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On the ψ-union of closed balls property

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Abstract

We provide a new analytical proof for a strengthened version of the variable radius form of the union of closed balls conjecture introduced in [15]. We also introduce a strong version of this conjecture and discuss its validity.

Key words: Exterior and interior sphere conditions, union of closed balls property, proximal analysis, nonsmooth analysis.

1. Introduction

Let $S \subset \mathbb{R}^n$ be a nonempty closed set. The regularity of the *minimal time function* associated to the target S, denoted here by $T(\cdot, S)$, is a widely studied topic in control theory. It is known that under a suitable controllability assumption known as *Petrov condition*, the function $T(\cdot, S)$ is locally Lipschitz continuous in its domain of definition. On the other hand, simple examples show that $T(\cdot, S)$ fails to be everywhere differentiable, in general. Differentiability results for $T(\cdot, S)$ have been proved for linear systems if the boundary of S is smooth, while Hölder continuity results have been obtained under weaker controllability assumptions. For more information about these results, see e.g. [2, Chapter 4] and the references therein.

In the paper [5] (see also the book [6]), Cannarsa and Sinestrari were interested in the semiconcavity property of the minimal time function which is an intermediate property between Lipschitz continuity and continuous differentiability. More precisely, semiconcave functions are essentially a C^2-perturbation of concave functions and therefore inherit several regularity properties from convexity, see [6] where several features of semiconcavity

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were thoroughly studied. After finding a connection between the dynamics and the target, Cannarsa and Sinestrari proved that if the target S satisfies a *uniform interior sphere condition* and the dynamics is smooth enough, then $T(\cdot, S)$ is semiconcave. In [4], Cannarsa and Frankowska generalized this semiconcavity result for the case where the dynamics satisfies the uniform interior sphere condition and not necessarily the target S. An analogous result was also found by Sinestrati in [17]. These regularity results are generalized to the non-Lipschitz case (that is, when the Petrov condition is not necessarily satisfied) in [8–10].

In [4] (see also [1,3]), the authors used the following as a definition for the uniform interior sphere condition (we follow this definition here): There exists $r > 0$ such that for any boundary point x in S one can find $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S,$$

where $\bar{B}(y_x; r)$ denotes the closed ball centered at y_x with radius r. In this case we say that S satisfies the interior r-sphere condition. On the other hand, in [5,6,17] the authors used the following as definition: There exists $r > 0$ such that for all $x \in S$ one can find $y_x \in S$ such that

$$x \in \bar{B}(y_x; r) \subset S.$$

This means that S is the *union of closed* r*-balls.* Clearly, if S is the union of closed r-balls then it satisfies the interior r-sphere condition. A simple example, see [14, Example 2], proves that the reverse implication is not necessarily true and then the two definitions (with same specific radius r) are *not equivalent.* More precisely, Nour, Stern and Takche gave in [14, Example 2] a set $S \subset \mathbb{R}^n$ which satisfies the interior r-sphere condition but fails to be the union of closed r'-balls for any $r' > \frac{nr}{2\sqrt{n^2-1}}$ (which is less than r). This lead Nour, Stern and Takche to introduce in [11,12,14] the following two versions of the *union of uniform closed balls conjecture*:

Conjecture 1 (Weak version) *Suppose that $S \subset \mathbb{R}^n$ is a nonempty closed set satisfying the interior r-sphere condition. Then there exists $r' \leq \frac{nr}{2\sqrt{n^2-1}}$ such that S is the union of closed r'-balls.*

Conjecture 2 (Strong version) *Suppose that $S \subset \mathbb{R}^n$ is a nonempty closed set satisfying the interior r-sphere condition. Then S is the union of closed $\frac{nr}{2\sqrt{n^2-1}}$-balls.*

In [11, Corollary 4.2], Nour, Stern and Takche proved the validity of Conjecture 1 under the assumption that S is *wedged* with compact boundary. Recall that a set S is said to be wedged (or *epi-Lipschitz*) if near any boundary point, S can be viewed, after application of an orthogonal transformation, as the epigraph of a Lipschitz continuous function. The proof employed a more general result which asserts that under the wedgedness and compactness hypotheses, *proximal smoothness* of $(\text{int}\, S)^c$ (the complement of the interior of S) and the interior sphere condition of S coincide; see [11, Corollary 3.12]. A proof for Conjecture 1 in the general case and for $r' = \frac{r}{2}$ was given, apparently for the first time, by the same authors in their recent paper [15]. We note that $\frac{r}{2}$ is the greatest radius r' (independent from n) that works for all spaces $\mathbb{R}^n$. Indeed, any such r' must satisfy (due to [14, Example 2]) the inequality $r' \leq \frac{nr}{2\sqrt{n^2-1}}$ for all n, and $\lim_{n \longrightarrow +\infty} \frac{nr}{2\sqrt{n^2-1}} = \frac{r}{2}$.

The proof of Conjecture 1 given in [15] was a geometric proof that uses some results from proximal analysis. Using this proof, Nour, Stern and Takche also presented in [15] a generalization of Conjecture 1 to the case in which the radius of the balls can be taken to be a continuous function, see [15, Theorem 3.1]. More precisely, Nour, Stern and Takche introduced two geometric properties, the *θ-interior sphere condition* and the *ψ-union of closed balls property*, that can be seen as a generalization to the variable radius case, for the uniform interior sphere condition and the union of uniform closed balls property, respectively. Then they showed that if S satisfies the θ-interior sphere condition then it is the ψ-union of closed balls with a formula relating ψ to θ.

The goal of the present article is to provide a strengthened version of [15, Theorem 3.1] (by giving a smaller function ψ) with a direct and analytical proof. As corollary, we deduce a new analytical proof for Conjecture 1. We also present a new conjecture that can be seen as a strong version of [15, Theorem 3.1] by involving, as in Conjecture 2, the dimension n in the formula of ψ. Moreover, we prove via counterexamples that the method used in [15] to deduce the variable radius form from the constant radius form does not work in the strong case. Therefore, a direct proof for the strong version of [15, Theorem 3.1] is needed.

The layout of this article is as follows. In the next section, we present some preliminaries from proximal analysis and provide new analytical characterizations for the θ-interior sphere condition and the ψ-union of closed balls property defined in [15]. Using these analytical characterizations, we give in Section 3 a new analytical proof for a strengthened version of [15, Theorem 3.1]. Section 4 is devoted to the introduction of a strong version of [15, Theorem 3.1] and to discuss its validity.

2. Preliminaries and some analytical characterizations

We denote by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, B and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B(x;\rho) := x + \rho B$ and $\bar{B}(x;\rho) := x + \rho\bar{B}$. For a set $S \subset \mathbb{R}^n$, S^c, $\operatorname{int} S$, ∂S, and $\operatorname{cl} S$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary, and the closure of S, respectively. We also denote by S' the complement of the interior of S, that is, $S' = (\operatorname{int} S)^c$. The distance from a point x to a set S is denoted by $d_S(x)$, and $\operatorname{proj}_S(x)$ denotes the set of closest points in S to x, that is, the set of points $s \in S$ satisfying $\|s - x\| = d_S(x)$.

Now we provide certain definitions from proximal analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [7]; see also [16]. Let S be a nonempty closed subset of $\mathbb{R}^n$. For $x \in S$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to S at x* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, s - x\rangle \leq \sigma\|s - x\|^2 \quad \forall s \in S. \tag{1}$$

The relation (1) is commonly referred to as the *proximal normal inequality.* No nonzero ζ satisfying (1) exists if $x \in \operatorname{int} S$, but this may also occur for $x \in \partial S$, as is the case when S is the epigraph of the function $f(z) = -|z|$ and $x = (0,0)$. For such point, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to S at x is a convex cone, and we denote it by $N_S^P(x)$. Now let $x \in \partial S$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$B\left(x + r\frac{\zeta}{\|\zeta\|}; r\right) \cap S = \emptyset. \tag{2}$$

Then ζ is a proximal normal to S at x and we say that ζ is *realized by an r-sphere.* Note that ζ is then also realized by an r'-sphere for any $0 < r' < r$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = \dfrac{1}{2r}$; that is,

$$\left\langle \frac{\zeta}{\|\zeta\|}, s - x \right\rangle \leq \frac{1}{2r}\|s - x\|^2 \quad \forall s \in S. \tag{3}$$

We proceed to define the ψ-union of closed balls property that can be seen as a generalization, to the variable radius case, of the uniform union of closed balls property.

Definition 3 *A nonempty closed set $S \subset \mathbb{R}^n$ is said to be the ψ-union of closed balls if there exists a function $\psi : S \longrightarrow [0, +\infty[$ such that:*

(i) ψ is upper semicontinuous on S and continuous on ∂S.

(ii) For all $x \in S$ there exists $y_x \in S$ such that:

- *$x \in \bar{B}\left(y_x; \frac{1}{2\psi(x)}\right) \subset S$, if $\psi(x) > 0$.*
- *$x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$, if $\psi(x) = 0$.*

If $\psi \equiv \psi_0$ is a positive constant then the ψ_0-union of closed balls property coincides with the union of closed $\frac{1}{2\psi_0}$-balls property. The following is a new analytical characterization for this property.

Proposition 4 *A nonempty closed set $S \subset \mathbb{R}^n$ is the ψ-union of closed balls if and only if there exists a function $\psi : S \longrightarrow [0, +\infty[$ such that:*

(i) ψ is upper semicontinuous on S and continuous on ∂S.

(ii) For all $x \in S$, one can find a unit vector ζ_x satisfying:

- *There exists $t \in \left[0, \frac{1}{2\psi(x)}\right]$ such that $\langle \zeta_x, z - x + t\zeta_x \rangle \leq \psi(x)\|z - x + t\zeta_x\|^2$ for all $z \in S'$, if $\psi(x) > 0$.*
- *$\langle \zeta_x, z - x \rangle \leq 0$ for all $z \in S'$, if $\psi(x) = 0$.*

Proof. Let $x \in S$ and consider the following two cases:

Case 1: $\psi(x) > 0$.

Then we need to prove the equivalence between the existence of $y_x \in S$ satisfying

$$x \in \bar{B}\left(y_x; \frac{1}{2\psi(x)}\right) \subset S,$$

and the existence of a unit vector ζ_x and $t \in \left[0, \frac{1}{2\psi(x)}\right]$ satisfying

$$\langle \zeta_x, z - x + t\zeta_x \rangle \leq \psi(x)\|z - x + t\zeta_x\|^2 \text{ for all } z \in S'.$$

For the first implication, assume that $x \in \bar{B}\left(y_x; \frac{1}{2\psi(x)}\right) \subset S$ for some $y_x \in S$. We define

$$t := \frac{1}{2\psi(x)} - \|y_x - x\|, \text{ and } \zeta_x := \frac{y_x - x}{\|y_x - x\|} \text{ if } y_x \neq x \text{ and any unit vector if } y_x = x.$$

Clearly we have $t \in \left[0, \frac{1}{2\psi(x)}\right]$ and $y_x = x + \left(\frac{1}{2\psi(x)} - t\right)\zeta_x$. On the other hand, for every $z \in S'$, we have $z \notin \operatorname{int} S$ and then

$$z \notin B\left(y_x; \frac{1}{2\psi(x)}\right) = B\left((x - t\zeta_x) + \frac{1}{2\psi(x)}\zeta_x; \frac{1}{2\psi(x)}\right).$$

Now using the proximal normal inequality we get that

$$\langle \zeta_x, z - x + t\zeta_x \rangle \leq \psi(x)\|z - x + t\zeta_x\|^2.$$

For the converse implication, let $t \in \left[0, \frac{1}{2\psi(x)}\right]$ and let ζ_x be a unit vector such that:

$$\langle \zeta_x, z - x + t\zeta_x \rangle \leq \psi(x)\|z - x + t\zeta_x\|^2 \text{ for all } z \in S'$$

Then $z \notin B\left((x - t\zeta_x) + \frac{1}{2\psi(x)}\zeta_x; \frac{1}{2\psi(x)}\right)$ for all $z \in S'$ and this implies that

$$B\left(x - t\zeta_x + \frac{1}{2\psi(x)}\zeta_x; \frac{1}{2\psi(x)}\right) \subset \operatorname{int} S.$$

Now if we define $y_x := x + \left(\frac{1}{2\psi(x)} - t\right)\zeta_x$ then we obtain that $\bar{B}\left(y_x; \frac{1}{2\psi(x)}\right) \subset S$. Moreover, $x \in \bar{B}\left(y_x; \frac{1}{2\psi(x)}\right)$ since

$$\|y_x - x\| = \left|\frac{1}{2\psi(x)} - t\right| \leq \frac{1}{2\psi(x)}.$$

Case 2: $\psi(x) = 0$.

Then we need to prove the equivalence between the existence of $y_x \in S$ satisfying

$$x \in \bar{B}(x + t(y_x - x); t) \subset S \text{ for all } t > 0,$$

and the existence of a unit vector ζ_x satisfying

$$\langle \zeta_x, z - x \rangle \leq 0 \text{ for all } z \in S'.$$

For the first implication, let $y_x \in S$ such that $x \in \bar{B}(x + t(y_x - x); t) \subset S$ for all $t > 0$. We claim that $\zeta_x := y_x - x$ is a unit vector. Indeed, $\|\zeta_x\| \leq 1$ since $x \in \bar{B}(x + t\zeta_x; t)$. Now assume that $\|\zeta_x\| < 1$. We shall derive a contradiction. For $s \in \operatorname{proj}_{\partial S}(x)$, we define

$$t_0 := \frac{1 + \|x - s\|}{1 - \|\zeta_x\|} > 0. \tag{4}$$

Since s is a boundary point, we have that $s \notin B(x + t_0\zeta_x; t_0) \subset S$. Then

$$t_0 \leq \|s - x - t_0\zeta_x\| \leq \|s - x\| + t_0\|\zeta_x\|. \tag{5}$$

If we combine (4) and (5) we get that $t_0 \leq t_0 - 1$ which gives the desired contradiction. Now for every $z \in S'$, we have $z \notin \operatorname{int} S$ and then $z \notin B(x + t\zeta_x; t)$ for every $t > 0$. This implies that $\langle \zeta_x, z-x\rangle \le \frac{1}{2t}\|z-x\|^2$ for every $t>0$. Therefore $\langle \zeta_x, z-x\rangle \le 0$. The first implication follows. For the converse implication, assume that there exists a unit vector ζ_x satisfying

$$\langle \zeta_x, z-x\rangle \le 0 \text{ for all } z \in S'. \tag{6}$$

Let $y_x := x+\zeta_x$. Then $\langle \zeta_x, y_x - x\rangle = \langle \zeta_x, \zeta_x\rangle = 1 > 0$. This gives, using (6), that $y_x \in \operatorname{int} S \subset S$. Now for $t>0$, let $z \in B(x+t\zeta_x;t)$. Then

$$\langle \zeta_x, z-x\rangle > \frac{1}{2t}\|z-x\|^2 > 0.$$

By (6) we deduce that $z \in \operatorname{int} S$. Hence

$$B(x+t\zeta_x;t) \subset \operatorname{int} S,$$

which gives that

$$x \in \bar{B}(x+t(y_x-x);t) \subset S.$$

Therefore

$$x \in \bar{B}(x+t(y_x-x);t) \subset S \text{ for all } t>0.$$

This terminates the proof of the converse implication and then the proof of the proposition. □

Next we introduce the θ-interior sphere condition. For more information about this property, we invite the reader to see [13] where the θ-interior sphere condition and its relation to *φ-convexity* were studied. We can also find in [13] some regularity results concerning Lipschitz continuous functions with epigraphs satisfying the θ-interior sphere condition. The non-Lipschitz case, with application to minimal time function, was studied in [10].

Definition 5 *A nonempty closed set $S \subset \mathbb{R}^n$ is said to satisfy the θ-interior sphere condition if there exists a continuous function $\theta : \partial S \longrightarrow [0,+\infty[$ such that for all $x \in \partial S$ one can find a point $y_x \in S$ satisfying:*
- *$x \in \bar{B}\left(y_x; \frac{1}{2\theta(x)}\right) \subset S$, if $\theta(x) > 0$.*
- *$x \in \bar{B}(x+t(y_x-x);t) \subset S$ for all $t>0$, if $\theta(x)=0$.*

Clearly the θ_0-interior sphere condition ($\theta_0 > 0$) coincides with the interior $\frac{1}{2\theta_0}$-sphere condition. The following is an analytical characterization for this property.

Proposition 6 *A nonempty closed set $S \subset \mathbb{R}^n$ satisfies the θ-interior sphere condition if and only if there exists a continuous function $\theta : \partial S \longrightarrow [0,+\infty[$ such that for all $x \in \partial S$ one can find a unit vector ζ_x such that*

$$\langle \zeta_x, z-x\rangle \le \theta(x)\|z-x\|^2 \quad \forall z \in S' = (\operatorname{int} S)^c. \tag{7}$$

Proof. Let $x \in \partial S$. As in the proof of the preceding proposition there are two cases. The case $\theta(x) > 0$ follows directly from the proximal normal inequality, and the case

$\theta(x) = 0$ is similar to the case $\psi(x) = 0$ of the proof of Proposition 4. The details are left to the reader. □

3. Main result

We begin this section by recalling the main result of [15, Section 3] which is an equivalence between the θ-interior sphere condition and the ψ-union of closed balls property. Clearly if S is the ψ-union of closed balls then it satisfies the θ-interior sphere condition with $\theta = \psi$ (on ∂S). The following is [15, Theorem 3.1] which proves that the converse implication holds with a relation relating ψ to θ.

Theorem 7 [15, Theorem 3.1] *Let $S \subset \mathbb{R}^n$ be a nonempty closed set which satisfies the θ-interior sphere condition. Then S is the $\widehat{\psi}$-union of closed balls where $\widehat{\psi}(\cdot)$ is defined by*

$$\widehat{\psi}(x) := 2\max\{\theta(s) : s \in \operatorname{proj}_{\partial S}(x)\} \quad \forall x \in S.$$

The main result of this section is the following strengthened version of Theorem 7. We provide a function ψ smaller than the function given in Theorem 7, see Remark 11 and Example 12. We also provide a direct and analytical proof for this theorem that uses the analytical characterizations of Section 2.

Theorem 8 *Let $S \subset \mathbb{R}^n$ be a nonempty closed set satisfying the θ-interior sphere condition. Then S is the $\widetilde{\psi}$-union of closed balls where $\widetilde{\psi}(\cdot)$ is defined by:*

$$\widetilde{\psi}(x) := \min\left\{\frac{1}{2d_{\partial S}(x)}, \frac{\widehat{\psi}(x)}{\sqrt{1+\widehat{\psi}(x)^2 d_{\partial S}(x)^2}}\right\} \quad \forall x \in S.$$

Here $\frac{1}{2d_{\partial S}(x)}$ is taken to be ∞ whenever x belongs to ∂S and $\widehat{\psi}(\cdot)$ is the function defined in Theorem 7.

Proof. Since $\widetilde{\psi}(\cdot) = \widehat{\psi}(\cdot) = 2\theta(\cdot)$ on ∂S, we have that $\widetilde{\psi}(\cdot)$ is continuous on ∂S. On the other hand, the upper semicontinuity of $\widetilde{\psi}(\cdot)$ on S follows directly from the continuity of $d_{\partial S}(\cdot)$, the upper semicontinuity of $\widehat{\psi}(\cdot)$ and the fact that the function $\frac{t}{\sqrt{1+t^2d^2}}$ is an increasing function with respect to t. The details are left to the reader.

Lemma 9 *For $x \in \operatorname{int} S$ and $s \in \operatorname{proj}_{\partial S}(x)$, there exists a unit vector $\zeta_s \in N^P_{S'}(s)$ such that*
(i) $\langle \zeta_s, z - s\rangle \le \theta(s)\|z - s\|^2 \ \ \forall z \in S'$, and
(ii) $\langle \zeta_s, x - s\rangle \ge 0$.

In order to prove this lemma, we first remark that for every integer $m \ge 1$ we have $\bar{B}\left(s; \frac{\|x-s\|}{2m}\right) \not\subset S$. Then for $z_m \in \bar{B}\left(s; \frac{\|x-s\|}{2m}\right) \cap S^c$ the segment joining the point $x \in S$ to the point $z_m \notin S$ intersects ∂S at a point s_m. Let $\zeta_m := \frac{s_m - x}{\|s_m - x\|} = \frac{z_m - x}{\|z_m - x\|}$, then

$$s_m = x + t_1\zeta_m \quad \text{and} \quad z_m = x + t_2\zeta_m,$$

where $\|x-s\| \le t_1 < t_2 \le \|x-s\| + \frac{\|x-s\|}{2m}$. We claim that

$$\|s_m - s\| \le \|z_m - s\| \le \frac{\|x-s\|}{2m}. \tag{8}$$

Indeed,

$$\begin{aligned}
&\|s_m - s\|^2 - \|z_m - s\|^2 \le 0\\
&\Longleftrightarrow \langle (s_m - s) - (z_m - s), (s_m - s) + (z_m - s)\rangle \le 0\\
&\Longleftrightarrow \langle s_m - z_m, s_m + z_m - 2s\rangle \le 0\\
&\Longleftrightarrow \langle (t_1 - t_2)\zeta_m, (t_1 + t_2)\zeta_m + 2x - 2s\rangle \le 0\\
&\Longleftrightarrow \langle \zeta_m, (t_1 + t_2)\zeta_m + 2x - 2s\rangle \ge 0\\
&\Longleftrightarrow t_1 + t_2 + \langle \zeta_m, 2x - 2s\rangle \ge 0,
\end{aligned}$$

which holds since

$$\begin{aligned}
t_1 + t_2 + \langle \zeta_m, 2x - 2s\rangle &\ge \|x-s\| + \|x-s\| + 2\langle \zeta_m, x - s\rangle\\
&\ge 2|x-s\| - 2\|x-s\| = 0.
\end{aligned}$$

Since S satisfies the θ-interior condition and s_m is a boundary point, there exists a unit vector $\zeta_{s_m} \in N^P_{S'}(s_m)$ such that

$$\langle \zeta_{s_m}, z - s_m\rangle \le \theta(s_m)\|z - s_m\|^2 \ \ \forall z \in S'. \tag{9}$$

Clearly we can assume that the sequence ζ_{s_m} is convergent to a certain unit vector ζ_s. Now by (9) and using the fact that $z_m \in S'$ we deduce that

$$\langle \zeta_{s_m}, z_m - s_m\rangle \le \theta(s_m)\|z_m - s_m\|^2 \ \ \forall m.$$

Therefore

$$\begin{aligned}
\langle \zeta_s, x - s\rangle = -\lim_{m\longrightarrow\infty}\langle \zeta_{s_m}, s_m - x\rangle &= -\lim_{m\longrightarrow\infty}\|s_m - x\|\langle \zeta_{s_m}, \zeta_m\rangle\\
&= -\lim_{m\longrightarrow\infty}\|s_m - x\|\left\langle \zeta_{s_m}, \frac{z_m - s_m}{\|z_m - s_m\|}\right\rangle\\
&\ge -\lim_{m\longrightarrow\infty}\|s_m - x\|\theta(s_m)\|z_m - s_m\|\\
&= 0,
\end{aligned}$$

where the sequences s_m and z_m both converge to s due to (8). To prove (i), it is sufficient to take $m \longrightarrow \infty$ in (9). This completes the proof of the lemma.

Lemma 10 *For every $x \in \operatorname{int} S$, we have: $\widehat{\psi}(x) d_{\partial S}(x) \ge \frac{1}{\sqrt{3}} \Longleftrightarrow \widetilde{\psi}(x) = \frac{1}{2d_{\partial S}(x)}$.*

The proof of this lemma follows directly from the definition of $\widetilde{\psi}(\cdot)$. The details are left the reader.

We proceed with the proof of Theorem 8 and we consider $x \in S$.

Case 1: $x \in \partial S$.

Then $\widetilde{\psi}(x) = \widehat{\psi}(x) = 2\theta(x)$. Since S satisfies the θ-interior sphere condition there exists a unit vector $\zeta_x \in N^P_{S'}(x)$ satisfying $\langle \zeta_x, z - x\rangle \leq \theta(x)\|z - x\|^2$ for all $z \in S'$. Now if $\widetilde{\psi}(x) = 0$ then $\theta(x) = 0$ and therefore $\langle \zeta_x, z - x\rangle \leq 0$ for all $z \in S'$. If $\widetilde{\psi}(x) > 0$ then $\theta(s) < \widetilde{\psi}(x)$ and hence

$$\langle \zeta_x, z - x\rangle \leq \theta(x)\|z - x\|^2 \leq \widetilde{\psi}(x)\|z - x\|^2 \;\; \forall z \in S'.$$

Taking $t := 0$ we get that

$$\langle \zeta_x, z - x + t\zeta_x\rangle \leq \widetilde{\psi}(x)\|z - x + t\zeta_x\|^2 \;\; \forall z \in S'.$$

Case 2: $x \in \operatorname{int} S$.

Let $s \in \operatorname{proj}_{\partial S}(x)$ such that $\widehat{\psi}(x) = 2\theta(s)$.

Case 2.1: $\widehat{\psi}(x) d_{\partial S}(x) \geq \frac{1}{\sqrt{3}}$.

Then by Lemma 10 we get that $\widetilde{\psi}(x) = \frac{1}{2d_{\partial S}(x)}$. Now for $t := \frac{1}{2\widetilde{\psi}(x)}$ and $\zeta_x := \frac{x-s}{\|x-s\|}$ we can easily get using the proximal normal inequality and the fact that $z - x + t\zeta_x = z - s$ that

$$\langle \zeta_x, z - x + t\zeta_x\rangle \leq \widetilde{\psi}(x)\|z - x + t\zeta_x\|^2 \quad \forall z \in S'.$$

Case 2.2: $\widehat{\psi}(x) d_{\partial S}(x) < \frac{1}{\sqrt{3}}$.

Then by Lemma 10 we get that

$$\widetilde{\psi}(x) = \frac{\widehat{\psi}(x)}{\sqrt{1 + \widehat{\psi}(x)^2 d_{\partial S}(x)^2}}, \quad \text{and then} \quad \widetilde{\psi}(x) \geq \frac{\sqrt{3}}{2}\widehat{\psi}(x) = \sqrt{3}\,\theta(s).$$

By Lemma 9, there exists $\zeta_s \in N^P_{S'}(s)$ such that

$$\langle \zeta_s, x - s\rangle \geq 0 \quad \text{and} \quad < \zeta_s, z - s\rangle \leq \theta(s)\|z - s\|^2 \;\; \forall z \in S'. \tag{10}$$

Now if $\widetilde{\psi}(x) = 0$ then $\widehat{\psi}(x) = 2\theta(s) = 0$. Hence for $z \in S'$ we get, using (10), that

$$\langle \zeta_s, z - x\rangle = \langle \zeta_s, z - s\rangle - \langle \zeta_s, x - s\rangle \leq 0.$$

So taking $\zeta_x := \zeta_s$ we find the desired inequality.
Now we assume that $\widetilde{\psi}(x) > 0$. This gives that $\widehat{\psi}(x) > 0$ and then one can define $c_s := s + \frac{\zeta_s}{\widehat{\psi}(x)}$. We claim that $\|c_s - x\| \leq \frac{1}{\widetilde{\psi}(x)}$. Indeed,

$$\|c_s - x\|^2 = \left\| s - x + \frac{\zeta_s}{\widehat{\psi}(x)} \right\|^2$$

$$
\begin{aligned}
&= \|s-x\|^2 + \frac{1}{\widehat{\psi}(x)^2} + \frac{2}{\widehat{\psi}(x)}\langle \zeta_s, s-x\rangle \\
&\le d_{\partial S}(x)^2 + \frac{1}{\widehat{\psi}(x)^2} \quad \text{(using (10))} \\
&= \frac{1}{\widetilde{\psi}(x)^2}.
\end{aligned}
$$

If $\|c_s - x\| \le \frac{1}{2\widetilde{\psi}(x)}$, then we define $t := \frac{1}{2\widetilde{\psi}(x)} - \|c_s - x\|$ and $\zeta_x := \frac{c_s - x}{\|c_s - x\|}$. For $z \in S'$, we can prove that

$$
z - x + t\zeta_x = z - c_s + \frac{1}{2\widetilde{\psi}(x)}\zeta_x.
$$

Therefore

$$
\begin{aligned}
&\|z - x + t\zeta_x\|^2 - \frac{1}{\widetilde{\psi}(x)}\langle \zeta_x, z - x + t\zeta_x)\rangle \\
&\qquad = \left\| z - c_s + \frac{1}{2\widetilde{\psi}(x)}\zeta_x \right\|^2 - \frac{1}{\widetilde{\psi}(x)} \left\langle \zeta_x, z - c_s + \frac{1}{2\widetilde{\psi}(x)}\zeta_x \right\rangle \\
&\qquad = \|z - c_s\|^2 + \frac{1}{4\widetilde{\psi}(x)^2} - \frac{1}{2\widetilde{\psi}(x)^2} \\
&\qquad \ge \|z - c_s\|^2 - \frac{1}{\widehat{\psi}(x)^2} \qquad \left(\text{since } \widetilde{\psi}(x) > \frac{\sqrt{3}}{2}\widehat{\psi}(x)\right) \\
&\qquad = \left\langle z - c_s - \frac{1}{\widehat{\psi}(x)}\zeta_s, z - c_s + \frac{1}{\widehat{\psi}(x)}\zeta_s \right\rangle = \left\langle z - s - \frac{2}{\widehat{\psi}(x)}\zeta_s, z - s \right\rangle \\
&\qquad = \|z - s\|^2 - \frac{1}{\theta(s)}\langle \zeta_s, z - s\rangle \ge 0,
\end{aligned}
$$

where the last inequality follows directly from Lemma 9.

If $\|c_s - x\| > \frac{1}{2\widetilde{\psi}(x)}$ then let us define $t := 0$ and $\zeta_x := \frac{c_s - x}{\|c_s - x\|}$. We need to prove that

$$
\langle \zeta_x, z - x\rangle \le \widetilde{\psi}(x)\|z - x\|^2 \ \ \forall z \in S'.
$$

The inequalities $\|c_s - x\| \le \frac{1}{\widetilde{\psi}(x)}$ and $\widehat{\psi}(x) d_{\partial S}(x) < \frac{1}{\sqrt{3}} < 1$ clearly give that

$$
\left(\|c_s - x\| - \frac{1}{\widetilde{\psi}(x)}\right)\left(\|c_s - x\| + \widetilde{\psi}(x)\left(\frac{1}{\widehat{\psi}(x)^2} - d_{\partial S}(x)^2\right)\right) \le 0.
$$

This gives, after expanding the product, that

$$
\begin{aligned}
\|c_s - x\|^2 - \frac{1}{\widehat{\psi}(x)^2} &\le \widetilde{\psi}(x)\|c_s - x\|\left(\frac{-1}{\widehat{\psi}(x)^2} + d_{\partial S}(x)^2 + \frac{1}{\widetilde{\psi}(x)^2}\right) - d_{\partial S}(x)^2 \\
&\le d_{\partial S}(x)^2(2\widetilde{\psi}(x)\|c_s - x\| - 1).
\end{aligned}
$$

Now for $z \in S'$ we get that

$$\|c_s - x\|^2 - \frac{1}{\widehat{\psi}(x)^2} \leq \|z - x\|^2 (2\widetilde{\psi}(x)\|c_s - x\| - 1),$$

and then

$$\|z - x\|^2 + \|c_s - x\|^2 - \frac{1}{\widehat{\psi}(x)^2} \leq 2\widetilde{\psi}(x)\|z - x\|^2 \|c_s - x\|. \tag{11}$$

On the other hand

$$\begin{aligned} 2\langle z - x, c_s - x\rangle &= \|z - x\|^2 + \|c_s - x\|^2 - \|z - c_s\|^2 \\ &= \|z - x\|^2 + \|c_s - x\|^2 - \left\| z - s - \frac{1}{\widehat{\psi}(x)}\zeta_s \right\|^2 \\ &= \|z - x\|^2 + \|c_s - x\|^2 - \|z - s\|^2 - \frac{1}{\widehat{\psi}(x)^2} + \frac{2}{\widehat{\psi}(x)}\langle \zeta_s, z - s\rangle \\ &\leq \|z - x\|^2 + \|c_s - x\|^2 - \frac{1}{\widehat{\psi}(x)^2} \qquad \text{(Lemma 9)} \\ &\leq 2\widetilde{\psi}(x)\|z - x\|^2 \|c_s - x\|. \quad \text{(from (11))} \end{aligned}$$

Then $\langle \zeta_x, z - x\rangle \leq \widetilde{\psi}(x)\|z - x\|^2$ which completes the proof of Theorem 8. □

Remark 11 We can easily prove that Conjecture 1 is a direct consequence of Theorem 8. Indeed, if S satisfies the interior r-sphere condition then it satisfies the θ_0-interior sphere condition with $\theta_0 = \frac{1}{2r}$. Then by Theorem 8 we get that S is the $\widetilde{\psi}$-union of closed balls with

$$\widetilde{\psi}(x) := \min\left\{ \frac{1}{2d_{\partial S}(x)}, \frac{2\theta_0}{\sqrt{1 + 4\theta_0^2 d_{\partial S}(x)^2}} \right\} \quad \forall x \in S.$$

Now it is sufficient to remark that $\widetilde{\psi}(\cdot) \leq 2\theta_0$ to deduce that S is the union of closed $\frac{r}{2}$-balls. On the other hand, it is true that Conjecture 1 is also a direct consequence of [15, Theorem 3.1] but here we obtain a better result. Indeed, we obtain that S is the union of closed balls with variable radius (upper semicontinous function) which is always greater than or equal to $\frac{r}{2}$. This can be seen in the following example that also shows that the function $\widetilde{\psi}(\cdot)$ is less than $\widehat{\psi}(\cdot)$.

Example 12 We consider the set S of [11, Example 4.1], that is, S is the closed region inside the three unit circles of Figure 1. Clearly this set satisfies the θ-interior sphere condition with $\theta = 1$. Now if we calculate the functions $\widehat{\psi}(\cdot)$ and $\widetilde{\psi}(\cdot)$ then we find that:

- $\widehat{\psi}(x) = 1$ for all $x \in S$, and
- $\widetilde{\psi}(x) = \min\left\{ \frac{1}{2d_{\partial S}(x)}, \frac{1}{\sqrt{1+d_{\partial S}(x)^2}} \right\}$ for all $x \in S$.

Clearly both functions coincide on ∂S but $\widetilde{\psi}(x) < \widehat{\psi}(x)$ for all $x \in \operatorname{int} S$. For example:

- At the point $\left(\frac{1}{2\sqrt{3}}, 0\right)$ we have $\widetilde{\psi}\left(\frac{1}{2\sqrt{3}}, 0\right) = \sqrt{\frac{12}{13}} < 1 = \widehat{\psi}\left(\frac{1}{2\sqrt{3}}, 0\right)$.

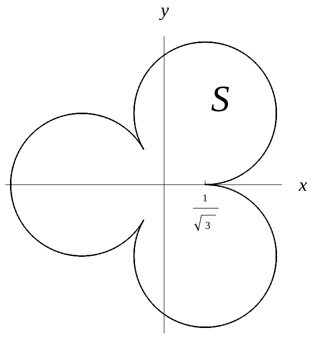

Fig. 1. Example 12

- At the origin, we have $\widetilde{\psi}(0,0) = \frac{\sqrt{3}}{2} < 1 = \widehat{\psi}(0,0)$. Notice that at the origin the function $\widetilde{\psi}(\cdot)$ gives the largest possible radius of a closed ball in S containing it.

4. Strong version and its validity

The goal of this section is to introduce a strong version of Theorem 8. More precisely, we will introduce as in Conjecture 2 the dimension n in the formula of $\widetilde{\psi}(\cdot)$. The idea is that when x is close to the boundary of S then it is better to use the closed ball of Conjecture 2 of radius $\frac{nr}{2\sqrt{n^2-1}}$. Then we obtain the following new conjecture:

Conjecture 13 *Let $S \subset \mathbb{R}^n$ be a nonempty closed set satisfying the θ_0-interior sphere condition. Then S is the $\widetilde{\psi}_n$-union of closed balls where $\widetilde{\psi}_n(\cdot)$ is defined by:*

$$\widetilde{\psi}_n(x) := \min \left\{ \frac{1}{2d_{\partial S}(x)}, \frac{2\theta_0}{\sqrt{1+4\theta_0^2\, d_{\partial S}(x)^2}}, \frac{2\theta_0\sqrt{n^2-1}}{n} \right\} \quad \forall x \in S.$$

Here $\frac{1}{2d_{\partial S}(x)}$ is taken to be ∞ whenever x belongs to ∂S.

Remark 14 In this remark we will prove that Conjecture 13 is in fact equivalent to Conjecture 2. The first implication follows directly from the fact that

$$\widetilde{\psi}_n(x) \leq \frac{2\theta_0\sqrt{n^2-1}}{n} = \frac{\sqrt{n^2-1}}{rn}.$$

For the converse implication, it is sufficient to remark that

$$\widetilde{\psi}_n(x) = \begin{cases} \dfrac{2\theta_0\sqrt{n^2-1}}{n} & \text{if } d_{\partial S}(x) \leq \dfrac{1}{2\theta_0\sqrt{n^2-1}} \\[2ex] \dfrac{2\theta_0}{\sqrt{1+4\theta_0^2\, d_{\partial S}(x)^2}} & \text{if } \dfrac{1}{2\theta_0\sqrt{n^2-1}} \leq d_{\partial S}(x) \leq \dfrac{1}{2\theta_0\sqrt{3}} \\[2ex] \dfrac{1}{2d_{\partial S}(x)} & \text{if } d_{\partial S}(x) \geq \dfrac{1}{2\theta_0\sqrt{3}} \end{cases}$$

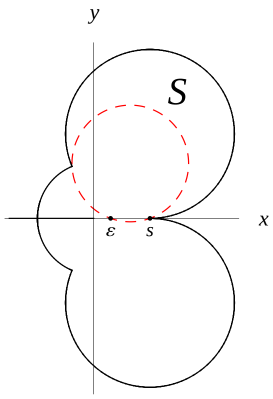

Fig. 2. Example 16

As we mentioned in the introduction, Nour, Stern and Takche used the proof of Conjecture 1 to prove [15, Theorem 3.1]. More precisely, the following geometric lemma, which is also the key lemma for the poof of Conjecture 1, played the central role in their proof.

Lemma 15 *Let* $S \subset \mathbb{R}^n$ *be a nonempty closed set and let* $x \in \operatorname{int} S$*. Assume that for an* $r > 0$ *there exist* $s \in \operatorname{proj}_{\partial S}(x)$ *and* $\rho > 0$ *such that for all* $s' \in B(s;\rho) \cap \partial S$*,* S' *has a proximal normal vector at* s' *realized by an* r*-sphere. Then there exists* $y_x \in S$ *such that*

$$x \in \bar{B}\left(y_x; \frac{r}{2}\right) \subset S \quad \text{and} \quad \|x - y_x\| = \frac{r}{2}.$$

Therefore we have the following diagram:

$$\text{Lemma 15} \Longrightarrow \text{Conjecture 1} \xrightarrow{\text{Lemma 15}} [15, \text{Theorem 3.1}] \tag{12}$$

A natural question follows:

Do we have a strong version of diagram (12)*? That is, can we find a strong version of* Lemma 15 *?*

The following example proves that the preceding question has a negative answer. In fact, we provide a set S and a point $x \in \operatorname{int} S$ satisfying the following:

- There exist $s \in \operatorname{proj}_{\partial S}(x)$ and $\rho > 0$ such that for all $s' \in B(s;\rho) \cap \partial S$, S' has a proximal normal at s' realized by an 1-sphere.
- We cannot find a closed ball inside S of radius $\frac{1}{\sqrt{3}}$ containing x.

Hence to prove Conjecture 2 (and then Conjecture 13), it is not sufficient to use the projections of an interior point x onto ∂S as in the proof of Conjecture 1. This will be, with finding a version of Conjecture 13 in which θ is not necessarily constant, a topic of future research.

Example 16 Let S be the union of the three balls: $\bar{B}((0,0);s)$ and $\bar{B}((s,\pm 1);1)$ where $s < \frac{1}{\sqrt{3}}$, see Figure 2. For $\varepsilon > 0$, we consider $x = (\varepsilon, 0)$. Clearly $(s,0)$ is the unique projection of x onto the boundary of S. Moreover, one can easily see the existence of a neighborhood O for $(s,0)$ such that at any point $s' \in O \cap \partial S$, S' has a proximal normal vector at s' realized by a 1-sphere. On the other hand, the largest closed ball in S containing x is the dashed ball in Figure 2 with radius

$$r' = \frac{1}{2}\sqrt{\left(1+\frac{\varepsilon}{d+\varepsilon}\right)^2 + d^2},$$

where $d = s - \varepsilon$ is the distance from x to the ∂S. For ε sufficiently small, we can easily prove that $r' < \frac{1}{\sqrt{3}}$.

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