & Sodden, 2000; Stylianou, 2002).

Thinking, as it is described by the V/A model, starts with a visualization step which can be the drawing of a picture, diagram, etc. followed by an analysis and an attempt to extract additional information from the representational system that students were not able to deduce from the statement of the problem (Greeno & Hall, 1997; Pithers & Sodden, 2000; Stylianou, 2002). The new information is then used during a few moments of mathematical elaboration where students combine all new and old information and manipulate them as needed to set new goals such as how to proceed in solving the problem (Pithers & Sodden, 2000; Stylianou, 2002).

Zazkis, Dubinsky & Dautermann (1996) emphasize that the act of visualization is a translation from external to mental or vice versa. On the other hand, an act of analysis is any mental manipulation of objects or processes with or without the aid of symbols (see in Stylianou,
2002). This process goes on until the student reaches the final solution of the problem (Stylianou, 2002).

Metacognition plays an important role in monitoring and reflecting the analysis process. It involves students’ awareness and self-regulation of mental processes and a variety of decisions and strategies including behaviors such as predicting, planning, revising, selecting, checking, guessing, and classifying (Pugalee, 2001). Thus, students’ problem solving behavior is influenced by metacognitive knowledge, beliefs and skills (Greeno & Hall, 1997; Pithers & Soden, 2000; VanLehn, 1996). Many research studies showed aspects of the metacognitive understanding of the students, which appeared to function as a vital element contributing to successful problem solving by allowing students to identify and work strategically (Ishida, 2002; Pugalee, 2001).

Strategies Used for Solving Problems Involving Equations

In the research done by Stacey and MacGregor (1999b) on strategies used by students in solving problem situations involving equations, interviews and written responses showed that when students choose a good mathematical strategy, they either value its efficiency and apply it to find a solution, or they do not appreciate its value and disregard it. However, students who choose poorer mathematical methods consider them easier to use and to understand. In the same research study, students were found to apply the following different routes while solving algebra problems: (a) non-algebraic route: arithmetic reasoning using backward operations, calculating from known numbers at every stage, (b) non-algebraic route: trial-and-error method using forward operations carried out in three ways: random, sequential, guess-check-improve, (c) superficially algebraic route: writing equations in the form of formulas representing the same reasoning as using arithmetic, (d) algebraic route: writing the equation, and (e) algebraic route:
solving the equation with the option of reverse operations or a flow chart, trial-and-error, and manipulation of symbols in a chain of deductive reasoning (Stacey & MacGregor, 1999b).

*Acquiring and Developing Algebraic Reasoning*

Research used “progressive formalization” to model how students move from informal reasoning (arithmetic) to formal conventional symbol-system (algebra). This is summarized in the work done by Brown & Herbert (1997), Ellis & Labato (2002), and Greeno & Hall (1997) who described the acquisition of algebraic reasoning as follows: (a) pattern seeking: extracting information from a given problem, (b) pattern recognition: mathematical analysis and mathematical representation of information in the form of diagrams, graphs, tables, equations, etc., and (c) generalization: interpreting and applying mathematical findings such as finding unknowns, identifying relationships... As a result, students can understand the power of algebraic thinking (Brown & Herbert, 1997; Ellis & Labato, 2002).

Another way for extending students’ thinking is to ask them to solve similar but more challenging problems to the previously solved ones (Brown & Herbert, 1997).

Teachers play a crucial role in assisting students in developing ways to construct and record their thinking (Yackel, 1997). Teachers must seek a balance between strengthening students’ arithmetic thinking and developing the powerful new patterns of reasoning provided by algebra. As such, they can promote algebraic problem solving by: (a) allowing students more time for thinking, (b) believing that students can solve problems, (c) listening carefully to students’ explanations, and (d) structuring an environment that values the work done by students (Buschman, 2003).

Algebraic problem solving has proven to be an invaluable tool in helping children develop mathematical and logical thinking skills. Not only it strengthens conceptual
understanding, but it also provides many other benefits, from reducing mathematics anxiety to increasing participation levels (Femiano, 2003).

For students, using algebra in solving problems represents an extra difficulty imposed by teachers for no obvious purpose. So to promote mathematical literacy in students, teachers should limit the use of algebra to problems that cannot be solved easily without algebra (Cai, 1998; Stacey & MacGregor, 1999b; Sutherland, 1989). And this measure is considered to be reactive.

The above literature emphasizes the power of algebraic thinking as well as the importance of algebraic problem solving in developing students' mathematical skills and thinking strategies. Despite this emphasis, research showed that most students resist the use of algebra and apply their own informal strategies rather than use the “difficult” formal algebraic methods taught in class (Stacey & MacGregor, 1999b). As mentioned by Sutherland (1989), students perceive a need to use algebraic thinking only when pre-algebraic thinking is very inefficient or no longer adequate to solve the problem at hand. Furthermore, one can notice that all studies done on the use of algebra for solving problems have been conducted on students who have been already introduced to the concept of equations. Thus, investigating students’ thinking strategies before the introduction of the concept of equations can lead to the development of proactive measures.
CHAPTER III
Methodology

This paper is an action research study. As defined by Burns (2000), an action research deals with an everyday problem situation with a future view of improving the circumstances due to such problem.

The study uses a descriptive analytical methodology. It utilizes "clinical interview" technique to analyze students' preferred problem solving methods. Moreover, it describes and compares the strategies that students employ successfully or unsuccessfully while solving a problem situation involving first-degree equations.

Sample

Twelve 7th grade students were selected to participate in this study: (a) two high achiever girls and two high achiever boys, referred to by: H1, H2, H3, H4, (b) two average girls and two average boys, referred to by: V1, V2, V3, V4, and (c) two low achiever girls and two low achiever boys, referred to by: L1, L2, L3, L4.

Participants are students of International College (IC), in Beirut. IC is a private, reputable, multicultural, liberal arts institution based on the concepts and precepts of American education. It offers three different instruction programs: the Lebanese program, the English College Preparatory Program, and the French program. The Lebanese program is taught in two languages, in different sections: the English and the French sections. The student population comes from middle and high socioeconomic class families, many of whom have lived abroad and/or acquired a second citizenship in addition to Lebanese.

All twelve subjects follow the Lebanese Curriculum in the French language, are distributed over two different sections, and are taught by the same teacher.
None of the students has received any formal instruction in algebra; none has been yet introduced to the concepts of algebraic expressions or first-degree equations.

Subjects in this study did not participate in any research study before. They were taught mathematics by the researcher, in the 6th grade level.

The twelve students were randomly selected from the three clusters of “high achievers”, “average students”, and “low achievers”, then interviewed with the permission of their school, teachers, and parents.

Students’ age ranges between 11 and 13 years old. Their 6th grade trimester and yearly mathematics averages were used to place them into high, average, and low achievement levels.

**Procedure**

*Implementation method*

The study was conducted in the first term of the school year when the 7th grade students are not yet introduced to the concept of equation.

In a clinical interview setting, students were faced with a geometry problem-solving situation, which was constructed in a way to satisfy certain specific conditions (refer to Problem Section). These interviews, during which students were observed, aimed at exploring the richness of students’ mathematical thinking by discovering and identifying their cognitive processes and evaluating their competences (Ginsburg, 1981).

After getting parents’ permission, clinical interviews were conducted at the researcher’s office upon appointments. They occurred during weekends and lasted no more than an hour.

In order to increase the validity of the adopted method, a pilot study was first conducted, whereby two average-ability students possessing the same characteristics of the sample students were clinically interviewed while solving the same problem. Accordingly, some modifications...
were taken into consideration in the statement of the problem, and a new version was added.

Two problems were then administered to students: problem A and problem B.

Each version of the given problem was solved by six students of different achievement levels (a high achiever boy, a high achiever girl, an average boy, an average girl, a low achiever boy, and a low achiever girl).

The purpose of this research, its procedures, and its educational implications were described to the students as well as to their parents.

Subjects were ensured that a high degree of confidentiality will be maintained so that any reader of the research would be unable to infer their identity.

A sheet stating the problem and a pencil were available to each student. Subjects were instructed not to use any other scratch paper and to write all their solving attempts and thinking on the given problem sheet.

Directions emphasized that the aim of the study is not to evaluate or grade students' work but rather to explore their thinking strategies while solving the given problem.

Each session was audio taped and video taped with a camera focusing on the student in order to record his/her facial expressions, his/her body movements, as well as his/her written work.

Students were informed that it is up to them to decide when the given task is accomplished, when their work on the problem is achieved, thus when the solution paper should be submitted.

Students were asked to think out loud while working. They were required to reflect on and develop written records of what strategies they used in order to solve the problem. Furthermore, students were encouraged to describe any difficulty they might have faced.
All along the interviews, the researcher-interviewer tried to create a relaxing and motivating atmosphere. However, care was taken in order not to interfere in the solution process or suggest any solution path. Thus, the researcher-interviewer was careful not to give hints about possible responses and refrained from expressing approval, surprise or shock at any of the respondent’s answers.

The researcher-interviewer was asking questions in a contingent manner and was requesting reflection on the part of the subject. He/she asked students “how” and “why” they approached the problem in a way or another, including questions such as:

- What difficulty (ies) are you facing?
- Why did you choose this particular method to solve the problem?
- Do you think the solution that you found is the right one? How do you know that?
- Do you think that your method is the best way for solving the problem? Why do you think that?

The researcher-interviewer was interested in the knowledge that students used and developed as they were working and answering questions. He/she was also interested in the participants’ strength of belief, whether they believe that their solutions and results are correct or not.

In addition, the researcher-interviewer was reminding each participant to “keep talking” while working on the problem in order to clarify his/her intentions, strategy (ies), and explanations. Thus, he/she was asking the student to verbalize his/her thoughts, to give reasons for his/her actions, and to reflect on what he/she has done.
In fact, the major purpose of those clinical interviews was to give insights into students’ reasoning and to explain much of the thinking underlying the written solutions that were collected from the sample study.

Data collection and analysis

Qualitative data were first collected through the transcriptions of problem solving sessions and interviews.

Then, students’ problem-solving process actions and approaches from the video/audiotapes, written solutions, and transcriptions were individually summarized and gathered in a rubric. This rubric mentions the major categories and sub-categories of the actions and approaches that emerged from students’ written work and answers to clinical interview questions (refer to Appendix).

Next, all data collected from the rubric, transcriptions, and students’ written work were analyzed and summarized under the following titles:

- Major difficulties faced by students while solving the problem
- Different strategies used by students for solving the problem, such as:
  1. Arithmetic logic reasoning: (a) performing arithmetic calculations (b) using backward or inverse operations by calculating from known numbers at every stage, and (c) using a diagram, flow chart or any other mathematical representation.
  2. Estimation / assumption: estimating the measures of the needed information.
  3. Guess-and-check: trying a single number as a solution and verifying that the given relationships are satisfied.
4. Trial-and-error: repeating process using forward operations inherent in the problem situation, testing different numbers in the statement of the problem. There are two types of trial-and-error method: (a) random trial-and-error and (b) sequential trial-and-error.

5. Inductive reasoning: trying to generalize a mathematical rule.

6. Writing a formula

7. Writing an algebraic equation: using the equality of two terms involving operating with unknown numbers.

8. Writing an arithmetic equation: writing the equality of two terms and representing the unknowns by their original names in the figure (such as BI or [BI]) or by their role (such as: perimeter).

At this stage, it was taken into consideration whether students have used one solution plan or have tried many in solving the problem, and whether they have adopted certain representational forms related to first-degree equations, such as the use of letters to designate unknowns.

• Different steps used by students in executing and verifying the solution plan

• Students’ motivation and desire to solve the problem

• Students’ strength of belief about the used method and obtained result

Finally, students’ actions and responses across all data were compared and contrasted. Triangulation of data from the observed actions, interviews, and students’ written solution plans contributed to the validation of this qualitative analysis.
Problems

The problems were given in French to students. Additionally, an English version will be provided in this section.

French versions administered to students

Problem A

Dans la figure ci-dessous, trouve la mesure du segment [BI] sachant que le périmètre du triangle ABC est le triple de la mesure du segment [BI].

Problem B

Dans la figure ci-dessous, trouve la mesure du segment [BI] sachant que le périmètre du triangle ABC est le triple de la mesure du segment [BI].
English versions of the given problems

Problem A

In the following figure, find the measure of the segment [BI] knowing that the perimeter of the triangle ABC is triple the measure of the segment [BI].

Problem B

In the following figure, find the measure of the segment [BI] knowing that the perimeter of the triangle ABC is triple the measure of the segment [BI].
Analysis of the problems

The above problems were carefully constructed. They are clear, free of grammatical and contextual features. Special attention was given to the choice of easy/straightforward wording of the problems.

The two problems are similar. They both represent the same given data and can be solved using the same strategy or procedures. However, the given measure of the segment \([BC]\) differs in problem \(A\) and problem \(B\).

In problem \(A\), the measure of the segment \([BC]\) was chosen in a way so that the obtained measure of the segment \([BI]\) is a whole number.

In problem \(B\), the measure of the segment \([BC]\) was chosen in a way so that the obtained measure of the segment \([BI]\) is a decimal number. In fact, this version of the problem was constructed in order to reduce the probability of using trial-and-error method since it is usually difficult for students to deal with decimal numbers mentally while testing different numbers in the statement of the problem. This way, it was hoped to direct students toward using other strategies for solving the problem and the probability of writing an equation that represents the problem situation might be increased.

The problems are not routine problems and can be solved using multiple strategies. To test their validity, they were independently reviewed by six experienced mathematics teachers who certified that the problems were mathematically and developmentally appropriate for 7th grade students.

To solve each of these problems, students are required to apply familiar geometric and arithmetic concepts, skills, or relationships that they have already acquired at the elementary level (such as operating with numbers, calculating the perimeter of a triangle, calculating the sum
and difference of segments, applying problem solving skills and strategies). An equation, which represents an algebraic relationship and requires the unknown to occur on both sides of the equal sign, is inherent in the structure of the problems.
CHAPTER IV

Results And Analysis

Difficulties that Students Faced while Solving Each of the Given Problems

Clinical interviews showed that 4 out of the 12 participants claimed not facing any difficulties in solving the given problem. Those students did not consider the given problem as representing a problem situation but rather viewed it as a simple exercise to which it was easy finding a solution. However, 2 out of those 4 students did not solve the problem correctly and two did reach a correct solution.

The study also revealed that other two subjects described the problem as easy, at first sight, because they understood the statement of the problem and were able to identify the given and needed information, but once involved in finding the measure of the segment [BI] or the perimeter of the triangle ABC, they changed their mind as they found themselves incapable of solving the problem.

On the other hand, data showed that the remaining six students faced major difficulties in finding the measure of the segment [BI] as well as the measure of the perimeter of the triangle ABC, in both versions of the problem A and B. They could not find the measure of the segment [BI] because the measure of the perimeter of the triangle ABC is not given in the statement of the problem, or find the measure of the perimeter of the triangle ABC because the measure of the segment [BI] is not provided. Thus, the fact that the two unknown measures ([BI] and perimeter) are related to each other (the perimeter is triple the measure of the segment [BI]) constituted an obstacle that prevented the students from solving the problem.

H1: “If we don’t know the measure of the perimeter, how to find the measure of the segment [BI]?”
V3: “If the measure of the perimeter is the triple of the measure of the segment [BI] and we don’t know the measure of [BI], how to find then the measure of the perimeter?”

A student also claimed that the difficulty of the problem remains in finding the measure of the segment [BA] whose measure could help in calculating the measure of the segment [BI] (since BI = BA – AI and [AI] being known to measure 2cm).

Some students judged the given problem difficult since it was the first time they were faced with such kind of problems especially in the absence of geometrical instruments.

Finally, students who used estimation, trial-and-error, or guess-and-check methods for finding the measure of the segment [BI] and the perimeter of the triangle ABC, faced difficulties in finding a formal mathematical method which justifies their results or their conjectures (V1: “why half the sum of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI]?”). Yet they believed that their strategies are informal, not mathematically acceptable.

Strategies Used by Students to Solve the Given Problem

In this study, the twelve 7th grade students who tried to solve the given problem showed remarkable ability to describe qualitatively the relationship between the measure of the segment [BI] and the perimeter of the triangle ABC rather than representing it symbolically. Even though, one student was able to write an arithmetic equation and two students were able to write an algebraic equation, none of them was able to solve it algebraically. In fact, they used the equation as a list of operations to be performed and when they were able to solve the problem, it was through trial-and-error method.

A summary of the different strategies used by students in trying to solve the given problem is represented in Table 1. However, it is important to mention that in addition to the
stated strategies all students have used arithmetic logic reasoning strategy by performing arithmetic operations and calculations. Moreover, the table shows whether students have reached a correct or incorrect result, whether they could justify their conjectures or not, and whether they were able to solve the written arithmetic or algebraic equation (if applicable).

Table 1.

<table>
<thead>
<tr>
<th>Student</th>
<th>Strategy (ies) for problem A</th>
<th>Strategy (ies) for Problem B</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1</td>
<td>RTE</td>
<td>RTE</td>
</tr>
<tr>
<td></td>
<td>A shift to STE</td>
<td>CR</td>
</tr>
<tr>
<td>H2</td>
<td>GC</td>
<td>RTE</td>
</tr>
<tr>
<td></td>
<td>CR</td>
<td>CR</td>
</tr>
<tr>
<td>V1</td>
<td>AIE with RTE</td>
<td>AO</td>
</tr>
<tr>
<td></td>
<td>A shift to IR</td>
<td>Return to AO</td>
</tr>
<tr>
<td></td>
<td>CR</td>
<td>CR</td>
</tr>
<tr>
<td>V3</td>
<td>ES with STE</td>
<td>AO</td>
</tr>
<tr>
<td></td>
<td>CR</td>
<td>IR</td>
</tr>
<tr>
<td>L1</td>
<td>ArE</td>
<td>AO with ES</td>
</tr>
<tr>
<td></td>
<td>Return to ArE</td>
<td>Return to AO with ES</td>
</tr>
<tr>
<td></td>
<td>NS</td>
<td>Return to AO with ES</td>
</tr>
<tr>
<td>L2</td>
<td>AO</td>
<td>RTE</td>
</tr>
<tr>
<td></td>
<td>IR</td>
<td>CR</td>
</tr>
</tbody>
</table>

Abbreviations: AO: arithmetic logic reasoning performing arithmetic operations; BO: arithmetic logic reasoning performing backward operations; AIE: writing an algebraic equation; ArE: writing an arithmetic equation; ES: estimation; GC: guess-and-check; IR: inductive reasoning; RTE: random trial-and-error; STE: sequential trial-and-error; CR: correct result; IC: incorrect result; JC: justified conjecture; NS: no solution.

Research findings showed the following results:

- Five out of the 12 participating students used random trial-and-error method to solve problem A or problem B. Those students considered different measures, in no particular order, for the segment \([BI]\). For each considered length, students calculated the perimeter of the triangle ABC. If the obtained measure of the perimeter was not
the triple of the considered measure of the segment [BI], then the number was rejected. This process continued until the given condition (the perimeter of the triangle ABC is the triple of the measure of the segment [BI]) was verified.

- Two students used sequential trial-and-error method for solving problem A. Those students considered the same procedures followed by students who used random trial-and-error method. The only difference is that the various measures considered for the segment [BI] were taken in sequence order.

- Only one student used guess-and-check method for solving problem A. This student considered a single measure of the segment [BI] (BI = 8cm) then calculated the perimeter of the triangle ABC and found that the given relationship (the perimeter of the triangle ABC is the triple of the measure of the segment [BI]) was verified.

- Two students relied on their visual perception and estimated the measure of the segment [BI] according to its length, as shown on the figure.

- One student used inductive reasoning for solving problem A. This student intuitively tried to generalize a mathematical rule justifying the following conjecture he made:

  "In a scalene triangle ABC, if a point I belongs to the side [BA] such that the perimeter of the triangle is the triple the measure of the segment [BI], then half the sum of the measures of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI]."

The student constructed a similar problem to the given one (Figure 1) but considered the following segment measures: 40cm as the measure of the segment [AI], 20cm as the measure of the segment [AC], and 30cm as the measure of the segment [BC]. For [BI] equal half the sum of the measures 40cm, 20cm, and 30cm, the student obtained
a perimeter of 135 cm. Since 135 is the triple of 45 then the student evaluated his
conjecture as justified thus as true for all such problem cases. However, since this
student is not used to such proving method, he considered his strategy as informal.

![Diagram](image)

*Figure 1.* Student's construction of a similar problem to the given one.

- Only one student wrote two arithmetic equations for calculating the measure of the
  segment [BI] (Figure 2(a) and Figure 2(b)). For each written equation, the student
  wrote an equality of two terms by representing the unknowns by their status and
  original names in the given figure. However, this student could not solve any of the
  written equations.

\[
\begin{align*}
[BI] &= \text{perimètre } - (AC + BC + AI) \\
[BI] &= \text{perimètre } - (5\text{ cm} + 8\text{ cm} + 2\text{ cm}) \\
[BI] &= \text{perimètre } - 16 \text{ cm}
\end{align*}
\]

*Figure 2 (a).* Arithmetic equation for calculating [BI] knowing the
perimeter of the triangle.
Equations, Problem Solving Strategies

- \[ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \]
- \[ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} A - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \]
- \[ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} A - 2 \text{ cm} \]

Figure 2 (b). Arithmetic equation for calculating [BI] knowing the measure of [BA].

- Two students wrote algebraic equations. One of the students represented the unknown (measure of the segment [BI]) by "?" (a question mark) and solved it using random trial-and-error method (Figure 3). The other student represented the unknown (measure of the segment [BI]) by "_" (a blank space), thought about using backward operations to solve the written equation but could not reach a solution (Figure 4).

\[ 15 = 3w + \text{sum of 6 cm} \]
\[ 2w + 5w + 5w + 5w = 2 \times 3 \]
\[ 2w + 5w + 9w + \text{6 cm + 8 cm} = 8 \times 3 \]
\[ \underline{2w + 5w + 9w + \text{6 cm} + \text{8 cm}} = \underline{2w} \]

Figure 3. Algebraic equation representing the unknown with a "?" (a question mark).

\[ B_1 = ( IA + AC + BC + \_ \_ \_ ) \div 3 \]

Figure 4. Algebraic equation representing the unknown with a "_" (a blank space).
In addition, the study revealed that half the students did not adopt only one solution method but rather followed multiple strategies for solving the given problem. In fact, 4 out of the 12 subjects used a combination of two different strategies such as:

- **V1** who wrote an algebraic equation then solved it using random trial-and-error method (Figure 3).

- **L4** who wrote an algebraic equation (Figure 4) then tried to solve it mentally using backward operations.

L4: “Solving this equality is similar to solving a riddle. Usually when we think about solving a riddle we consider inverse operations to the used one. The inverse operation of addition is subtraction and the inverse operation of division is multiplication, but I don’t know how to solve: \( BI = (IA + AC + BC + \_\_) \div 3 \).”

- **V3** who estimated the length of the segment \([BI]\) (“I estimated, it seems that the length of the segment \([BI]\) is equal to 8 cm”) then justified the estimated measure using sequential trial-and-error method (“If I try \( BI = 7 \text{ cm} \), the sum of \([BI]\), \([AI]\), \([AC]\), and \([BC]\) will be 23 which is not divisible by 3”).

- **L3** who used arithmetic logic reasoning by performing arithmetic operations to calculate the measure of the segment \([BI]\) (“We find the sum of \([AI]\), \([AC]\), and \([BC]\) then \([BI]\) will be the triple of the obtained sum. \([BI]\) is the triple of 15 thus we have to calculate \( 15 \times 15 \times 15 \) ... “We calculate \( AC + BC = 13 \text{ cm} \). Now, I calculate 13 cm - \( AI \). The result is 11 cm, then \( BI = 11 \times 11 \times 11 \) ... “I will try now to calculate \( AI + AC = 2 \text{ cm} + 5 \text{ cm} = 7 \text{ cm} \), then \( BI = 7 \times 7 \times 7 \)”) then rejected each of the obtained results after comparing them to some estimated lengths of the segment \([BI]\) (“The result is illogic. \([BI]\) should be equal to 8 cm or 9 cm according to the figure”...
"Maybe [BI] is equal to 10cm... "It can't be equal to the obtained result").

However, three other students used multiple unrelated strategies while solving the given problem. Those students shifted from a certain type of strategy to other different methods. Findings showed that two of the successful students who used random trial-and-error method for solving problem A, tried to shift to another method. In fact, they were searching for a formal mathematical way for solving the given problem since they considered their trial-and-error strategy as being an informal, a non mathematical, and a non acceptable way for solving the problem, for calculating the measure of the segment [BI] and the perimeter of the triangle ABC, thus for proving their obtained results. Nevertheless, a shift to another type of strategy was necessary to a third student who used random trial-and-error method for solving problem B. In fact, the use of such strategy proved to be helpless in finding the decimal length of the segment [BI] in problem B, more so because this student was considering only whole numbers.

On the other hand, the study showed that 3 out of the 12 participants returned to previous applied methods while they were solving the given problem. For example,

- L1 wrote an arithmetic equation in order to calculate the measure of the segment [BI] knowing the measure of the perimeter of the triangle ABC (Figure 2 (a)). However, when L1 realized that he cannot solve the written equation because the measure of the perimeter is not given, he returned to the same previous strategy but wrote, this time, another arithmetic equation in order to calculate the measure of the segment [BI] knowing the measure of the segment [BA] (Figure 2(b)).

- V2 started using arithmetic logic reasoning by performing arithmetic operations with angle measures. V2 wanted to prove that the triangle BCI is isosceles having two equal angles in order to deduce that the length of the segments [BI] and [BC] are
equal. When V2 realized that he couldn’t prove that the triangle BCI is isosceles having two equal angles, he thought he should change his method but when he couldn’t find any other, he returned to the same previous type of strategy (arithmetic logic reasoning by performing arithmetic operations). However, this time, V2 performed arithmetic operations using segment measures (“I will multiply each given measure by 3. Thus BC = 24, AC = 15, and AI = 6. To find the measure of [BI] we do 24-15-6”… “If I calculate 8-5-2 then multiply the result by 3, I might obtain the measure of [BI]”).

- L3 returned more than once to the same arithmetic logic reasoning and estimation strategies. L3 started first by performing arithmetic operations for calculating the measure of the segment [BI] (“we find the sum of [AI], [AC], and [BC] then [BI] will be the triple of the obtained sum. [BI] is the triple of 15 thus we have to perform 15×15×15”). When L3 obtained results that are not equal or close to the estimated measures she considered for the segment [BI], she decided to change her strategy (“The result is illogic. [BI] should be equal to 8cm or 9cm according to the figure”… “Maybe [BI] is equal to 10cm”… “It can’t be equal to 3475 cm. I will try to find another method to calculate the measure of the segment [BI]”). However, L3 unconsciously returned to the same previous method by performing exactly the same procedures and computations. When L3 realized that she’s always obtaining the same result (BI = 3475 cm), she continued using the same strategies (arithmetic logic reasoning and estimation) but changed her calculation techniques and used different operations (“We calculate AC+BC=13 cm. Now, I calculate 13 cm - AI. The result is
11 cm, then $BL = 11 \times 11 \times 11$"... "No, the result is illogic"... "I will try now to 
calculate $AI + AC = 2 \text{ cm} + 5 \text{ cm} = 7 \text{ cm}$, then $BI = 7 \times 7 \times 7$.

*Steps Used by the Students in Executing and Verifying the Solution Plan*

All students, with no exceptions, were able to identify the given and the needed 
information of the given problem. Students used to re-read the problem and re-observe the figure 
several times throughout the orientation, organization, and/or verification steps of their plan. 
Such re-reading and re-observation occurred more often simultaneously. Students reconsidered 
their actions by re-reading the problem for finding alternative plans and strategies to reach a 
correct solution. Thus, if they could not find the solution quickly and if they did not have a plan 
for the solution, they would re-read the problem statement and try to find a clue to assist them in 
order to re-orient themselves, to monitor their actions, and to reconsider the processes used in 
solving the problem.

Few were the students who asked themselves out loudly process oriented questions such 
as "how" and "why", rather than just stating their thinking strategies, the procedures they 
followed, the calculations they performed, etc...

While solving the problem, most students relied on the figure’s shape as well as on their 
visual perceptions in order to come up with conclusions that might help them find the measure of 
the segment $[BI]$ and the perimeter of the triangle $ABC$. Nevertheless, those students ended up 
attributing some non-valid properties to the elements of the given figure. For instance, certain 
students considered the triangle $ABC$ as being isosceles since the lengths of the sides $[AB]$ and 
$[BC]$ seemed equal, others considered the triangle $ABC$ as being right angled at $C$ because the 
angle $ACB$ seemed to be equal to 90 degrees. Furthermore, some participants considered the 
segment $[CI]$ as being the height relative to the side $[AB]$ in the triangle $ABC$ or the base of the
isosceles triangle BIC, or even the bisector of the angle ACB. However, those students were totally aware of the necessity of proof. Therefore, when they realized that none of their assumptions could be proved, they reconsidered the triangle ABC as being scalene.

On the other hand, data showed that certain students thought about some irrelevant concepts or existing relationships / properties while they were trying to make certain conjectures in order to find the needed information. For instance, in order to calculate the measure of the segment [BI], some subjects started thinking about angle measures, about the distance between two parallel lines, about congruent triangles as well as other irrelevant mathematical properties such as:

- The height relative to a side in a triangle divides this side into two parts such that the measure of the first part is four times the measure of the second part.
- Since BI = 8 cm and AI = 2 cm then the perimeter of the triangle BIC is four times the perimeter of the triangle AIC.
- If the triangle ABC is right angled at C then the sum of the sides [BC] and [AC] is equal to the length of the side [AB].

Some students such as H1 and V1 noticed that half the sum of the measures of the segments [AI], [AC], and [BC] is equal to the measure of the segment [BI] but did not know why and were not able to prove it, hence considered such conclusion as an irrelevant idea that occurred to their mind.

In addition, clinical interviews and written solutions showed that certain students did not have a clear understanding of the meaning of "perimeter", "length", and "segment". Such students used to confuse between these three notions and misused their terminologies. For example, a student wrote "the measure of the segment ABC" to describe "the perimeter of the
triangle ABC" and considered, along with another student, the sum of the measures of the segments [AI], [AC], and [BC] as representing the perimeter of the triangle ABC.

On the other hand, the study revealed that the majority of the participants performed mentally most of their calculations. Few were those who showed or wrote all their computations on the solution paper.

However, 4 out of the 12 subjects performed certain irrelevant calculations that are not related to the problem’s situation. Those students were just trying to translate, word by word, the given information of the problem into different simple arithmetic computations. Data indicated that such calculations were due to the following two reasons: (a) a search for some plausible data, where certain students were trying to find reasonable acceptable values that might help them find a reasonable acceptable measure for the segment [BI], or (b) a lack of understanding of the notions of “perimeter” and “triple”, where certain students considered the perimeter of the triangle ABC equal to the sum of the three given segment measures or translated the notion of “triple” into “an exponent of 3”.

In addition, findings showed that almost all students checked their calculations mentally, in writing, or orally in order to avoid calculation mistakes.

When students solved the problem incorrectly but had confidence in their answer, they did not check if their result is correct. Such participants were so sure of their solution plan that they did not verify whether their answer is reasonable, although some of them re-checked their strategic plan. However, when students solved the problem correctly, they often checked their results, the reasonableness of their answer, as well as their solution plan but not necessarily checked whether they have answered the problem’s question or whether they have stated or written the final result on the solution paper.
Motivation and Desire to Solve the Problem

Written solutions and clinical interviews showed that most students, in general, did not give up easily the problem and did not allow their frustration to influence their persistence in seeking a solution. Nevertheless, 5 out of the 12 participating students, who tried unsuccessfully to calculate the measure of the segment [BI] and the perimeter of the triangle ABC using one or multiple solution methods, decided to stop their work as they found themselves incapable of solving the problem or unable of reaching a correct solution. Two of those students used arithmetic logic reasoning with arithmetic operations but could not continue due to the unreasonable results they obtained. Another student who used random trial-and-error method could not reach a right answer because she was considering mainly different whole measures for the segment [BI] in problem B rather than a variation of decimal numbers. As for the last two subjects, one of them found himself incapable of solving his algebraic equation using backward operations, and the other could not solve any of his arithmetic equations since, according to him, the unknowns’ measures were not provided.

In addition, data showed that four of the subjects who used random or sequential trial-and-error techniques considered themselves unable of solving the problem at the beginning but when they decided to continue they managed and reached a correct result. However two of those students viewed their trial-and-error method as informal and tried to search for another formal mathematical way for calculating the measure of the segment [BI] and the perimeter of the triangle ABC but failed to achieve their objective.

On the other hand, the three remaining students considered the given problem easy to solve and claimed not facing any difficulty while calculating the measure of the segment [BI] and the perimeter of the triangle ABC. However, one of those students only was capable of
reaching the correct result through guess-and-check method and not arithmetic logic reasoning, which was used by the other subjects who obtained wrong answers.

*Strength of Belief about the Obtained Result*

Research findings showed that 7 out of the 12 participating students solved the given problem and believed that their final answer is correct. However, two of those students did not, in reality, obtain a correct result even though they used arithmetic logic reasoning and were confident that their solutions were reasonable. Contrary to the others, those two subjects solved the problem without checking the correctness or the reasonableness of their results.

In addition, data showed that 2 out of the 5 students who decided to stop solving the problem after several trials, did not present a final solution because they believed that the results they got were incorrect. In fact, they claimed obtaining unreasonable or wrong measures for the segment [BI].

As for the remaining three students, they could not decide whether their result is correct or not since they did not actually solve the problem or reach any final answer.

*Strength of Belief about the Used Method*

The study revealed that 9 out the 12 participating students believed that there must be another different method to solve the problem than the one they used. In addition, one of the subjects who wrote an algebraic equation and could not solve it declared that there must be a way for calculating the unknown measure of the segment [BI]. However, all those students claimed that they were unable to find this “other method” because they don’t know it and didn’t learn it. Some of those subjects referred to this “other method” by an easier or shorter or more precise method and others referred to it by a formal mathematical way for proving their results or calculating the length of the segment [BI] and the perimeter of the triangle ABC.
On the other hand, one student only considered her method (estimation with sequential trial-and-error) as being the best way for solving the given problem since it was the only strategy which helped her reach the final correct result.

Concerning the student who wrote two arithmetic equations, he was not sure whether there exist any methods to solve those equations. This student claimed that the problem is unsolvable and believed that the problem’s given information is incomplete since the measure of the segments \([BI]\) and \([AB]\) as well as the measure of the perimeter of the triangle \(ABC\) were not provided.

**Between Arithmetic and Algebra**

Written samples and clinical interviews showed that students’ mathematical thinking was diverse. Students brought a variety of ideas and interpretations and generated many different solution plans that varied in sophistication and technique.

Most students in our sample demonstrated capabilities in comprehending and solving the given problem by non-algebraic methods. Analyses of students’ solution strategies have revealed widespread use of arithmetic logic reasoning as well as other informal arithmetic solution methods such as: random or sequential trial-and-error, guess-and-check, and estimation. The most commonly used strategy was trial-and-error. Nevertheless, many students did not identify this method as a problem solving strategy. They rather viewed it more as an approach of guessing or trying to find out the result in their own non-mathematical ways.

Few were the students who formulated an equation to represent the problem situation. Only 3 out of the 12 participating subjects wrote equations. However, although these students began along the algebraic route but when they started solving their equations they switched to an arithmetic method. Whether they succeeded or not to reach a final answer, those students tried to
use a sequence of arithmetic calculations to find or to guess the value of the unknown, the measure of the segment [BI].

One student wrote two arithmetic equations (Figures 2 (a) and 2(b)) in order to calculate the measure of the segment [BI] in two different ways (knowing the measure of the perimeter of the triangle ABC or the measure of the segment [BA]) but could not proceed. This student viewed the problem as unsolvable and considered its given information incomplete since the measures of the perimeter of the triangle ABC or the segment [BA] are missing. Therefore, students are not yet used to deal, at the same time, with two related unknown numbers.

Another participant tried to apply, mentally, the working-backwards approach that students usually use in arithmetic problem solving but did not know how to continue in order to solve his first-degree algebraic equation. In fact, the presence of the unknown on both sides of the equal sign, as shown in figure 4, made it difficult for this student to proceed and find the measure of the segment [BI].

Only one student succeeded in solving his algebraic equation and finding the measure of the segment [BI] (the unknown) but through arithmetic trial-and-error method (Figure 3), which he judged being an informal non-mathematical way for solving a word problem.

In addition, it was found that none of the written equations introduced the unknown by an alphabetical letter. The measure of the segment [BI] was presented by its original name "[BI]" (Figures 2 (a) and 2 (b)), by a "?" (Figure 3) or by a "_" (Figure 4).

The study shows that even if students get to write an arithmetic or algebraic equation representing a problem situation, before being introduced to the technique of solving the equation, they usually proceed arithmetically with the solution plan. They combine their algebraic way of thinking about unknowns with their arithmetic procedural conception of
numbers and operations. In fact, it is yet unfamiliar for those students who represented the problem's mathematical content with a mathematical equation form to continue and proceed algebraically. For such students, translating a word problem into an equation is tantamount to translating it into a strange language with which they don't know yet how to deal. For example, students whose works are shown in figures 2(a), 2(b), 3, and 4, succeeded in writing an equation representing the given problem but did not and could not operate on their equations. The equation helped those students to structure the problem but was not part of the solution process. Even though the unknown number was an integral part of students' reasoning, its original name ([BI]) or the symbol ("?" or "_") representing it was not used along the students' calculation process.

Therefore, the study shows that, prior to formal instruction in algebra, students do not function in a pure algebraic mode. They are always deflected from the algebraic route by reverting to thinking grounded in arithmetic problem solving methods. Students might transform their prior knowledge in arithmetic into building algebraic equations but they return and proceed arithmetically and informally in order to solve it.

Interestingly, although these methods such as trial-and-error, guess-and-check, or backwards strategies are arithmetic and viewed as being informal, they may be also considered proto-algebraic methods in that they contain some of the essential conceptual underpinnings of the formal symbol-manipulation methods. Such methods of reasoning and symbolizing constitute a way for students to facilitate the transition from arithmetic to an algebraic mode of problem solving. Trial-and-error or guess-and-check method uses an iterative approach by substituting the unknown with a certain numerical value until a solution, which satisfies the quantitative constraints of the problem, is reached. Such strategies instantiate the algebraic concept of
unknowns and show equations as mathematical structures. On the other hand, working backwards strategy embodies the notion of inverting operations, hence stresses the process of “transposing” known by “bringing a number from one side of the equal sign to another and changing the operation sign”.

As mentioned earlier, most students in this study solved the given problem using intuitive, non-algebraic methods. Very few were those who used algebraic symbolism and presented the problem by a first-degree equation. However, in both cases, findings demonstrated that students were unlikely to switch from an arithmetic approach. At this stage a question arises: “Would it be possible for students to construct algebraic ways of solving equations without any prior instruction or intervention from their teacher?”

Nevertheless, introducing the concept of equation through problem solving was essential to this project in that it created relevance and need for algebra. In fact, the given word problem allowed students to grasp the concept of equation intuitively and implicitly in their own ways before being formally introduced to it thus, before it becomes symbolically formalized. Moreover, this research showed that by applying their arithmetic knowledge, students constructed explicitly a certain meaning for algebraic equation. Such construction anchors the concept of equation in arithmetic thereby making the notion meaningful later on, when students learn to operate algebraically with letters and when they are taught the formal transposing strategy for solving first-degree equations.
CHAPTER V

Conclusions and Recommendations

In this study, examining students’ different responses to the same problem gave a robust qualitative description of their solutions or their pre-instructional use of equations in solving such a problem situation.

It was demonstrated that prior to formal instruction in algebra, few students might be able to construct an algebraic equation representing a problem situation but when it comes to solving it they proceed logically and arithmetically by using their own informal mathematical strategies such as trial-and-error.

This project has set out to show how students deal with a new logic of thinking based on their prior arithmetic knowledge for solving problems involving first-degree equations, prior to formal instruction in algebra. For this purpose, some equation examples in which informal notations deviated from conventional algebra syntax (such as the inconsequent use of letters and the pseudo-absence of the unknown in solving equations) were displayed and described. In reality, this fact brings new considerations for future research and raises the following question: “How can teachers bridge the gap between students’ intuitive and meaningful notations and the more formal level of conventional symbolism and use of strategies for solving first-degree equations?”

Although a limitation of the study remains in the inability to generalize the results to a wider population of students of different schools following different programs in various language sections, it is hoped that this project can feed into the development of interventions that improve instruction in the teaching of first-degree equations. One way that this work can contribute is that the presented results and analysis can help understand how students develop
their symbolic reasoning abilities, how they approach a problem involving first-degree equation in their own ways before being introduced to the concept itself. This could potentially help teachers diagnose misconceptions and adjust their existing instruction in order to avoid difficulties that students may encounter later. Thus, teachers will know some of what to expect as the class proceeds: what kind of inventions or alternative reasoning methods students might offer, which inventions offer productive avenues to follow, and how to help guide students' work in these productive directions. Therefore, if teachers are provided with a reasonably good idea of what to expect, their burden will be reduced. They will be able to address students' concerns and make better instructional decisions.

Finally, the presented analysis is expected to support the hypothesis and to help develop proactive measures to prevent or reduce the difficulties encountered by students in writing and solving first-degree equations.
REFERENCES


APPENDIX

Rubric Listing Students’ Problem-Solving Process Actions and Approaches.
### CATEGORIES / SUB-CATEGORIES

#### I- Understanding the Problem:

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<thead>
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<th>L1</th>
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<td>Re-reads the problem when into solution</td>
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<td>Observes the figure</td>
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<td>Asks continuous process oriented questions</td>
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#### II- Solving the Problem:

##### A- Strategies:

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<td>Arithmetic logic reasoning: using backward operations</td>
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<td>Guess-and-check</td>
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<td>Random trial-and-error: guessing answers in no particular order</td>
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<td>Sequential trial-and-error: trying numbers in sequence</td>
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<td>Writing an arithmetic equation</td>
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<td>Writing an equation and representing the unknown by a question mark</td>
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<td>Inductive reasoning: generalizing a mathematical rule</td>
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##### B- Carrying out a Plan

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<td>Performs computations mentally</td>
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<td>Uses operations related to the problem situation</td>
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<td>Uses irrelevant concepts, not related to the problem situation</td>
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<td>Modifies the solution plan</td>
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### III- Motivation:

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<td>Expresses inability to solve the problem and desire to stop</td>
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<td>Expresses inability to solve the problem but still continues</td>
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### IV- Strength of Belief:

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<td>Expresses satisfaction, belief that the answer is correct</td>
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<td>Expresses dissatisfaction, belief that the answer is incorrect</td>
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<td>Expresses belief that the used method is the best for solving the problem</td>
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<td>Expresses belief that there is a better method for solving the problem</td>
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<td>Expresses belief that the problem cannot be solved</td>
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### V- Verification

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<td>Checks solution plan</td>
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<td>Justifies assumptions or estimations</td>
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<td>Verifies the reasonableness of the result</td>
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<td>Verifies the result</td>
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<td>Verifies that the question was answered</td>
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