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Improving understanding in ordinary differential equations through writing in a dynamical environment

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Research on writing in mathematics has shown that students learn more effectively in an environment that promotes this skill and that writing is most beneficial when it is directed at the learning aspect. Writing, however, necessitates proficiency on the part of the students that may not have been developed at earlier learning stages. Research has indicated though that the burden placed on teachers and learners to master this skill is compensated by the mathematical learning in such an environment. Techniques to successfully integrate writing in the mathematics classroom can be varied. This study is conducted on students in an introductory differential equations class in which a reformed approach is adopted be it in the topics discussed, the textbook used, the technology employed or the assignments/exams given. More precisely, the article explores the effect of writing on improving student understanding of particular topics in differential equations and investigates the development of the students' writing skills.

1. Introduction

In the NCTM (National Council of Teachers of Mathematics, 2000) standards, mathematical representations were recognized as symbolic, verbal, graphical and numeric. The Standards advocated using multiple representations within a problem-solving situation. It is without any doubt that every particular representation carries strengths and weaknesses (Dugdale, 1993; Keller & Hirsch, 1998; Knuth, 2000). Thus, and as argued by Keller and Hirsch, multiple representations must lead to increases in student conceptual understanding. Dugdale also believes that when students are not constrained to one representation, problem-solving skills are strengthened. In fact, multiple representations are important because of the connections they create between the various viewpoints of one particular concept (Romberg *et al*., 1993).

Calls for using multiple representations in mathematics teaching came along with calls for integrating writing across the curriculum. It has been suggested that the skill of writing assists students to learn more effectively and more deeply. Rose (1989) considers writing in mathematics as a learning experience that deepens mathematical understanding. Shepard (1993) adds that writing in mathematics extends student thinking. Freitag (1997) argues that writing in mathematics requires a solid understanding of the numeric, symbolic, graphical and verbal representations of a given mathematical concept and of their interconnections. Consequently, mathematical writing can be simultaneously considered as a single representation and the conjoining of all representations. As stated in the Standards (p. 60), 'students who have opportunities, encouragement and support for speaking, writing, reading and listening in mathematics classes reap dual benefits: they communicate to learn mathematics, and they learn to communicate'. Thus, the goal for increasing writing in mathematics must be increasing mathematical learning and indeed the real benefits for using writing to learn mathematics is due, not to the actual activity of writing, but rather to the fact that it requires students to spend time thinking about mathematical ideas and then communicating these ideas to others.

To achieve the best results, the selection of appropriate classrooms tasks and assignments is essential. As with the various approaches to learning mathematics, students need to be trained to write mathematically (Shibli, 1992; Moore, 1993). Acquiring this skill depends on the learning experience provided by the teacher. Therefore, it is critical that teachers show their students the importance of writing in the mathematics classroom (Aiken, 1997; Blanton, 1991). Needless to say that student interest in writing can be further promoted through assignments and exams.

2. The experiment

This research article explores student understanding of key concepts in an introductory differential equations class through writing assignments and also investigates the improvement of their writing skills. The course offered at the Lebanese American University in Beirut, Lebanon, is a sophomore level course mainly offered for engineering students. The instructor of the course (I) has been experimenting with various instruction methods in this particular course with an eye on promoting multiple representations of concepts (Habre, 2000). There has been a particular emphasis throughout though on the visual representation that has recently become more learner-friendly because of the availability of dynamical software programs. Traditionally, differential equations were taught in a very mechanical way: equations are usually classified and for each class, a method of solution is presented. Since differential equations are widely used in engineering and the physical sciences, this mechanical approach has defeated the purpose of the course as an aid to understanding real life problems (such as the harmonic oscillator, predator–prey models and others). As a result, many educators came to conclude that teachers and students alike are losing sight of the practical value of differential equations. In fact, how such an equation is written is not of much importance; the importance lies in what the equation tells us about the situation being modelled.

To achieve the learning objectives of this course, strategies and special lesson plans have been devised in order to highlight the visual aspect of some concepts, especially the meaning of a solution to a first-order ordinary differential equation (ode). In this reformed setting, the instructor usually highlights the geometric meaning of a solution to the differential equation $\frac{dy}{dt} = f(t, y)$ as the curve whose rate of change at a given point (t_0, y_0) is equal to $f(t_0, y_0)$. The direction field is therefore a key concept for understanding this notion and the use of dynamical specialized software is critical for an effective teaching and learning environment. In this context, the dynamical program ODE Architect (1999) has been adopted for demonstration purposes in the classroom.

When working with direction fields of first-order odes the possibilities are wide. Not only do we require students to draw solutions but also to investigate their behaviour, consequently to explore the dependence of solutions on initial conditions. Below is one example: (see Fig. 1).

As the figure suggests, the behaviour of solutions for some odes can vary considerably if initial conditions change. Of course, such variations may not be obvious when solutions are only presented

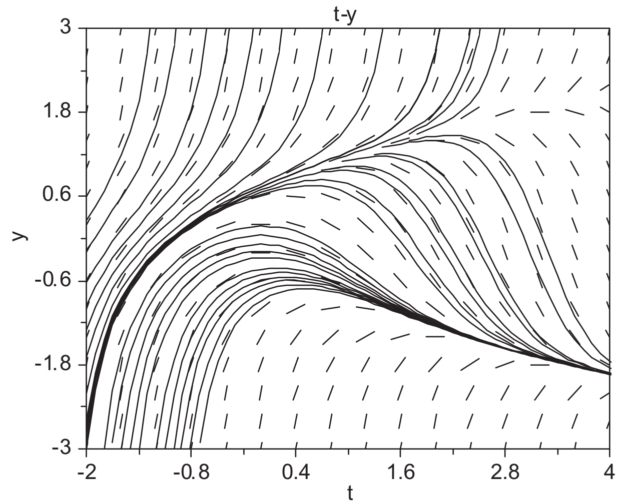


FIG. 1. The direction field of $y' = y^2 - t$.

symbolically, hence the importance of the graphical representation. Indeed, a solid understanding of first-order odes often requires the learner to collate the symbolic with the visual aspects of the problems at hand. As research suggests writing can be used to conjoin these two representations; for instance, Habre (2003) study showed that most subjects of the study argued that writing was essential in such a course. Some reasoned that writing complements the geometrical approach while others thought that it was necessary for enhancing the learning.

The experiment, object of this study, was conducted in the spring of 2010. The book (Boyce & DiPirma, 2009) seemed to serve the purpose of the reform approach because its authors combine the quantitative (symbolic) and qualitative (visual) approaches for solving odes and often outcomes are analyzed in writing. Consequently, during the course of the semester, the instructor emphasized the role of writing or using prose as a tool to communicate the solution behaviour of differential equations and also to relate the symbolic with the graphical representation of a solution whenever possible.

Initially, 43 students were enrolled in the course out of which only two dropped and another three failed. As the chart below suggests (Fig. 2), the students were academically strong with 72% earning a final grade above 70, 25% earning a final grade above 80 and 17% earning a final grade above 90. The experiment consisted primarily of unguided writing exercises on exams. In a few cases, the same exercises were assigned as take-home projects and guidance was given as to the teacher's expectations. More particularly, students were asked on Exam 1 to discuss in writing the shape of a solution satisfying an initial value problem; in another problem, they were to match differential equations with direction fields and to use prose to justify their choice. Also on Exam 1, students were to write a short paragraph explaining the concept of an ordinary differential equation. On Exam 2, a writing problem required the discussion of the motion of a door while opening/shutting based on graphs of solutions to odes. The Final Exam included a matching problem similar to what was given on Exam 1. Copies of the relevant exam problems were collected and grouped according to one basic criterion: Improvement in the student's writing skills.

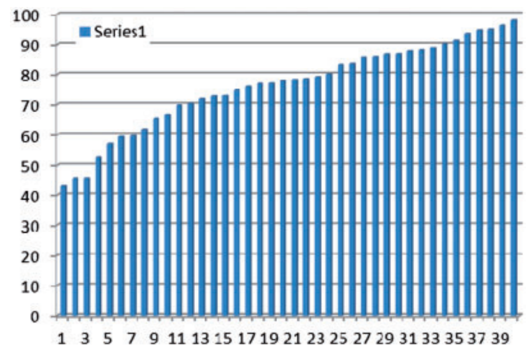


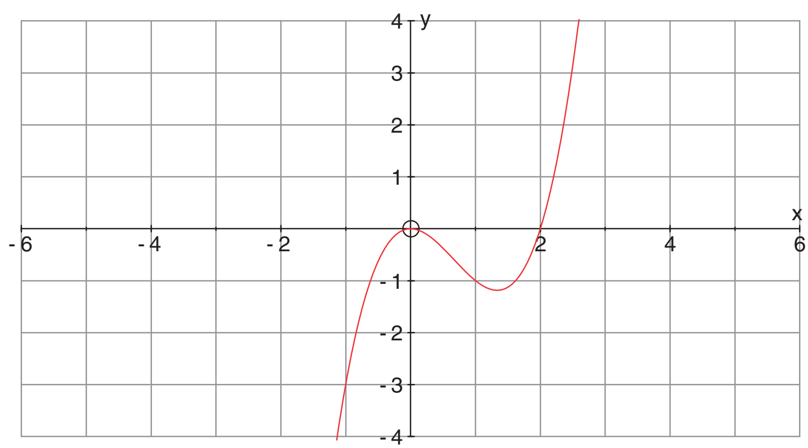
FIG. 2. Students' final averages. This figure appears in colour in the online Version of *Teaching Mathematics and its Applications*.

3. Results

3.1 Writing questions—exam 1

In line with the reformed curriculum of odes and in an attempt to highlight the different representation methods as advised by the Standards, 'solving' a first-order ode has taken a new meaning. In a non-traditional classroom, pupils are taught that 'solving' can mean graphing the solution function to an ode $\frac{dy}{dt} = f(t, y)$ because the shape of the solution and its behaviour (short or long term) may sometimes be more significant than its analytical form. As a result, the importance of a direction field for first-order odes takes on a new dimension as it plays a critical role in describing solutions graphically. For this reason, a considerable amount of time is spent discussing such fields, and there are various types of questions one can ask about a direction field and consequently about the shape of the solutions to an ode. Part of the discussion of solutions is done in writing and requires conjoining the physical properties of $f(t, y)$ to those of the solutions to the ode. In this context, the following was asked on Exam 1:

Consider the differential equation $\frac{dy}{dt} = f(y)$, where the graph of $f(y)$ takes the shape:



Draw the phase line of this autonomous equation, identify its equilibrium solutions, classify them and then discuss the shape of the solution satisfying the initial condition $y(0) = 1$.

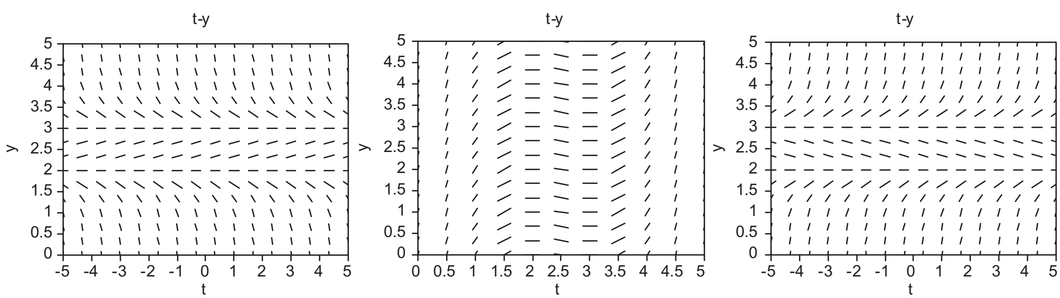
Based on the phase line, one concludes that the solution to this initial value problem is decreasing and that its long-term behaviour is asymptotic to the equilibrium solutions $y \equiv 0$ and $y \equiv 2$. The main reasons for the asymptotic behaviour are: (1) uniqueness of solutions (consequently, no two solutions can meet) and (2) the closer the solution is to any of the two equilibrium ones, the closer $\frac{dy}{dt}$ is to zero (consequently, the solution is more and more horizontal). These properties necessitate that the solution changes concavity from concave down to concave up.

The second writing question on Exam 1 reads as follows:

Consider the following first - order odes

$$\frac{dy}{dt} = (2-y)(3-y); \quad \frac{dy}{dt} = (y-2)(3-y); \quad \frac{dy}{dt} = (2-t)(3+t); \quad \frac{dy}{dt} = (2-y)(3+t)$$

Assign the direction fields below to the corresponding ode and write a short paragraph to justify your choices.



In this problem, the form of $f(t, y)$ is critical. In the first two cases, $f(t, y) = f(y)$ (the differential equation is then labelled autonomous), consequently the ode possesses equilibrium solutions and generally solutions are horizontal translates of each other; in the third case, $f(t, y) = f(t)$ and solutions are vertical translates. In the fourth case, solutions are neither. It is exactly these properties that students are expected to discuss. In addition, and in the case of the first two odes, pupils have to discuss the corresponding phase lines since both have the same equilibrium solutions; the directions on the phase lines are different though. There are other characteristics that students can explore such as $\lim_{y \to y_0} f(y)$, where y_0 is an equilibrium solution for the autonomous ode.

The third writing question on the first exam was more like essay writing and it aimed at examining the students' level of understanding of the concept of ordinary differential equations. Here is what it said:

Imagine yourself standing in front of an audience with minimal calculus background. Your task is to introduce to your audience in the simplest way the concept of a (ordinary) differential equation. Elaborate in a short paragraph how you would complete this task. Support your ideas with examples and describe/explain the various approaches to solve such equations.

Following the first exam, and upon their approval, students' work on these exams was photocopied and was divided into two main categories: satisfactory and unsatisfactory. Although writing was a key ingredient when preparing the questions, yet the grade assigned specifically on the first two questions depended mainly on the mathematical knowledge revealed in the solutions. This strategy was implemented because the teacher's expectations of the students' writing skills were not high.

When categorizing their work, writing skills were assessed as unsatisfactory if students barely used any prose, and thus the work presented was purely mathematical. Twenty nine students belonged to this category. Here is a sample by Ahmad (Fig. 3(a–c)).

Even though the shape of the solution in Fig. 3a is correct indicating that the student has acquired the basic knowledge needed to solve the given problem analytically, yet Ahmad has failed to describe the solution using prose. In particular, the student has not spoken of the uniqueness of solutions causing the asymptotic behaviour. Also much of the mathematics in Fig. 3b is correct: Ahmad spoke of horizontal mini-tangents (to highlight the existence of the horizontal equilibrium solutions) and of the $\lim_{y\to y_0} f(y)$, but in his discussion, he does not mention vertical or horizontal translates. Figure 3c is however the strongest evidence that Ahmad's appreciation of writing in mathematics is marginal: even though the problem is to introduce to an audience with minimal calculus background the concept of an ordinary differential equation, the student emphasized a quantitative meaning for differential equations, a method that could be alien to such an audience.

The work of another student Ali with unsatisfactory writing skills is also presented below. Except for writing question 3, the style of Ali is very similar to that of Ahmad's. It is shown here (Fig. 4(a–c)) because the performances of the two students differ on the second and final exams.

In writing question 3, we notice that Ali's presentation is clearer than Ahmad's. In this writing assignment, Ali enlightened the audience that a differential equation 'relates a function to its rate of change', a key idea to explain the concept of the derivative. His discussion reveals a slightly better understanding of the importance of differential equations than Ali's.

The writing skills of the remaining 11 students were categorized as satisfactory. The students in this group, however, needed not have all three writing exercises on the first exam expanded in the most

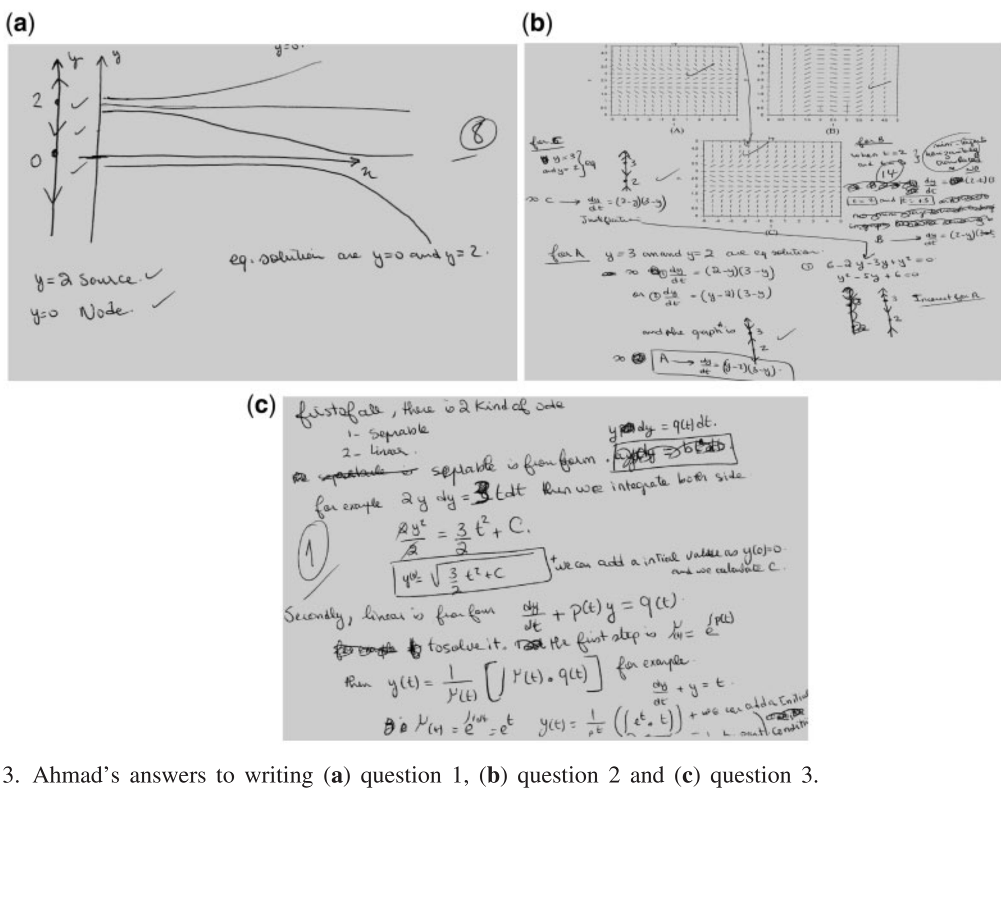


FIG. 3. Ahmad's answers to writing (**a**) question 1, (**b**) question 2 and (**c**) question 3.

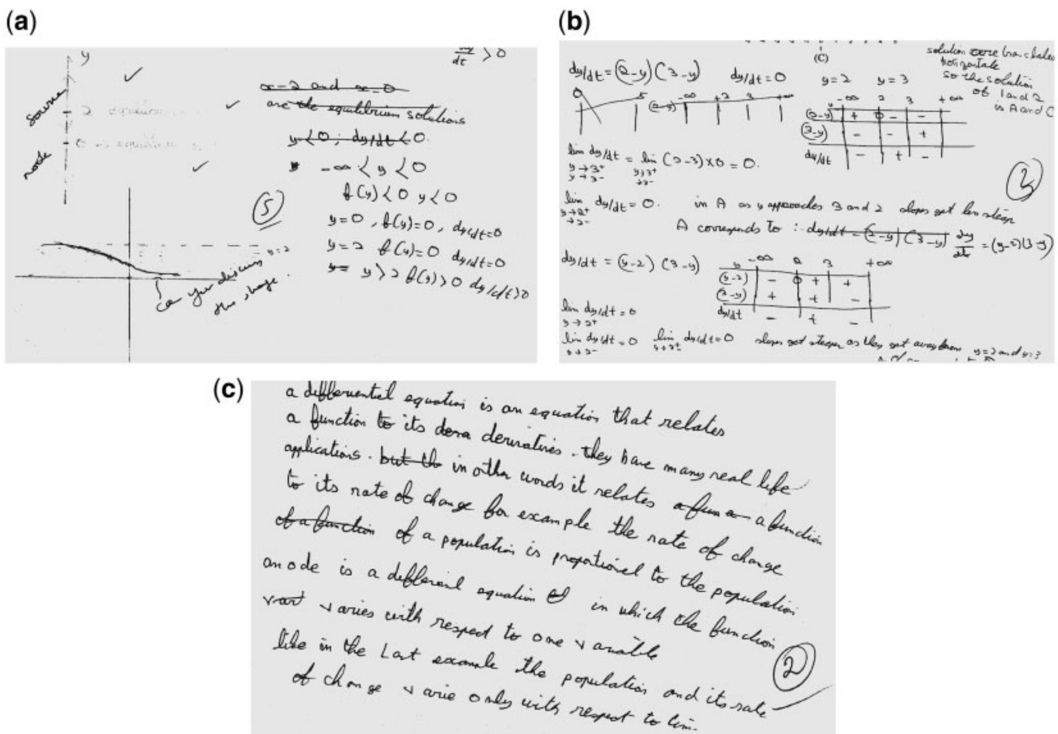
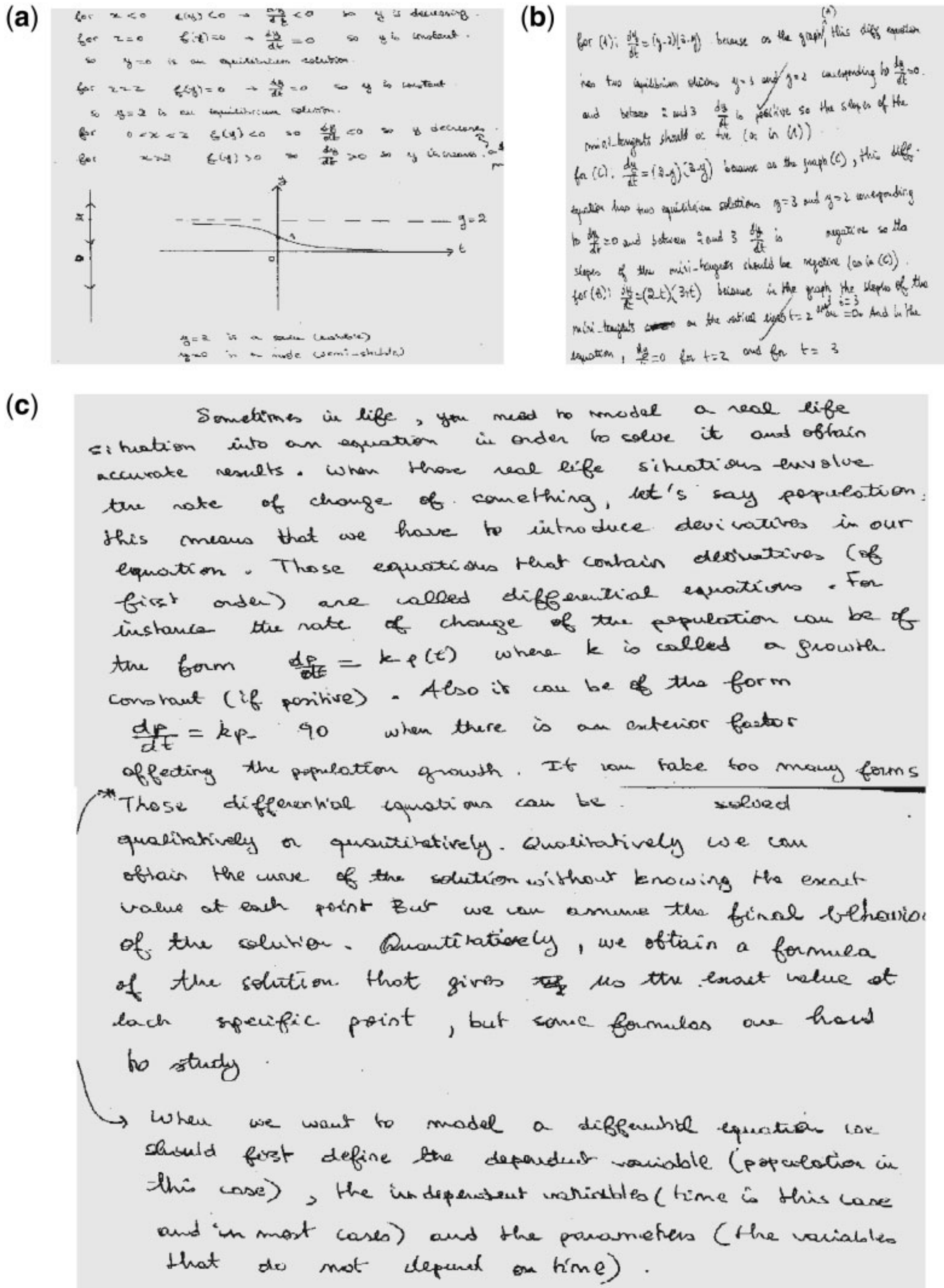


FIG. 4. Ali's answers to writing (**a**) question 1, (**b**) question 2 and (**c**) question 3.

comprehensive way. In the sample below (Fig. 5(a–c)), Cathy did not discuss, in detail, the shape of the solution satisfying the initial condition as requested, but when associating differential equations to slope fields she was more detailed. Although she did not mention specifically horizontal or vertical translates of solutions, her investigation of equilibrium solutions for the first two differential equations is an indication that in her mind she is thinking about autonomous equations (whose solutions are horizontal translates). For the third equation, she emphasizes that the mini-tangents of the slope field are vertical for specific values of t. This is one property for odes of the form $\frac{dy}{dt} = f(t)$. But, again this is an indication of her thinking strategies. The most details were presented in introducing the topic of differential equations to a non-specialized audience. Cathy explained the meaning of the derivative, gave a real-life example and discussed the various forms of solutions (qualitative *vs*. quantitative): 'when those real life situations involve the rate of change of something, let's say population, this means that we have to introduce derivatives to our equation .... Quantitatively, we can obtain the curve of the solution without knowing the exact value at each point but we can assume the final behaviour of the solution ...'.

3.2 Writing questions—exam 2 and final exam

On the second exam, only one writing question was presented to the students, and on the final exam, another problem requesting associating differential equations with slope fields was given. Here is the writing question on Exam 2:



(a)

for x < 0 f(y) < 0 → dy/dt < 0 so y is decreasing.
for x = 0 f(y) = 0 → dy/dt = 0 so y is constant.
so y = 0 is an equilibrium solution.
for x = 2 f(y) = 0 → dy/dt = 0 so y is constant
so y = 2 is an equilibrium solution.
for 0 < x < 2 f(y) < 0 so dy/dt < 0 so y decreases.
for x > 2 f(y) > 0 so dy/dt > 0 so y increases.

y = 2 is a source (unstable)
y = 0 is a node (semi-stable)

(b)

for (A): dy/dt = (y-2)(3-y) because as the graph (*), this diff equation has two equilibrium solutions y = 3 and y = 2 corresponding to dy/dt = 0. and between 2 and 3 dy/dt is positive so the slopes of the mini-tangents should be +ve (as in (A))
for (C): dy/dt = (2-y)(3-y) because as the graph (C), this diff. equation has two equilibrium solutions y = 3 and y = 2 corresponding to dy/dt = 0 and between 2 and 3 dy/dt is negative so the slopes of the mini-tangents should be negative (as in (C)).
for (B): dy/dt = (2-t)(3-t) because in the graph the slopes of the mini-tangents on the vertical lines t = 2 and t = 3 are = 0. And in the equation, dy/dt = 0 for t = 2 and for t = 3

(c)

Sometimes in life, you need to model a real life situation into an equation in order to solve it and obtain accurate results. When those real life situations involve the rate of change of something, let's say population, this means that we have to introduce derivatives in our equation. Those equations that contain derivatives (of first order) are called differential equations. For instance the rate of change of the population can be of the form dp/dt = k p(t) where k is called a growth constant (if positive). Also it can be of the form dp/dt = kp - 90 when there is an exterior factor affecting the population growth. It can take too many forms
*Those differential equations can be solved qualitatively or quantitatively. Qualitatively we can obtain the curve of the solution without knowing the exact value at each point But we can assume the final behavior of the solution. Quantitatively, we obtain a formula of the solution that gives us the exact value at each specific point, but some formulas are hard to study.
→ When we want to model a differential equation we should first define the dependent variable (population in this case), the independent variables (time is this case and in most cases) and the parameters (the variables that do not depend on time).

FIG. 5. Cathy's answers to writing (**a**) question 1, (**b**) question 2 and (**c**) question 3.

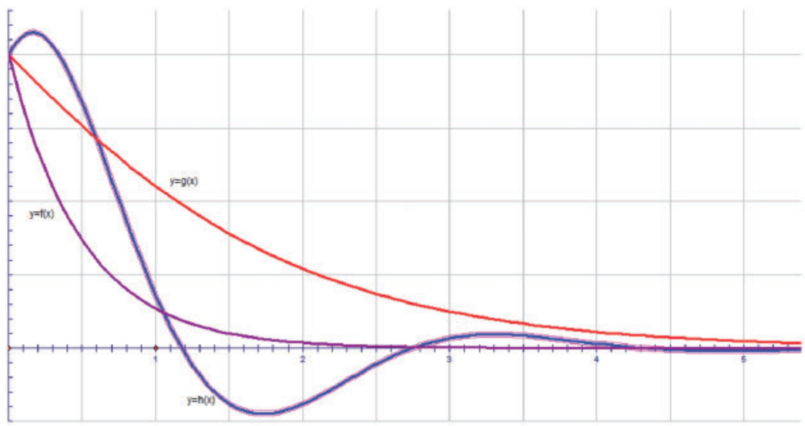
In class we discussed doors and how they would close. The modelling of such a problem is through a second-order differential equation with constant coefficients that are positive. Some solutions take one of the following forms:

$$f(t) = Ae^{-at}; \quad g(t) = Ae^{-at} + Bte^{-at}; \quad a,\ b, A, B > 0,$$

while others take the form:

$$h(t) = e^{-at}(k_1 \cos(bt) + k_2 \sin(bt))$$

Below are sample graphs of such functions; discuss the differences in which these doors actually come to rest. Also discuss what happens in case $h(t) = \cos(bt)$ i.e. $a = k_2 = 0$. Is there a best model for this problem?

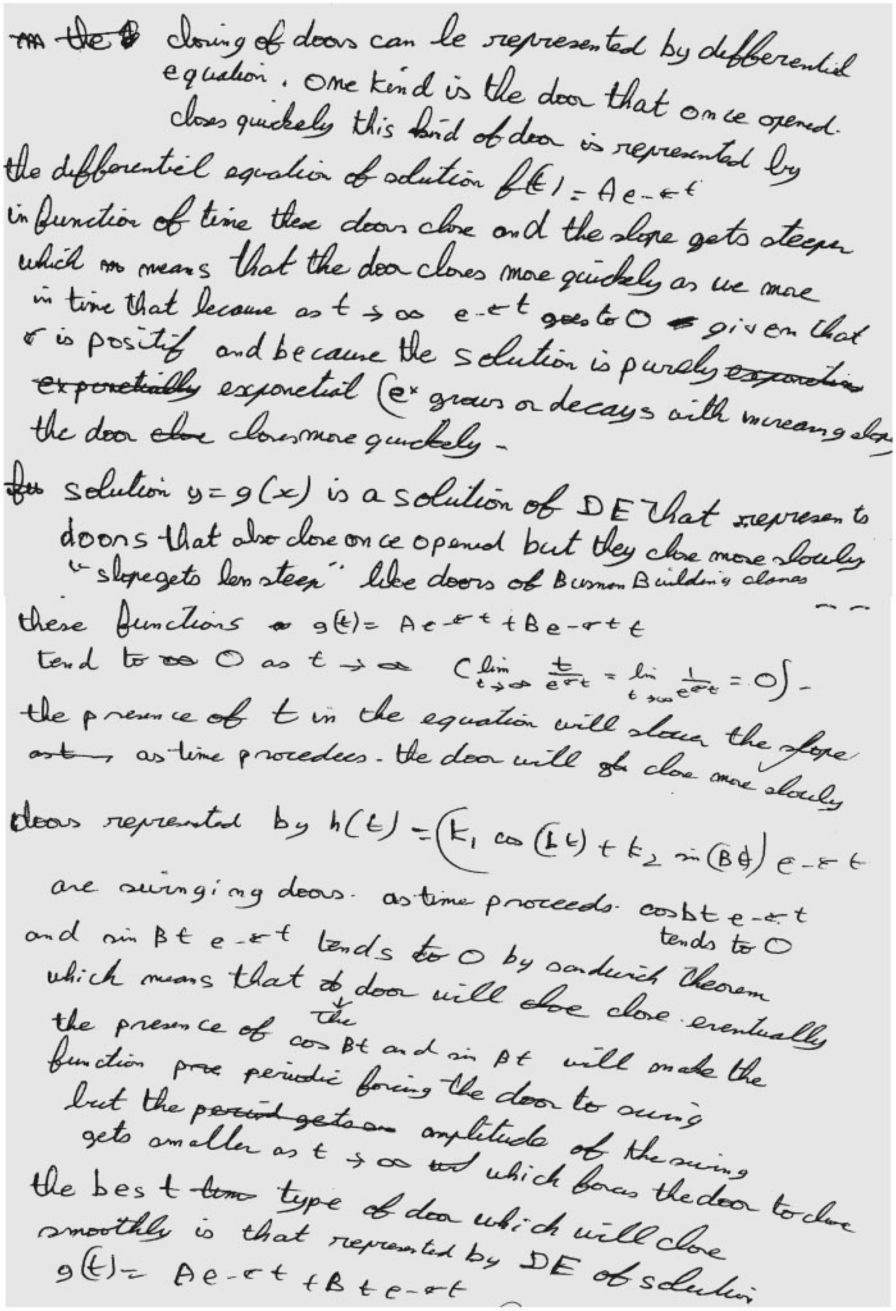


In this problem, students are expected to observe that the decrease of the function $f(t) = Ae^{-at}$ to its equilibrium state (i.e. the door comes to rest)is rather sharp; whereas in the case of $g(t) = Ae^{-at} + Bte^{-at}$, the function approaches the t-axis more slowly, hence the door shuts in a smoother way. The function $h(t)$ represents a swinging door and hence the door oscillates in and out before coming to rest. Clearly, in the case of non-swinging doors, $g(t)$ is the best model.

Only 10 out of the 29 students who were categorized as having unsatisfactory writing skills showed improvement in these skills; student Ali was one of them. I begin by presenting Ali's work (Fig. 6).

This work shows a great improvement in Ali's writing skills. His detailed answer reveals a deep understanding of the problem. His discussion emphasized the importance of a door closing quickly versus a door closing more slowly. He was also successful in associating the motion of the door to the mathematical equations: '... the slope gets steeper which means that the door closes more quickly as we move in time... because as $t \to \infty e^{-at}$ goes to zero given that a is positive...', while 'the presence of t in the equation [of $g(t)$] will slower the slope as time proceeds'. Ali also discussed, in detail, the swinging door represented by $h(t)$. He went the extra mile of using the Sandwich theorem to justify the graphical behaviour of $h(t)$ and hence the motion of the door.

Ali's work on the final exam is yet another testimony of the great improvement of his writing skills. A simple comparison between his work on the problem in Exam 1 requesting the student to associate slope fields to differential equations and the similar problem on the final exam shows progress (Fig. 7). Indeed, Ali noticed that on the top graph, solutions are vertical translates while on the bottom one, solutions are neither vertical nor horizontal translates. He moved on with his discussion to looking at



~~the~~ closing of doors can be represented by differential equation. One kind is the door that once opened closes quickely this kind of door is represented by the differentiel equation of solution $f(t) = Ae^{-\sigma t}$ in function of time these doors close and the slope gets steeper which means that the door closes more quickely as we move in time that because as $t \to \infty$ $e^{-\sigma t}$ goes to 0 given that σ is positif and because the solution is purely ~~exponentially~~ exponetial (e^x grows or decays with increasing slope) the door ~~close~~ closes more quickely.

~~the~~ solution $y = g(x)$ is a solution of DE that represents doors that also close once opened but they close more slowly "slope gets less steep" like doors of Bustani Building classes

these functions $g(t) = Ae^{-\sigma t} + Be^{-\sigma t}t$ tend to 0 as $t \to \infty$ ($\lim_{t\to\infty} \frac{t}{e^{\sigma t}} = \lim_{t\to\infty} \frac{1}{\sigma e^{\sigma t}} = 0$).

the presence of t in the equation will slow the slope as time procedees. the door will close more slowly

doors represented by $h(t) = (k_1 \cos(\beta t) + k_2 \sin(\beta t))e^{-\sigma t}$ are swinging doors. as time proceeds $\cos\beta t\, e^{-\sigma t}$ tends to 0 and $\sin \beta t\, e^{-\sigma t}$ tends to 0 by sandwich theorem which means that the door will close eventually

the presence of $\cos \beta t$ and $\sin \beta t$ will make the function periodic forcing the door to swing but the ~~period gets~~ amplitude of the swing gets smaller as $t \to \infty$ which forces the door to close

the best type of door which will close smoothly is that represented by DE of solution $g(t) = Ae^{-\sigma t} + Bte^{-\sigma t}$

FIG. 6. Ali's work on exam 2.

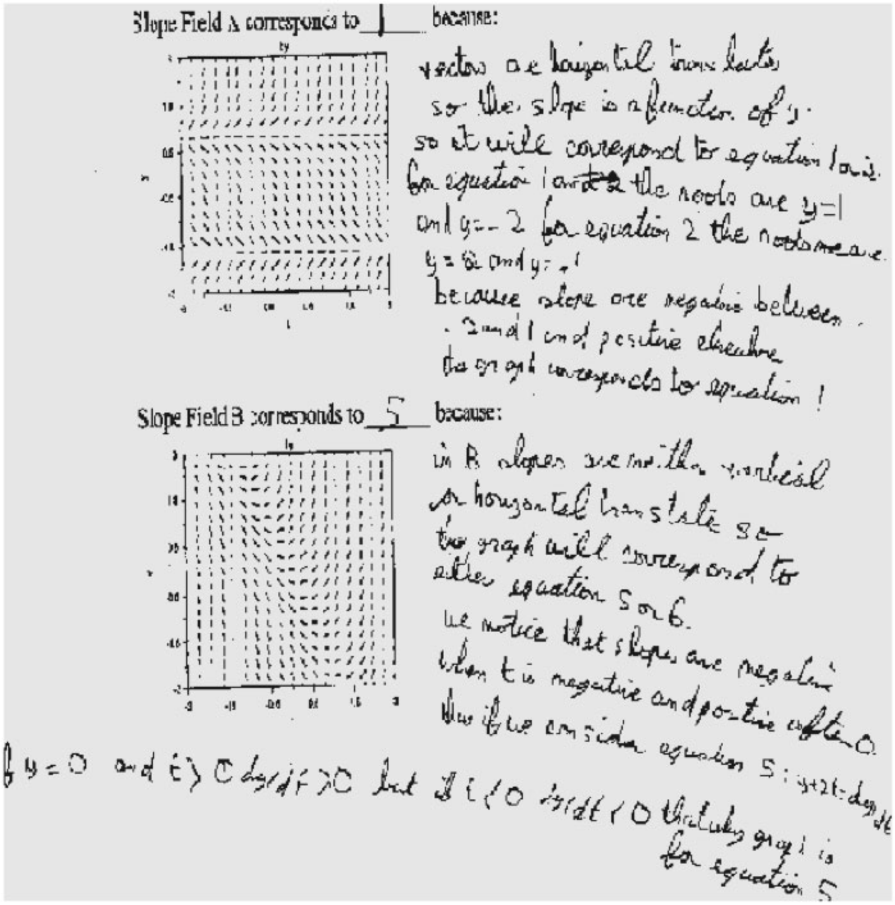


FIG. 7. Ali's work on final exam.

the slopes of the mini-tangents (positive *vs*. negative) and at the equilibrium solutions (in the case of the top graph) in order to do the association with the differential equations.

In the case of Cathy, her writing skills on Exam 2 and on the Final exam were consistent with the skills she showed on the first exam. This was true for almost all who were placed within the same category.

4. Discussion and conclusions

Introducing a writing component in this introductory differential equations course came naturally with the reformed teaching approach adopted in this class. The inclusion of a qualitative element in the course side-by-side with the traditional quantitative element necessitates that the learner is able to combine the two components, understand how they are related and how they complement each other. Writing is thus a natural way to achieve this goal. In addition, since the class is intended for future engineers, graduates are not expected only to generate solutions to practical problems, but rather in many cases, they are required to explain in layman's prose what these solutions mean for the people who will then have to turn the mathematical solutions into active practice.

As observed earlier, each particular representation has its strengths and weaknesses. In the case of an introductory differential equations course, the main strength of the quantitative approach is the ability to find the exact form of the solution of such an equation. For a first-order differential equation, the student is taught to classify the ode into exact, linear . . . and then coached to solve it is a purely mechanical way. In the case of a second-order differential equation, the quantitative discussion is limited to linear equations. It is exactly these limitations that constitute the weaknesses of this approach. The field of differential equations is an applied one and as mentioned earlier, the course as offered at the Lebanese American University is intended for engineering students. Thus, when used to solve real-life problems, the quantitative approach may not be useful; rather the qualitative one which permits the student to sketch a solution without having its closed form is more central. In addition, if not taught properly, both the quantitative and qualitative approaches may not contribute to achieving the learning outcomes of the class unless complemented by an analysis of the solutions such as increase, decrease, rate of increase/decrease and long-term behaviour. This analysis is sometimes not possible in the quantitative approach because of the complexity of the solution. In the qualitative approach however, sketching the solution through the help of technology allows such an analysis. Hence, it is only natural in this case to request that the student discusses in writing the behaviour of the solutions and extracts all its necessary characteristics.

The teaching of Mathematics in Lebanon is still very traditional, be it at the school or college level. Students are exposed only rarely to a reformed approach. This applies to the teaching environment at LAU as well. Only in a few cases, and upon personal initiatives of the instructor, are non-conventional teaching approaches implemented. It is only natural therefore to expect a large number of students with weak writing skills in mathematics. In the case of this study, 67% (28 students out of 43) were initially found to have unsatisfactory such skills. And, even those who were categorized otherwise, not all their work was satisfactory. For instance, Cathy did not justify properly the shape of the solution in Fig. 5a. Her work however in Fig. 5b was slightly more elaborated and very detailed in Fig. 5c. As seen in Fig. 2, the overall academic performance of the students, subjects of this study, was of good quality. Cathy who earned a solid A on the course had not been exposed to a writing component in a mathematics course before. In her mind, drawing the correct shape of the solution in Fig. 5a was a satisfactory answer irrespective of the question that specifically asked for justification. In Fig. 5c, however, her answer was very comprehensive perhaps because she was not required to ‘solve’ any mathematical problem but rather explain a particular mathematical topic.

Following the first exam, the writing exercises on the exam were given again as an optional take home assignment, but this time my expectations as an instructor were made clearer. For instance, in the first writing question, students were asked in particular to discuss the existence and uniqueness of solutions, their increase, decrease, concavity, long-term and asymptotic behaviour. Such details showed to be important since they served as a scoring rubric for the learner. It is a well-known fact that scoring rubrics when shared with students provide a guide for the teachers’ expectations. In the case of this research project, even though only few people returned this assignment (and therefore, its results will not be presented here), discussing it in class however must have contributed to some enhancement of the writing skills on exam 2 and on the final exam. This explains perhaps Ali’s improvement on the second exam and on the final. As for the general class performance, 10 additional students were classified as possessing satisfactory writing skills so that almost 50% of the entire class ended up in this category. In addition, the improvement observed on the second exam such as Ali’s is an evidence of the importance of conjoining the various mathematical modes of learning. In Ali’s case, for instance, the student was successful in analyzing the solutions provided and specifically to associate the motion of the door to mathematical equations.

In conclusion, writing skills can be improved if clearly emphasized by the instructor as an important means for communicating mathematical ideas and for intertwining these ideas. Better results may be achieved if the teachers' expectations are clearly spelled out in the writing assignments. In addition, the inclusion of a writing component whenever appropriate may lead sometimes to a better understanding of concepts. Thus, even though students initially view the idea of writing in mathematics as very alien to them and may sometimes show resistance to it, yet with time many of them will reap its benefits through learning more, learning better and acquiring new skills.

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