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CONSTRUCTION OF SOLUTIONS TO A GLOBAL EIKONAL EQUATION

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Abstract. We give a new and simple proof to the main result of [8] in which we derived a geometric necessary and sufficient condition for the existence of solutions to a global eikonal equation.

1. **Introduction.** We consider in this paper the following global eikonal equation:

$$1 + h_C(\partial_P \varphi(x)) = 0 \quad \forall x \in \mathbb{R}^n, \quad \varphi(S) = 0, \qquad \text{(e)}$$

where $C \subset \mathbb{R}^n$ is a convex body containing zero in its interior, $S \subset \mathbb{R}^n$ is a nonempty compact set with an empty interior, $h_C(\xi) := \min\{\langle \xi, c\rangle : c \in C\}$ is the *lower support function* of C, and ∂_P is the *proximal subdifferential* (see [3]). A solution of (e) means a lower semicontinuous function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ such that $\varphi(S) = 0$ and for every $x \in \mathbb{R}^n$, for every $\zeta \in \partial_P\varphi(x)$ (if any), we have $1 + h_C(\zeta) = 0$. This concept of solution is called *proximal solution* (see [2]) and coincides, in the context of this paper, with the viscosity solution concepts (see [6]).

Now let g_C be the (Minkowski) *gauge function* associated to the set C defined by $g_C(x) := \min\{\lambda \geq 0 : \frac{x}{\lambda} \in C\}$. It is well-known, using the value function method, that if we replace $\mathbb{R}^n$ by $\mathbb{R}^n \setminus S$ in (e), then the function $T(x) := \min_{s\in S} g_C(x - s)$ becomes a solution of (e) (we note the the function $T(\cdot)$ coincides with the well known minimal time function if the dynamic is taken to be the set C, see [8]). But the function $T(\cdot)$ can never be a solution of (e) (on S) since $0 \in \partial_P T(x)$ for all $x \in S$ (global minimum at any points of S) and $h_C(0) = 0$.

In [8], we studied (apparently for the first time) the existence of solutions to the equation (e). Let us recall the main result of this paper. We begin with some geometric definitions. Let $A \subset \mathbb{R}^n$ be a nonempty closed set and let $\alpha \in A$. We say that a vector $v \in \mathbb{R}^n$ is an exterior vector to A at α if the vector $\alpha + tv$ is not in the interior of A for all $t \geq 0$. The set of all exterior vectors is called *exterior cone* to A at α and denoted by $\mathrm{Ext}_A(\alpha)$. We note that when $\alpha \in \operatorname{int} A$ then we set $\mathrm{Ext}_A(\alpha) := \{0\}$. We say that the set S is an *exterior set* to C if for all $\alpha \in S$ there exists β in the boundary of C such that $(\alpha - S) \subset \mathrm{Ext}_C(\beta)$ (see [8] for more information about these definitions). Now we can state the main result of [8].

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Theorem 1 ([8], Theorem 14)**.** *The eikonal equation* (e) *admits a solution if and only if* S *is an exterior set to* C.

To prove the preceding theorem, we used the main results of [4] and [7] where Clarke and Nour studied the Hamilton-Jacobi equation of the minimal time function (for a nonlinear system) in a domain which contains the target set. The purpose of this paper is to give a direct and more simple proof to Theorem 1. Our new (and self-contained) proof uses the properties of the proximal subdifferential and an analytical construction of solutions. The plan of the paper is as follows. In the next section, we present our notations and hypotheses. The proof of Theorem 1 is given in Section 3.

2. **Notations and hypotheses.** We denote by $\|\cdot\|$ the Euclidean norm and by $\langle , \rangle$ the usual inner product. For $\rho > 0$ we denote by

$$B(0;\rho) := \{x \in \mathbb{R}^n : \|x\| < \rho\} \text{ and } \bar{B}(0;\rho) := \{x \in \mathbb{R}^n : \|x\| \le \rho\}.$$

The open (resp. closed) unit ball in $\mathbb{R}^n$ is denoted by B (resp. $\bar{B}$). For a set $A \subset \mathbb{R}^n$, $\operatorname{int} A$ and $\operatorname{bdry} A$ are the interior and the boundary of A, respectively.

We assume throughout this paper that C is a convex body (and then compact and convex) with $0 \in \operatorname{int} C$ and that S is a compact set with $\operatorname{int} S = \emptyset$. We note that, to prove the existence of solutions to the equation (e), we must assume that $\operatorname{int} S = \emptyset$. This assumption is related to the nature of the semicontinuous solution chosen here. Indeed, if $\operatorname{int} S \neq \emptyset$, then a solution φ of (e) vanishes on $\operatorname{int} S$ and then $0 \in \partial_P \varphi(x)$ for all $x \in \operatorname{int} S$, which gives a contradiction since $h_C(0) = 0$.

3. **Proof of Theorem 1.** First we recall some usual properties of g_C.

Proposition 1. *The gauge function* g_C *satisfies the following:*

(*i*) *The gauge* g_C *is positively homogeneous, that is,* $g_C(rx) = rg_C(x)$ *for all* $x \in \mathbb{R}^n$ *and* $r \ge 0$.
(*ii*) *The gauge* g_C *is subadditive, that is,* $g_C(x+y) \le g_C(x) + g_C(y)$ *for all* $x, y \in \mathbb{R}^n$.
(*iii*) *We have*
- $g_C(x) = 0$ *if and only if* $x = 0$.
- $x \in C$ *if and only if* $g_C(x) \le 1$.
- $x \in \operatorname{bdry} C$ *if and only if* $g_C(x) = 1$.
- $x \in \operatorname{int} C$ *if and only if* $g_C(x) < 1$.

(*iv*) *The gauge function is globally Lipschitz on* $\mathbb{R}^n$.

Proof. See [1] and [5]. □

In the following lemma, we prove that any solution to the equation (e) is globally Lipschitz on $\mathbb{R}^n$.

Lemma 1. *Let* $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function such that* $\varphi(S) = 0$.

(*i*) *If* φ *is a solution to the equation* (e) *then* φ *is globally Lipschitz on* $\mathbb{R}^n$.
(*ii*) *The function* φ *is a solution to the equation* (e) *if and only if it is a solution to the equation* (e) *by replacing the proximal subdifferential* ∂_P *by the limiting subdifferential* ∂_L *(see* [3] *for the definition).*

Proof. (*i*) It is sufficient to prove, using [3, Theorem 1.7.3], the existence of a constant $K > 0$ such that

$$\forall \alpha \in \mathbb{R}^n\ \forall \zeta \in \partial_P \varphi(\alpha) \quad \|\zeta\| \leq K.$$

Using the fact that $0 \in \operatorname{int} C$ we get the existence of a $K > 0$ such that $\bar{B}(0, \frac{1}{K}) \subset C$. Since φ is a solution to the equation (e) we have

$$\forall \alpha \in \mathbb{R}^n\ \forall \zeta \in \partial_P \varphi(\alpha) \quad \min_{v \in C} \langle \zeta, v \rangle = -1.$$

Taking $v := \frac{-\zeta}{K\|\zeta\|}$ we get that

$$\forall \alpha \in \mathbb{R}^n\ \forall \zeta \in \partial_P \varphi(\alpha) \quad \|\zeta\| \leq K.$$

(*ii*) Follows from the continuity of h_C and from the fact that ∂_L is constructed from ∂_P by a limiting process. □

The following lemma gives necessary and sufficient condition for a given globally Lipschitz function φ to be a solution to the equation (e).

Lemma 2. *Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a globally Lipschitz function such that $\varphi(S) = 0$. Then φ is a solution of* (e) *if and only if*

(C1) $\forall \alpha \in \mathbb{R}^n\ \forall v \in C$ *we have*

$$\varphi(\alpha + sv) + s \leq \varphi(\alpha + tv) + t \quad \forall s \leq t \in [0, +\infty[.$$

(C2) $\forall \alpha \in \mathbb{R}^n\ \exists v \in \operatorname{bdry} C$ *such that*

$$\varphi(\alpha) \geq \varphi(\alpha + tv) + t \quad \forall t \in [0, +\infty[.$$

Proof. It is sufficient to prove that

- (C1) $\Longleftrightarrow 1 + h_C(\partial_P(\varphi(x)) \geq 0$ for all $x \in \mathbb{R}^n$, and
- (C2) $\Longleftrightarrow 1 + h_C(\partial_P(\varphi(x)) \leq 0$ for all $x \in \mathbb{R}^n$.

The preceding two equivalences follows from the definition of h_C, the preceding lemma (to replace ∂_P by ∂_L and then use the chain rule of ∂_L, see [3, Chapter 1]) and using the characterization of the monotonicity of a real function by the proximal subdifferential (see [3, Chapter 1]). The details are left to the reader. □

Now we begin the proof of the necessary condition of Theorem 1. We assume that the equation (e) has a solution φ (which is globally Lipschitz by Lemma 1). We need to prove that S is an exterior set to C. Let $\alpha \in S$. By Lemma 2, there exists $v \in \operatorname{bdry} C$ such that

$$\varphi(\alpha + tv) + t = 0 \quad \forall t \in [0, +\infty[. \tag{1}$$

We claim that $\alpha - S \subset \operatorname{Ext}_C(v)$. Indeed, let $\alpha' \in S$. By (C1) and using the realtion $\alpha + tv = \alpha' + g_C(\alpha + tv - \alpha')\frac{\alpha + tv - \alpha'}{g_C(\alpha + tv - \alpha')}$, we get that

$$0 = \varphi(\alpha') \leq \varphi(\alpha + tv) + g_C(\alpha + tv - \alpha'). \tag{2}$$

Combining (1) and (2), we find that

$$g_C(\alpha + tv - \alpha') \geq t \quad \forall t \in [0, +\infty[.$$

Then

$$g_C(v + t(\alpha - \alpha')) \geq 1 \quad \forall t \in [0, +\infty[,$$

which gives by Proposition 1 that

$$v + t(\alpha - \alpha')) \notin \operatorname{int} C \quad \forall t \in [0, +\infty[.$$

Therefore, S is an exterior set to C. The necessary condition follows.

For the sufficient condition, assume that S is an exterior set to C. We need to prove that the equation (e) has a solution. We consider the function φ defined on $\mathbb{R}^n$ by

$$\varphi(\alpha) := \liminf_{\substack{t\longrightarrow+\infty \\ v\in \operatorname{bdry} C}} [t - \min_{\alpha'\in S} g_C(\alpha + tv - \alpha')].$$

We claim that φ is a solution to the equation (e). Using the inequality (follows by the subadditivity of g_C)

$$-\min_{\alpha'\in S} g_C(\alpha - \alpha') \le t - \min_{\alpha'\in S} g_C(\alpha + tv - \alpha') \le \min_{\alpha'\in S} g_C(\alpha' - \alpha)$$

we get that

- $\varphi(S) \ge 0$ and
- $\varphi(\alpha)$ is finite for all $\alpha \in \mathbb{R}^n$.

To prove that $\varphi(S) \le 0$, we consider $\alpha \in S$. Since S is an exterior set to C, there exists $v_\alpha \in \operatorname{bdry} C$ such that $(\alpha - S) \subset \operatorname{Ext}_C(v_\alpha)$ and then (by Proposition 1) $g_C(\alpha + tv_\alpha - \alpha') \ge t$ for all $\alpha' \in S$. This gives that $\varphi(\alpha) \le 0$. Therefore, $\varphi(S) = 0$. Now we prove that φ is globally Lipschitz on $\mathbb{R}^n$. Using the subadditivity of g_C we can easily prove that

$$\varphi(\alpha) \le \varphi(\beta) + g_C(\beta - \alpha) \quad \forall \alpha, \beta \in \mathbb{R}^n.$$

then (K is the Lipschitz constant of g_C)

$$-K\|\alpha - \beta\| \le -g_C(\alpha - \beta) \le \varphi(\alpha) - \varphi(\beta) \le g_C(\beta - \alpha) \le K\|\alpha - \beta\|,$$

which gives that φ is globally Lipschitz on $\mathbb{R}^n$. Let us show that φ satisfies the conditions (C1) and (C2). We begin by (C2).
Let $\alpha \in \mathbb{R}^n$. By the definition of $\varphi(\alpha)$ there exist two sequences $t_i > 0$ and $v_i \in \operatorname{bdry} C$ such that $t_i \longrightarrow +\infty$ and

$$\varphi(\alpha) = \lim_{i\longrightarrow+\infty} [t_i - \min_{\alpha'\in S} g_C(\alpha + t_i v_i - \alpha')].$$

We can assume that the sequence v_i converges to a vector $v \in \operatorname{bdry} C$. For $t \ge 0$ we have

$$\begin{aligned} \varphi(\alpha + tv) &\le \lim_{i\longrightarrow+\infty} [(t_i - t) - \min_{\alpha'\in S} g_C(\alpha + tv + (t_i - t)v_i - \alpha')] \\ &= \lim_{i\longrightarrow+\infty} [t_i - \min_{\alpha'\in S} g_C(\alpha + t(v - v_i) + t_i v_i - \alpha')] - t \\ &= \lim_{i\longrightarrow+\infty} [t_i - \min_{\alpha'\in S} g_C(\alpha + t_i v_i - \alpha')] - t \\ &= \varphi(\alpha) - t. \end{aligned}$$

Therefore,

$$\varphi(\alpha + tv) + t \le \varphi(\alpha) \quad \forall t \ge 0.$$

The condition (C2) follows.
For the condition (C1), let $\alpha \in \mathbb{R}^n$, $v \in \operatorname{bdry} C$ and $s \le t \in [0, +\infty[$. We need to prove that $\varphi(\alpha + sv) + s \le \varphi(\alpha + tv) + t$. By the definition of $\varphi(\alpha + tv)$, there exist two sequences $t_i > 0$ and $v_i \in \operatorname{bdry} C$ such that $t_i \longrightarrow +\infty$ and

$$\varphi(\alpha + tv) = \lim_{i\longrightarrow+\infty} [t_i - \min_{\alpha'\in S} g_C(\alpha + +tv + t_i v_i - \alpha')].$$

We have

$$\begin{aligned}
\varphi(\alpha+tv) &= \lim_{i\longrightarrow+\infty}[t_i - \min_{\alpha'\in S} g_C(\alpha++tv+t_iv_i-\alpha')] \\
&= \lim_{i\longrightarrow+\infty}[t_i - \min_{\alpha'\in S} g_C(\alpha+tv+t_iv_i-\alpha'+sv-sv)] \\
&\geq \lim_{i\longrightarrow+\infty}[t_i - \min_{\alpha'\in S} g_C(\alpha+sv+t_iv_i-\alpha') - g_C((t-s)v)] \\
&= \lim_{i\longrightarrow+\infty}[t_i - \min_{\alpha'\in S} g_C(\alpha+sv+t_iv_i-\alpha')] - t + s \\
&\geq \varphi(\alpha+sv) - t + s.
\end{aligned}$$

The condition (C1) follows. This ends the proof of the theorem.

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