ANZIAM J. 45(2003), 295–302

DECOMPOSITION METHOD FOR SOLVING A NONLINEAR BUSINESS CYCLE MODEL

ELIAS DEEBA[1], GHASSAN DIBEH[2], SUHEIL KHURI[3] and SHISHEN XIE[1]

(Received 15 November, 2001; revised 13 February, 2002)

Abstract

In this paper we present a Kaleckian-type model of a business cycle based on a nonlinear delay differential equation. A numerical algorithm based on a decomposition scheme is implemented for the approximate solution of the model. The numerical results of the underlying equation show that the business cycle is stable.

1. Introduction

This paper deals with a class of nonlinear differential equations that arises in the study of economic dynamics and business cycles. Models of business cycles that include time lags represent a growing field of study in economic dynamics (see [3] and [10]). Economic environments are modelled either as discrete dynamical systems [12] or as continuous dynamical systems [19]. Discrete dynamical systems give rise to difference equations and have been used in most applied and theoretical formulations of economic models (see [17] and [18]). Continuous dynamical systems give rise to differential equation models and have been used in formulations of the business cycle as in the Keynesian tradition [11] and in structural macro economics [20].

In this paper we shall consider the Kaleckian model (see [4]) given by

$$\dot{K}(t) = \left(\frac{a}{\tau} - n\right) K(t) - \left(\frac{a}{\tau} + m\right) K(t-\tau) - \epsilon K^3(t), \tag{1.1}$$

where $K(t)$ represents capital stock, τ represents the gestation period in production and ϵ is small. The triplet (a, n, m) represents economic parameters: a represents

[1]Department of Computer and Mathematical Sciences, University of Houston-Downtown, Houston, Texas 77002, USA; e-mail: deebae@dt.uh.edu and xies@dt.uh.edu.
[2]Department of Economics, Lebanese American University, Byblos, Lebanon; e-mail: gdibeh@lau.edu.lb.
[3]Department of Computer Science, Mathematics and Statistics, AUS, UAE.

the proportion of profits that is reinvested by firms in the economy and usually ranges from 0.8 to 0.95, n represents the degree of responsiveness of investment to the level of capital stock $K(t)$ in the economy (the higher the capital stock, the less profitable the investment) and m is the same as n but for the nonlinear capital stock.

There are four main assumptions that form the fundamental core of Kaleckian theory. First, investment is the main driver of business cycle activity and hence any model of the business cycle should be a model of investment dynamics. Second, Kalecki assumed that in a capitalist economy the main motive for the decision to invest is the profit motive. Third, the capital stock forms a depressing factor on investment. Fourth, in modern economics where investment involves the installation of machinery, there exists a time lag between the decision to invest and the actual realisation of investment. This is known in the modern literature as the time-to-build ([3] and [16]).

Equation (1.1) is a consequence of the assumed relation between investment (I), profits (π) and capital stocks (K) as given in [13]:

$$I = f(\pi, K), \tag{1.2}$$

where $\partial I/\partial \pi > 0$ and $\partial I/\partial K < 0$. Since there is a time lag, τ, between the decision to invest and the actual delivery and installation of investment equipment, (1.2) becomes $I = f(\pi(t-\tau), K(t-\tau))$. A possible choice of a nonlinear dependence given in [4] is $G = G(K(t), K(t-\tau))$ with τ being the time lag. The function G represents the effect on capital accumulation from $-\tau$ up to the present. In [4] G was given by

$$G(t) = nK(t) + mK(t-\tau) + \epsilon K^3(t). \tag{1.3}$$

The nonlinearity ϵK^3 captures the increasing negative effect of capital stock on profitability. As the capital stock becomes larger, its effect on profit increases disproportionally. Substituting (1.3) in (1.2) and using the identity (see [14])

$$\pi(t) = (K(t+\tau) - K(t))/\tau$$

we get (1.1) and when $\epsilon = 0$ we get the linear version of (1.1), (see [4])

$$\dot{K}(t) = \left(\frac{a}{\tau} - n\right) K(t) - \left(\frac{a}{\tau} + m\right) K(t-\tau). \tag{1.4}$$

It has been argued in [4] that the nonlinear term $\epsilon K^3(t)$ captures the increasing negative effect of capital stock on profitability and leads to a stable cycle. In this paper we will present a numerical scheme based on a decomposition method to solve (1.1) and show that the nonlinearity leads to a stable cycle.

The balance of the paper is as follows. In Section 2, we will give a brief description of the method, while in Section 3 we will solve (1.1) using the decomposition method and show that the numerical results lead to a stable cycle for a large range of the parameters a, n and m.

2. The method

In this section we first briefly describe the Adomian decomposition method as it is applied to general nonlinear ordinary differential equations. For a detailed description of the decomposition method, we refer the reader to [1, 2, 5, 6, 8, 9].

Let L represent the highest-ordered differential operator and R the remainder of the ordinary differential operator, including the nonlinear term. For example, L is d/dt and R acting on K yields the expression on the right-hand side of (1.1). Using this notation, a general nonlinear ordinary differential equation can be rewritten as

$$LK = R(K) + g, \tag{2.1}$$

where g represents a known function. In (1.1), $LK = \dot{K}(t)$, $R(K) = \left(\frac{a}{\tau} - n\right) K(t) - \left(\frac{a}{\tau} + m\right) K(t-\tau) - \epsilon K^3(t)$ and $g = 0$.

For problems like (1.1), we define the inverse operator L^{-1} of $L = d/dt$ as the integration operator. For example, in (1.1), $L = d/dt$ implies that $L^{-1} = \int_0^t$ and hence $L^{-1}LK = K(t) - C$, with C being the integration constant.

Applying L^{-1} to (2.1), we obtain

$$K(t) = C + L^{-1}R(K). \tag{2.2}$$

The decomposition method assumes a series solution for K given by

$$K = \sum_{n=0}^{\infty} K_n = K_0 + K_1 + K_2 + \cdots, \tag{2.3}$$

with K_0 identified as C in (2.2). Assuming $R(K)$ is analytic, we can write

$$R(K) = \sum_{n=0}^{\infty} R_n(K_0, K_1, \ldots, K_n), \tag{2.4}$$

where R_n are the specially generated Adomian polynomials that depend only on the K_0 to K_n components. To be more specific, we define the order of the component K_j^m to be jm, and the order of $K_i^m K_j^n$ to be $im + jn$. Then the Adomian polynomial R_0 depends upon K_0 with order 0, R_1 depends upon K_0 and K_1 with order 1, *etc.* For example, if $R(K) = K^3$, then the expansion of $R(K)$ is

$$\begin{aligned} R(K) = {} & K_0^3 + K_1^3 + K_2^3 + K_3^3 + \cdots + 3K_0^2K_1 + 3K_0^2K_2 + 3K_0^2K_3 + \cdots \\ & + 3K_1^2K_0 + 3K_1^2K_2 + 3K_1^2K_3 + \cdots + 3K_2^2K_0 + 3K_2^2K_1 + 3K_2^2K_3 \\ & + \cdots + 6K_0K_1K_2 + 6K_0K_1K_3 + 6K_0K_1K_4 + \cdots \\ & + 6K_1K_2K_3 + 6K_1K_2K_4 + \cdots . \end{aligned}$$

Therefore the first five Adomian polynomials in (2.4) for $R(K)$ are:

$$R_0 = K_0^3(t), \quad R_1 = 3K_0^2(t)K_1(t),$$
$$R_2 = 3K_0^2(t)K_2(t) + 3K_0(t)K_1^2(t),$$
$$R_3 = K_1^3(t) + 3K_0^2(t)K_3(t) + 6K_0(t)K_1(t)K_2(t),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Substituting (2.3) and (2.4) into (2.2), we have

$$K_0 + K_1 + K_2 + \cdots = C + L^{-1}R_0 - L^{-1}R_1 - L^{-1}R_2 + \cdots .$$

If the series solution is convergent, then we can determine each term of the series $\sum_{n=0}^{\infty} K_n$ recursively:

$$\begin{aligned} &K_0 = C, \quad K_1 = -L^{-1}R_0(K_0), \\ &K_2 = -L^{-1}R_1(K_0, K_1), \\ &\dots\dots\dots\dots\dots\dots \\ &K_n = -L^{-1}R_{n-1}(K_0, K_1, \dots, K_{n-1}), \\ &\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{2.5}$$

The algorithm in (2.5) determines the K_i's and hence the solution K will be known.

The convergence of the series solution has been established in [7]. The two hypotheses that are needed to prove convergence of Adomian's algorithm as given in [7] are:

(1) The nonlinear functional equation (2.1) has a series solution (2.3) such that $\sum_{n=0}^{\infty}(1+\epsilon)^n|K_n| < \infty$ where ϵ may be very small.

(2) The nonlinear operator R is analytic and can be developed in the series $R(K) = \sum_{n=0}^{\infty} \alpha_n K^n$.

These two hypothesis are usually satisfied in physical problems.

3. Adapting the decomposition algorithm to the business cycle equations

We will in this section adapt the decomposition method to solve (1.1), with the initial condition

$$K(0) = C. \tag{3.1}$$

For simplicity, we let $\alpha = a/\tau - n$ and $\beta = a/\tau + m$. Then (1.1) becomes

$$\dot{K}(t) = \alpha K(t) - \beta K(t-\tau) - \epsilon K^3(t). \tag{3.2}$$

In this problem, the operator L is the differential operator d/dt, and its inverse L^{-1} is the integral over the interval $[0, t]$. Writing (3.2) in operator form, we have

$$L[K(t)] = \alpha K(t) - \beta K(t-\tau) - \epsilon K^3(t). \tag{3.3}$$

Applying L^{-1} to both sides of (3.3) yields

$$K(t) - K(0) = \alpha L^{-1}[K(t)] - \beta L^{-1}[K(t-\tau)] - \epsilon L^{-1}[K^3(t)].$$

After using the initial condition (3.1) the above expression becomes

$$K(t) = C + \alpha L^{-1}[K(t)] - \beta L^{-1}[K(t-\tau)] - \epsilon L^{-1}[K^3(t)]. \tag{3.4}$$

As discussed in Section 2, the decomposition method assumes that $K(t)$ has a series solution $K(t) = \sum_{i=0}^{\infty} K_i(t) = K_0 + K_1 + K_2 + \cdots$, and the expansion of $K^3(t)$ is $K^3(t) = A_0(t) + A_1(t) + A_2(t) + \cdots$, where A_i $(i = 1, 2, \dots)$ are Adomian polynomials with

$$\begin{aligned}
&A_0 = K_0^3(t), \qquad A_1 = 3K_0^2(t)K_1(t),\\
&A_2 = 3K_0^2(t)K_2(t) + 3K_0(t)K_1^2(t),\\
&A_3 = K_1^3(t) + 3K_0^2(t)K_3(t) + 6K_0(t)K_1(t)K_2(t) + \cdots,\\
&A_4 = 3K_0^2(t)K_4(t) + 3K_0(t)K_2^2(t) + 3K_1^2(t)K_2(t) + 6K_0(t)K_1(t)K_3(t) + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3.5}$$

Substituting into (3.4) we obtain

$$\begin{aligned}
K_0(t) &+ K_1(t) + K_2(t) + \cdots\\
&= C + \alpha L^{-1}[K_0(t) + K_1(t) + K_2(t) + \cdots]\\
&\quad - \beta L^{-1}[K_0(t-\tau) + K_1(t-\tau) + K_2(t-\tau) + \cdots]\\
&\quad - \epsilon L^{-1}[A_0(t) + A_1(t) + A_2(t) + \cdots].
\end{aligned}$$

We can now determine each term of the series recursively:

$$\begin{aligned}
K_0(t) &= C,\\
K_1(t) &= \alpha L^{-1}[K_0(t)] - \beta L^{-1}[K_0(t-\tau)] - \epsilon L^{-1}[A_0(t)],\\
K_2(t) &= \alpha L^{-1}[K_1(t)] - \beta L^{-1}[K_1(t-\tau)] - \epsilon L^{-1}[A_1(t)],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
K_{n+1}(t) &= \alpha L^{-1}[K_n(t)] - \beta L^{-1}[K_n(t-\tau)] - \epsilon L^{-1}[A_n(t)],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3.6}$$

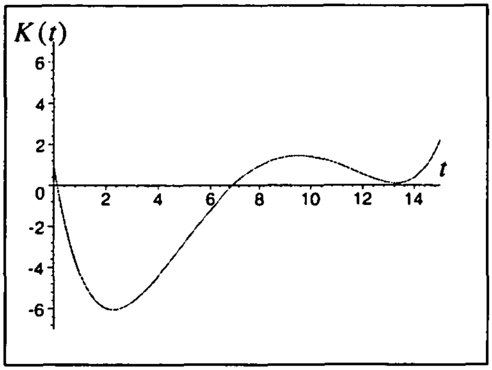

FIGURE 1. The capital stock $K(t)$ oscillates for $0 \leq t \leq 15$ in Case 1.

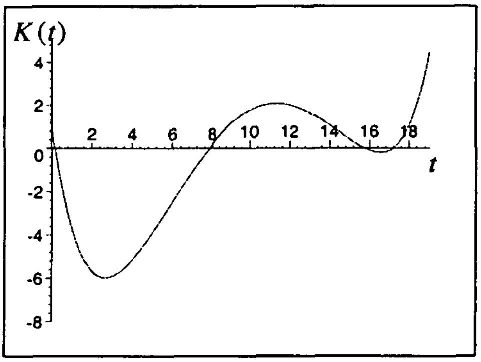

FIGURE 2. The capital stock $K(t)$ oscillates for $0 \leq t \leq 18$ in Case 2.

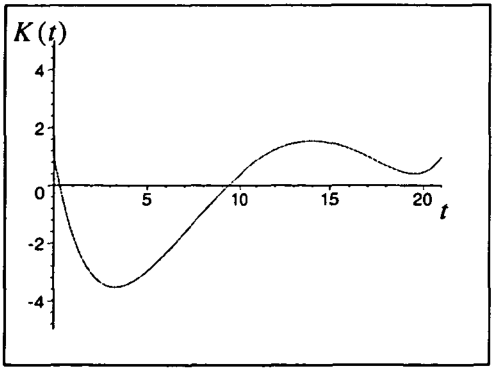

FIGURE 3. The capital stock $K(t)$ oscillates for $0 \leq t \leq 20$ in Case 3.

with $A_i(t)$ $(i = 0, 1, 2, \dots)$ specified in (3.5). Applying $L^{-1} = \int_0^t$ to the K_i's in (3.6), we can use the algorithm in these equations to determine the K_i's and hence the solution K.

Applying $L^{-1} = \int_0^t$ to the right-hand side expressions of (3.6), we can determine an explicit expression for the K_i's:

$$K_0(t) = C, \qquad K_1(t) = (\alpha - \beta)Ct - \epsilon C^3 t,$$
$$K_2(t) = (\alpha - \beta)Ct^2/2 - \epsilon C^3 t^2/2 - \left[\beta(\alpha - \beta)C + \epsilon\beta C^3\right](t^2/2 - bt)$$
$$- 3\epsilon C^2\left[(\alpha - \beta)Ct^2/2 - \epsilon C^3 t^2/2\right],$$

....

Without loss of generality, C is assumed to be 1, the time delay $\tau = 1$ and $\epsilon = 0.01$. In the computation, we use the first nine terms to approximate $K(t)$, that is,

$$K(t) \approx K_0(t) + K_1(t) + \cdots + K_8(t).$$

We investigated the following three different cases:

Case	a	n	m	T (years)	Stability
1	0.95	0.1	0.15	15	stable
2	0.95	0	0.2	18	stable
3	0.95	0	0.15	20	stable

The graphs of these three cases are shown in Figures 1, 2 and 3, respectively. In these figures we can see the behaviour of the business cycles, which represent oscillations in the real economy. The limit cycles are generated for large ranges of the parameters and are stable for $0 \le t \le T$, as listed in the above table. For larger T's we can add more terms in the approximation of $K(t)$.

In this paper we have presented an alternative method to study the stability of the nonlinear delay equation (1.1). Since the method approximates the exact solution by a series representation, we can only show that the bounded fluctuations in the business cycle occur locally.

References

[1] K. Abbaoui and Y. Cherruault, "The decomposition method applied to the Cauchy problem", *Kybernetes* **28** (1999) 68–74.

[2] G. Adomian, "A new approach to nonlinear partial differential equations", *J. Math. Anal. Appl.* **102** (1984) 420–434.

[3] P. K. Asea and P. Zak, "Time-to-build and cycles", *J. Econom. Dynam. Control* **23** (1999) 1155–1175.

[4] A. C. Casal and G. Dibeh, "Functional differential equations modeling in economics and environment", in *Environment, economics and their mathematical models* (eds. J. I. Diaz and J. L. Lions), (Masson, Paris, 1994) 19–27.

[5] Y. Cherruault, *Modeles et methodes mathematiques pour les sciences du vivant* (Presses Universitaires de France, Paris, 1998).

[6] Y. Cherruault, *Optimisation: methodes locales et globales* (Presses Universitaires de France, Paris, 1999).

[7] Y. Cherruault, G. Saccomandi and B. Some, "New results for convergence of Adomian's method applied to integral equations", *Math. Comput. Modelling* **16** (1992) 85–93.

[8] Y. Cherruault and Y. Seng, "The resolution of non-linear integral equations of the first kind using the decompositional method of Adomian", *Kybernetes* **26** (1997) 198–206.

[9] E. Y. Deeba, S. A. Khuri and S. Xie, "An algorithm for solving a nonlinear integro-differential equation", *Appl. Math. Comput.* **115** (2000) 123–131.

[10] G. Dibeh, "Time delays and business cycles: Hilferding's model revisited", *Rev. Political Econ.* **13** (2001) 329–341.

[11] M. Dore, *The macrodynamics of business cycles* (Blackwell, Cambridge, 1993).

[12] R. E. A. Farmer, *The macroeconomics of self-fulfilling prophecies*, 2nd ed. (MIT Press, Cambridge, 1999).

[13] N. Kaldor, "A model of the trade cycle", *Econom. J.* **50** (1940) 78–92.

[14] M. Kalecki, *Collected Works, Vol. 1: Capitalism* (Oxford Press, New York, 1990).

[15] S. Khelifa and Y. Cherruault, "New results for the Adomian method", *Kybernetes* **29** (2000) 332–355.

[16] F. E. Kydland and E. C. Prencott, "Time to build and aggregate fluctuations", *Econometrica* **50** (1982) 1345–1369.

[17] R. E. Lucas, "An equilibrium model of the business cycles", *J. Political Econom.* **83** (1975) 1113–1144.

[18] C. Plosser, "Understanding real business cycles", *J. Econom. Perspect.* **3** (1989) 51–78.

[19] J. E. Stiglitz, "Methodological issues and the new Keynesian economics", in *Macroeconomics: A survey of research strategies* (eds. A. Vercelli and N. Dimitri), (Oxford Univ. Press, New York, 1992) 38–86.

[20] L. Taylor, *Income distribution, inflation and growth* (MIT Press, Cambridge, 1991).