THE FREDHOLM ALTERNATIVE FOR SECOND-ORDER LINEAR ELLIPTIC SYSTEMS WITH VMO COEFFICIENTS

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ABSTRACT. The family of second-order linear elliptic operators on the complex plane forms an open set in $\mathbb{C}^6$ with exactly six components. Let $E(\Delta)$ denote the component consisting of operators that are deformable to the Laplacian. The objective of this paper is the establishment of the Fredholm alternative for the equation

\[ SW(z) = g(z), \quad W \in W_0^{2,p}(\Omega), \quad g \in L^p(\Omega). \]

Using the Hilbert Transforms, the problem reduces to studying $L\omega = g(z)$, where $L$ is an integral operator and $\omega \in L^p(\Omega)$. We show that, if the coefficients of $S$ are functions of vanishing mean oscillation, then $L$ is a Fredholm operator with index zero for every $p \in (1, \infty)$.

0. Introduction and Statement of Results.

The general second-order linear differential operator on the complex plane takes the form:

\begin{equation}
S = \alpha \frac{\partial^2}{\partial z \partial z} + \beta \frac{\partial^2}{\partial z \partial \bar{z}} + \gamma \frac{\partial^2}{\partial z \partial z} + \delta \frac{\partial^2}{\partial z \partial \bar{z}} + \mu \frac{\partial^2}{\partial z \partial z} + \nu \frac{\partial^2}{\partial z \partial \bar{z}} + \text{lower order terms}
\end{equation}

where $\alpha, \beta, \gamma, \delta, \mu$ and $\nu$ are complex-valued functions. We shall examine the Dirichlet boundary value problem for $S$.

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Given $g \in L^p(\Omega)$, where $\Omega$ is a domain in the complex plane with smooth boundary, find functions $W \in W_0^{2,p}(\Omega)$ such that:

$$ SW(z) = g(z); \quad z \in \Omega $$

The homogeneous boundary condition for $W$ is understood in the sense of distribution and is included in the assumption that $W \in W_0^{2,p}(\Omega)$.

It is a well-known fact that ellipticity of the operator $S$ does not necessarily imply that the Dirichlet problem for Systems is well-posed. As illustrated by A.V. Bitsadze [2] in 1948, the Dirichlet problem:

$$ \frac{\partial^2 W}{\partial z \partial \overline{z}} = 0; \quad |z| < 1 $$

$$ W(z) = 0; \quad |z| = 1, $$

has infinitely many solutions. For instance, $W(z) = (1 - |z|^2)f(z)$, where $f(z)$ is a holomorphic function in the unit disk, are such solutions.

A natural question then arises: For what elliptic operators is the Dirichlet boundary value problem well-posed? This question has led to a classification of elliptic systems.

The operator $S$ with constant coefficients can be identified with a point $(\alpha, \beta, \gamma, \delta, \mu, \nu) \in \mathbb{C}^6$. In 1959, B. Bojarski [1] showed that elliptic operators form an open set in $\mathbb{C}^6$ consisting of exactly six components. More precisely, he showed that any second-order elliptic system on $\mathbb{C}$ is deformable (continuously) to exactly one of the following basic operators:

$$ \frac{\partial^2}{\partial z \partial z'} \frac{\partial^2}{\partial z \partial \overline{z}'} \frac{\partial^2}{\partial z \partial \overline{z}} \frac{\partial^2}{\partial z' \partial \overline{z}'} \frac{\partial^2}{\partial z' \partial \overline{z}} \frac{\partial^2}{\partial \overline{z} \partial \overline{z}'}.$$

In view of Bitsadze's discovery, it is clear that the only components where we expect to have a well-posed Dirichlet boundary value problem are those represented by the Laplacian $\Delta = 4 \frac{\partial^2}{\partial z \partial \overline{z}}$ or its complex conjugate.

Let $\mathcal{E}(\Delta)$ denote the component containing the Laplacian. Bojarski [1] showed that the principal part of an elliptic system of class $\mathcal{E}(\Delta)$ takes the form:

$$ S = \frac{\partial^2}{\partial z \partial \overline{z}} + \alpha(z) \frac{\partial^2}{\partial z \partial z'} + \beta(z) \frac{\partial^2}{\partial z \partial \overline{z}'} + \gamma(z) \frac{\partial^2}{\partial z' \partial \overline{z}} + \delta(z) \frac{\partial^2}{\partial \overline{z} \partial \overline{z}'}. $$

(0.3)
In the special case when $\gamma = \delta = 0$ ($S$ would become linear under complex numbers), the operator

\[ S_0 = \frac{\partial^2}{\partial z \partial \overline{z}} + \alpha(z) \frac{\partial^2}{\partial z \partial \overline{z}} + \beta(z) \frac{\partial^2}{\partial z \partial \overline{z}} \]

belongs to $\mathcal{E}(\Delta)$ if and only if:

\[ \alpha(z) = \frac{a(z)}{1 + a(z)b(z)} \text{ and } \beta(z) = \frac{b(z)}{1 + a(z)b(z)} \]

with complex functions $a(z)$ and $b(z)$ such that $|a(z)| < 1$ and $|b(z)| < 1$. $S_0$ is said to be uniformly elliptic if:

\[ \lambda |a(z)| \leq q_0 < 1 \text{ and } |b(z)| \leq q_0 < 1, \]

for almost every $z \in \mathbb{C}$.

For the general operator $S = S_0 + \gamma(z) \frac{\partial^2}{\partial z \partial \overline{z}} + \delta(z) \frac{\partial^2}{\partial z \partial \overline{z}}$ of class $\mathcal{E}(\Delta)$, (uniform) ellipticity is expressed in terms of the coefficients as follows:

\[ \left| \frac{1 + \alpha(z)\xi + \beta(z)\overline{\xi}}{\gamma(z)\overline{\xi} + \delta(z)\xi} \right| \geq \lambda > 1, \text{ for } |\xi| = 1, \]

where $\lambda$ is a constant independent of $z$ and $\xi$. In addition, it is required that $S_0$ is uniformly elliptic, that is (0.5) and (0.6) are satisfied.

In contrast with the single equations, many important and interesting questions regarding linear elliptic systems of second-order on the complex plane remain unanswered.

The objective of this paper is to establish the Fredholm alternative for uniformly elliptic operators in $\xi(\Delta)$. This problem was the focus for many researchers in the middle of the century. Unfortunately, no major breakthrough was accomplished.

In the case of $\Omega = \mathbb{C}$ the problem reduces to showing that a singular integral operator:

\[ L = I + \alpha T + \beta R + \gamma \overline{T} + \delta \overline{R} : L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C}) \]
is Fredholm. Here, \( \alpha, \beta, \gamma \) and \( \delta \) are the coefficients of the differential operator \( S \) while \( T \) and \( R \) are the complex Hilbert transforms on \( \mathbb{C} \) (see the next section for the definitions). We do not discuss domains other than \( \mathbb{C} \). However, we point out that for \( \Omega \) a "regular" domain, such as a disk, it is possible to extend the boundary value problem (0.2) on \( \Omega \) to the whole of \( \mathbb{C} \). Therefore, our results obtained for elliptic operators on \( \mathbb{C} \) are easily carried over to the operators on \( \Omega \).

Assuming first that \( \alpha, \beta, \gamma \) and \( \delta \) are constant functions on \( \mathbb{C} \), we show that \( L \) is in fact invertible. In case the coefficients are only measurable, we construct an operator with a non-trivial kernel, thus not invertible (see Proposition 4.1).

If \( \alpha, \beta, \gamma \) and \( \delta \) are uniformly continuous on \( \mathbb{C} \), it can be shown that \( L \) is Fredholm with index zero.

Results of this kind for elliptic systems with measurable coefficients do not exist. Our main theorem solves this problem for the case of \( \mathrm{VMO} \)-coefficients.

**Theorem 1.** Let \( L : L^p(\mathbb{C}) \to L^p(\mathbb{C}), 1 < p < \infty, \) be the integral operator (0.8) corresponding to the uniformly elliptic differentiable operator \( S \) of class \( \mathcal{E}(\Delta) \). Suppose that the coefficients of \( S \) are \( \mathrm{VMO} \)-functions on the complex plane. Then \( L \) is a Fredholm operator with index zero.

1. **Basic Notation.**

Throughout this paper we shall be concerned with the following spaces of functions and distributions defined on an open subset \( \Omega \) of the complex plane:

- \( L^p(\Omega), 1 \leq p \leq \infty; \) those are the usual \( L^p \)-spaces with respect to the Lebesgue measure \( d\sigma(z) = \frac{1}{2i}dz \wedge d\overline{z} = dx dy \) for \( z = x + iy \).
- \( \mathcal{C}^\infty(\Omega) \): the complex space of infinitely differentiable functions on \( \Omega \).
- \( \mathcal{C}_0^\infty(\Omega) \): that subspace of \( \mathcal{C}^\infty(\Omega) \) consisting of functions whose support is compact in \( \Omega \).
- \( L^p_k(\Omega), 1 \leq p \leq \infty, k = 0, 1, 2, \ldots; \) this is the space of locally integrable functions whose \( k \)-th derivatives are in \( L^p(\Omega) \). This space is equipped with the semi-norm:

\[
\|W\|_{L^p_k(\Omega)} = \left( \int_\Omega \int (\sum_{|\alpha|=k} |D^\alpha W(z)|^2)^{p/2} d\sigma(z) \right)^{1/p}.
\]
$W^{m,p}(\Omega) = \bigcap_{k=0}^{m} L^p_k$: this is the space of distributions whose partial derivatives up to order $m$ belong to $L^p(\Omega)$, $1 \leq p < \infty$, and $m = 0, 1, 2, \ldots$. $W^{m,p}(\Omega)$ is commonly called the Sobolev space and is equipped with the norm:

$$\|W\|_{W^{m,p}(\Omega)} = \left(\sum_{k=0}^{m} \|W\|^p_{L^p_k(\Omega)}\right)^{1/p}.$$

The completion of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$. A function in $W^{m,p}(\Omega)$ is said to vanish on the boundary in the distributional sense if it belongs to $W_0^{m,p}(\Omega)$. For more facts concerning Sobolev spaces, see [6].

Let $Q = Q(x,r)$ denote the cube in $\mathbb{R}^n$ centered at $x$ with sides parallel to the coordinate axes and with length $r$. Let $f$ be a measurable function and let $f_Q = \frac{1}{|Q|} \int_Q f(t) \, dt$ denote the average of $f$ over the cube $Q$.

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the mean oscillation of $f$ in $Q$ is:

$$f^\sharp_Q = \frac{1}{|Q|} \int_Q |f(t) - f_Q| \, dt.$$  

Then, $f$ is said to be of bounded mean oscillation (BMO) if its sharp maximal function:

$$f^\sharp(x) = \sup\{f^\sharp_{Q(x,r)}; r > 0\}$$

belongs to $L^\infty(\Omega)$. In this case we set $\|f\|_{BMO} = \|f^\sharp\|_{\infty}$.

The closure of uniformly continuous functions with respect to the $BMO$ semi-norm is called the space of functions of vanishing mean oscillation ($VMO$).

Recall now the Cauchy-Riemann derivatives:

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial z} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

where $z = x + iy$. We shall often use in the paper the singular integral operators of Calderón-Zygmund type:

$$T \omega(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \int \frac{\omega(\xi)}{(z - \xi)^2} \, d\sigma(\xi)$$
and its inverse (also adjoint):

$$R\omega(z) = -\frac{1}{\pi} \int_C \int \frac{\omega(\xi)}{(\overline{z} - \xi)^2} d\sigma(\xi).$$

The importance of these operators lies in the following formulas connecting the Cauchy-Riemann derivatives:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \overline{z}} \circ T = T \circ \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z} \circ R = R \circ \frac{\partial}{\partial z}.$$

2. Reduction to Integral Equation.

We are concerned with the equation

$$(2.1) \quad SW = g; g \in L^2(\mathbb{C})$$

for $W \in L^2(\mathbb{C})$ - the space of complex functions whose second-order derivatives are in $L^2(\mathbb{C})$. This does not ensure that $W$ and its first derivatives are square integrable on $\mathbb{C}$.

Notice that every $W$ in $L^2(\mathbb{C})$ is a continuous function and may be represented by

$$W(z) = \int_C \int G(z, \xi) \omega(\xi) d\sigma(\xi)$$

where $G(z, \xi) = \frac{2}{\pi} \log|z - \xi|$ is Green's function for the complex plane and $\omega \in L^2(\mathbb{C})$ is an unknown function called the density function.

It is the property of Green's function that

$$(2.2) \quad \frac{\partial^2 W}{\partial z \partial \overline{z}} = \omega(z)$$

$$\begin{align*}
\frac{\partial^2 W}{\partial z \partial \overline{z}} &= -\frac{1}{\pi} \text{p.v.} \int_C \int \frac{\omega(\xi)}{(z - \xi)^2} d\sigma(\xi) = T\omega(z) \\
\frac{\partial^2 W}{\partial z \partial \overline{z}} &= -\frac{1}{\pi} \text{p.v.} \int_C \int \frac{\omega(\xi)}{(\overline{z} - \xi)^2} d\sigma(\xi) = R\omega(z).
\end{align*}$$

Here, the initials p.v. signify the principal value of the integrals and they will be dropped for the remainder of this paper.
The operators $T$ and $R$ are special cases of the Riesz transforms of order $k$. The latter are given explicitly by the singular integrals of convolution type:

$$\left( R_k \omega \right)(z) = \frac{|k|}{2\pi} \int_C \int \frac{(z - \xi)^k}{|z - \xi|^{k+2}} \omega(\xi)d\sigma(\xi), \quad k \neq 0 \text{ and } R_0 = 1.$$  

These formulas follow from Hecke's identities, see [8].

Letting $k = 2$ and $k = -2$ we obtain respectively,

$$R = -R_2 \text{ and } T = -R_2.$$  

Consequently, for any integer $m$,

$$R^m = R \circ R \circ \cdots \circ R = (-1)^m R_{2m} \quad \text{and} \quad T^m = T \circ T \circ \cdots \circ T = (-1)^m R_{-2m}.$$  

With the aid of identities (2.2) the differential equation (2.1) becomes an integral equation $L \omega(z) = g(z)$, where $L : L^2(\mathbb{C}) \rightarrow L^2(\mathbb{C})$ is given by:

$$L = I + \alpha T + \beta \overline{T} + \gamma \overline{R} + \delta \overline{R}.$$  

Therefore, the problem of finding $W(z) = L^2(\mathbb{C})$ for which $SW = g$, $g \in L^2(\mathbb{C})$, reduces equivalently to finding $\omega \in L^2(\mathbb{C})$ such that $L \omega = g$.

Calderón-Zygmund theory of singular integrals establishes the boundedness of the operator $L : L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})$ for all $1 < p < \infty$. In what follows, new estimates for the $L^p$-norms of $R_k$ that depend linearly on $k$ are essential. For this, we examine the iterated complex Hilbert transforms $R^m$ and $T^m$. These are easily decomposed into operators with odd kernels, namely

$$R^m = (-1)^m R_{2m-1} R_1 \quad \text{and} \quad T^m = (-1)^m R_{-2m-1} R_1.$$  

Applying the familiar method of rotation to $R_{2m-1}, R_{-2m-1}$ and $R_1$ (see [3]), we obtain:
Theorem 2.1. The $L^p$-norms of the iterated complex Hilbert transforms satisfy $\|T^m\|_p = \|R^m\|_p \leq C_p |m|$ for all integers $m \neq 0$, where $C_p$ is a constant independent of $m$. □

For more details about this result see [5]. Notice here that for $p = 2$, the operators $T$ and $R$ are isometries, so $\|T^m\|_2 = \|R^m\|_2 = 1$.

3. The Case of Constant Coefficients.

In this section we assume that the operator $L$ corresponding to $S \in \mathcal{E}(\Delta)$ has constant coefficients $\alpha, \beta, \gamma$ and $\delta$. We show that in this case $L : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is invertible for every $p \in (1, \infty)$.

The main idea is to associate with $L$ an operator $\hat{L} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ given by:

\[
\hat{L} = I + \beta T + \alpha R - \gamma \overline{T} - \delta \overline{R}.
\]

Indeed, it was this discovery first made that had set the foundations for the case of variable coefficients. Similar ideas were used in [4].

It is readily seen that $\hat{L}$ corresponds to a differential operator $\hat{S}$ of class $\mathcal{E}(\Delta)$ as well. Lengthy but elementary calculations show that:

\[
\hat{L}L = (\overline{\beta} \alpha - \delta \overline{\gamma})T^2 + (\alpha + \overline{\beta})T + (1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)I
\]

\[(3.2) \quad (\overline{\alpha} + \beta)R + (\beta \overline{\alpha} - \overline{\delta} \gamma)R^2.
\]

Equivalently, $(\hat{L}L)T^2 = (\overline{\beta} \alpha - \delta \overline{\gamma})T^4 + (\alpha + \overline{\beta})T^3

\[(3.3) \quad (1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)T^2 + (\overline{\alpha} + \beta)T + (\beta \overline{\alpha} - \overline{\delta} \gamma)I.
\]

This last identity suggests that we should examine the fourth order polynomial:

\[
Q(\lambda) = (\overline{\beta} \alpha - \delta \overline{\gamma})\lambda^4 + (\alpha + \overline{\beta})\lambda^3
\]

\[+(1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)\lambda^2 + (\overline{\alpha} + \beta)\lambda + (\beta \overline{\alpha} - \overline{\delta} \gamma); \lambda \in \mathbb{C}.
\]
Ellipticity condition (0.7) shows that $Q(\lambda)$ has no roots on the unit circle. Moreover, by a symmetry between the coefficients of this polynomial, we see that if $q$ is a non-zero root then so is $\frac{1}{q}$. Because of the interconnection between $Q(\lambda)$ and $LL$, a factorization of the former will yield a similar factorization of the latter.

Let $q_1, q_2, q_1^{-1}$ and $q_2^{-1}$ represent the roots of $Q(\lambda)$ where $0 \leq |q_1|, |q_2| < 1$. In this notation, it is understood that $\frac{1}{q}$ is not present if $q = 0$.

The factorization of $LL$ takes then the following form:

**Proposition 3.1.** The following three cases are possible

(a) $LL = (a - \beta\gamma)q_1^{-1}q_2^{-1}(I - q_1T)(I - q_2T)(I - \bar{q}_1\bar{R})(I - \bar{q}_2\bar{R})$;

(b) $LL = (a + \beta)q_1^{-1}(I - \bar{q}_1\bar{R})(q_1T - I); a - \beta\gamma = 0; a + \beta \neq 0$.

(c) $LL = (1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)I; \beta\alpha - \delta\gamma = \alpha + \beta = 0$.

Observe that, by the ellipticity condition (0.7), $LL$ as given in (c) is non-zero (see (5.12)).

The tools for proving the main result of this section are now at hand. The invertibility of $L$ reduces to the invertibility of each of the corresponding factors of $LL$.

**Theorem 3.1.** The operator $L : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ with constant coefficients is invertible for all $p$, $1 < p < \infty$.

**Proof.** According to Proposition 3.1 it is enough to examine the invertibility of $I - qT$ (or $I - qR$), for $|q| < 1$. The inverse of $I - qT$ will be found from the formula:

$$(I - qT)^{-1} = \sum_{k=0}^{\infty} (qT)^k = I + \sum_{k=1}^{\infty} q^k T^k.$$

The latter equality is justified because $q$ is a constant.

We show now that the series converges absolutely in the operator norm topology. By Theorem 2.1, $\|q^k T^k\|_p \leq C_p k \|q\|^k$, for $k \neq 0$. Hence, by the comparison test,

$$\sum_{k=1}^{\infty} \|q^k T^k\|_p \leq \frac{2C_p}{(1 - |q|^2)}.$$
4. Some Examples.

The purpose of this section is to show that the operator $L$ in (2.5) with variable coefficients need not be invertible. For this we shall examine two examples.

Example 4.1. Consider the following differential equation on the unit disk $D$:

\[(4.1) \quad (1 + a(z)b(z))w_{zz} + a(z)w_{zz} + b(z)w_{zzz} = 0; \quad z \in D.\]

where $a(z) = -\frac{5z}{6z^2}$ and $b(z) = \frac{(3-2 \ln |z|)^2}{(9-22 \ln |z|)^2}z$.

Clearly, $\|a\|_{\infty} = \frac{5}{6} < 1$ and $\|b\|_{\infty} = \frac{1}{3} < 1$. Therefore, equation (4.1) is elliptic and is deformable to the Laplacian. However the corresponding homogeneous Dirichlet problem admits a non-trivial solution $w(z) \in L^2_\Omega(D)$, namely $w(z) = \frac{2z \ln |z|}{1-2 \ln |z|}$. Indeed, by a straightforward calculation, we find that

\begin{align*}
(4.2) \quad w_{zz} &= \frac{3 - \ln |z|^2}{z(1 - \ln |z|^2)^3}; \\
&\quad w_{zzz} = \frac{z(1 + \ln |z|^2)}{z^2(1 - \ln |z|^2)^3}; \\
&\quad w_{zzz} = \frac{3 - \ln |z|^2}{z(1 - \ln |z|^2)^3}.
\end{align*}

Equation (4.1) is now readily verified.

It follows from (4.2) that $w(z) \in L^p_\Omega(\mathbb{C})$ for all $p$, $1 \leq p \leq 2$. This is a consequence of the fact that $|z|^{-1} \ln^{-2} |z|$ is $L^p$-integrable near the origin if and only if $1 \leq p \leq 2$.

The second example deals with a homogeneous Dirichlet problem outside the unit disk.

Example 4.2. Consider the equation:

\[(4.3) \quad (1 + \tilde{a}(z)\tilde{b}(z))\tilde{w}_{zz} + \tilde{a}(z)\tilde{w}_{zz} + \tilde{b}(z)\tilde{w}_{zz} = 0; \quad z \in \mathbb{C} - D,\]

where $\tilde{a}(z) = \frac{z}{2z^2}$ and $\tilde{b}(z) = \frac{\frac{z}{z^2} \ln |z|^2}{z^2(4 + 3 \ln |z|^2)}$. 

Therefore, $I - qT : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is invertible for every $1 < p < \infty$. $\square$
Clearly, $\|a\|_\infty = \frac{1}{2} < 1$ and $\|b\|_\infty = \frac{1}{3} < 1$. The function

\begin{equation}
\tilde{w}(z) = z \ln(1 + 2 \ln |z|)
\end{equation}

vanishes on the unit circle and is $C^\infty$-smooth.

To show that $\tilde{w}(z)$ solves (4.3), we compute its second derivatives

\begin{equation}
\begin{aligned}
\tilde{w}_{zz} &= \frac{2 \ln |z|}{z(1 + 2 \ln |z|)^2}; \\
\tilde{w}_{z\bar{z}} &= \frac{2 \ln |z|}{z(1 + 2 \ln |z|)^2}; \\
\tilde{w}_{\bar{z}z} &= \frac{z(2 + 2 \ln |z|)}{z^2(1 + 2 \ln |z|)^2}.
\end{aligned}
\end{equation}

Elementary computations verify then (4.3). It is of particular importance to notice that the second derivatives of $\tilde{w}(z)$ are of type $|z|^{-1} \ln^{-1} |z|$ at infinity and hence they belong to $L^p(\mathbb{C} - D)$ for all $p$ with $2 \leq p < \infty$, but not for $p < 2$.

Now, we are in a position to construct an integral operator with a non-trivial kernel. Let

\begin{equation}
L_0 = 1 + \alpha T + \beta R : L^2(\mathbb{C}) \to L^2(\mathbb{C}).
\end{equation}

With the notation of examples 4.1 and 4.2, we define:

\begin{equation}
\begin{aligned}
\alpha(z) &= \begin{cases}
\frac{a}{1 + ab}; & |z| \leq 1 \\
\frac{a}{1 + \bar{a}b}; & |z| > 1
\end{cases} \\
\beta(z) &= \begin{cases}
\frac{b}{1 + ab}; & |z| \leq 1 \\
\frac{b}{1 + \bar{a}b}; & |z| > 1
\end{cases} \\
\omega(z) &= \begin{cases}
w_{zz}; & |z| \leq 1 \\
\tilde{w}_{z\bar{z}}; & |z| > 1.
\end{cases}
\end{aligned}
\end{equation}

**Proposition 4.1.** The operator $L_0 : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ corresponds to a uniformly elliptic differential operator $S_0 \in \mathcal{E}(\Delta)$, but fails to be invertible. The function $\omega(z)$ defined in (4.7) belongs to the kernel of $L_0$. Moreover, $\omega \in L^p(\mathbb{C})$ only for $p = 2$.

**Proof.** Define the following function on $\mathbb{C}$:

\begin{equation}
W(z) = \begin{cases}
w(z); & |z| \leq 1 \\
\tilde{w}(z); & |z| \geq 1.
\end{cases}
\end{equation}
The first order derivatives of \( W \) are easily verified to be continuous on the whole of \( \mathbb{C} \). The second derivatives \( W_{zz}, W_{z\bar{z}} \) and \( W_{\bar{z}z} \) coincide with the corresponding second derivatives of \( w \) and \( \bar{w} \). Thus, they belong to \( L^2(\mathbb{C}) \). This means precisely that \( W \in L^2_p(\mathbb{C}) \) for \( p = 2 \), and therefore by (2.2) \( T\omega = W_{zz} \) and \( R\omega = W_{\bar{z}z} \). That \( L_0\omega = 0 \) is now obvious.

Since \( \omega \in L^p(D) \) only for \( 1 \leq p \leq 2 \) and \( \omega \in L^p(\mathbb{C} - D) \) only for \( 2 \leq p < \infty \), it follows that \( \omega \in L^p(\mathbb{C}) \) only for \( p = 2 \). \( \square \)

**Remark.** For differential operators with measurable coefficients, the regularity theory is concerned with the degree of integrability of the derivatives. For single homogeneous equations it is known that the \( L^2 \)-solutions actually belong to \( L^p_{2,\text{loc}} \) for some \( p > 2 \), the so-called higher integrability result. Proposition 4.1 shows that this regularity theory fails for uniformly elliptic systems even when they are linear over complex numbers and deformable to the Laplacian. \( \square \)

**5. The Fredholm Alternative.**

We focus our attention for now on the operator:

\[
L_0 = I - \alpha T - \beta R : L^p(\mathbb{C}) \to L^p(\mathbb{C})
\]

corresponding to the differential operator \( S_0(W) = W_{zz} - \alpha W_{z\bar{z}} - \beta W_{\bar{z}z} \in \mathcal{E}(\Delta) \). We assume that the coefficients \( \alpha \) and \( \beta \) belong to \( VMO(\mathbb{C}) \). We recall the uniform ellipticity of \( S_0 \) (see (0.6)):

\[
\alpha(z) = \frac{a(z)}{1 + a(z)b(z)} \quad \text{and} \quad \beta(z) = \frac{b(z)}{1 + a(z)b(z)}
\]

where \( a(z) \) and \( b(z) \) are measurable functions satisfying

\[
|a(z)| \leq q_0 < 1 \quad \text{and} \quad |b(z)| \leq q_0 < 1,
\]

for almost every \( z \in \mathbb{C} \).

We shall show that \( a(z) \) and \( b(z) \) are in fact functions of vanishing mean oscillation on \( \mathbb{C} \).

The identities in (5.2) imply that

\[
1 - 4\alpha\beta = \left( \frac{1 - ab}{1 + ab} \right)^2.
\]
For \( z \in \mathbb{C} \), let \( \zeta = a(z)b(z) \). Notice then that \( |\zeta|^2 \leq q_0^2 < 1 \). The function \( \zeta = \left( \frac{1 - \zeta}{1 + \zeta} \right)^2 \) maps conformally the disk \( |\zeta| \leq q_0^2 \) into a domain, say \( G \subset \mathbb{C} - \{0\} \). In other words, the range of the function \( 1 - 4\alpha(z)\beta(z) \) is contained in the simply connected region \( G \) not containing the origin. We choose a continuous branch of the square root function in \( G \) such that \( 1^{1/2} = 1 \). It is now clear that \( (1 - 4\alpha\beta)^{1/2} \) is a \( VMO \)-function with range a compact subset of \( \mathbb{C} - \{-1\} \).

On the other hand, (5.2) and (5.4) yield:

\[
a = \frac{2\alpha}{1 + (1 - 4\alpha\beta)^{1/2}} \quad \text{and} \quad b = \frac{2\beta}{1 + (1 - 4\alpha\beta)^{1/2}}.
\]

Since the space of \( VMO \)-functions is closed under these algebraic operations, we conclude that \( a \) and \( b \in VMO(\mathbb{C}) \), as desired.

All the prerequisites for the proof of Theorem 1 are now at hand.

Let us assume first that the coefficients of \( L \) are constant. We shall define sequences \( \{S_k\}_{k=1}^{\infty} \) and \( \{P_k\}_{k=1}^{\infty} \) of integral operators \( L^p(\mathbb{C}) \). For \( k = 1 \), we recall identity (3.2) and we put:

\[
S_1 = \hat{L}L = (\beta\alpha - \delta\gamma)T^2 + (\alpha + \bar{\beta})T + (1 + |\alpha|^2 + \beta^2 - |\gamma|^2 - |\delta|^2)I
+ (\bar{\alpha} + \beta)R + (\beta\bar{\alpha} - \delta\gamma)R^2.
\]

Proposition (3.1) shows that

\[
S_1 = \rho(I - q_1T)(I - q_2T)(I - \bar{q}_1R)(I - \bar{q}_2R), \quad \text{where}
\]

\[
(5.5) \quad \rho = \frac{1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2}{1 + |q_1q_2|^2 + |q_1 + q_2|^2}
\]

Changing the signs of the roots \( q_1 \) and \( q_2 \), a new operator is obtained:

\[
P_1 = \rho(I + q_1T)(I + q_2T)(I + \bar{q}_1R)(I + \bar{q}_2R).
\]

Therefore,

\[
P_1 = (\beta\alpha - \delta\gamma)T^2 - (\alpha + \bar{\beta})T + (1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2)I
- (\bar{\alpha} + \beta)R + (\beta\bar{\alpha} - \delta\gamma)R^2.
\]
For simplicity, we introduce a notation for the coefficients of $S_1$:

$$
A_1 = e q_1 q_2 = \bar{\beta} \alpha - \delta \gamma;
$$

$$
B_1 = -\rho(q_1 + q_2 + q_1 q_2 \bar{q}_2 + q_2 q_1 \bar{q}_1) = \alpha + \bar{\beta};
$$

$$
C_1 = \rho(1 + |q_1|^2 + |q_2|^2 + q_1 \bar{q}_2 + \bar{q}_1 q_2 + |q_1 q_2|^2)
= 1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2.
$$

Accordingly, $S_1 = A_1 T^2 + B_1 T + C_1 I + \bar{B}_1 R + \bar{A}_1 R^2$ and $P_1 = A_1 T^2 - B_1 T + C_1 I - \bar{B}_1 R + \bar{A}_1 R^2$. Notice that another way to obtain $P_1$ is by simply changing the sign of $B_1$ in $S_1$. Next, we look at the product:

$$
S_2 = S_1 P_1 = \rho_2(I - q_1^2 T^2)(I - q_2^2 T^2)(I - q_1^{-2} R^2)(I - q_2^{-2} R^2)
= A_2 T^4 + B_2 T^2 + C_2 I + \bar{B}_2 R^2 + \bar{A}_2 R^4,
$$

where,

$$
A_2 = \rho^2 q_1^2 q_2^2 = A_1^2;
B_2 = -\rho^2(q_1^2 + q_2^2 + q_1^2 q_2^2 - \bar{q}_1 q_2^2 q_1^2 - q_2^2 q_1^2 q_1^2) = 2A_1 C_1 - B_1^2;
C_2 = \rho^2(1 + |q_1|^4 + |q_2|^4 + q_1^2 q_2^2 - q_1^{-2} q_2^2 + q_1^{-2} q_2^2 + |q_1 q_2|^4)
= 2|A_1|^2 - 2|B_1|^2 + C_1^2.
$$

As in the previous step, the corresponding operator $P_2$ will be defined by changing the signs of $q_1^2$ and $q_2^2$, that is,

$$
P_2 = \rho^2(I + q_1^2 T^2)(I + q_2^2 T^2)(I + q_1^{-2} R^2)(I + q_2^{-2} R^2)
= A_2 T^4 - B_2 T^2 + C_2 I - \bar{B}_2 R^2 + \bar{A}_2 R^4.
$$

which also differs from $S_2$ by the sign of $B_2$.

This construction is carried on and we obtain for each positive integer $k$, the following operators:

$$
S_{k+1} = \rho^{2k}(I - q_1^{2k} T^{2k})(I - q_2^{2k} T^{2k})(I - \bar{q}_1^{2k} R^{2k})(I - \bar{q}_2^{2k} R^{2k})
= A_{k+1} T^{2k+1} + B_{k+1} T^{2k} + C_{k+1} I + \bar{B}_{k+1} R^{2k} + \bar{A}_{k+1} R^{2k+1};
$$

and

$$
P_{k+1} = \rho^{2k}(I + q_1^{2k} T^{2k})(I + q_2^{2k} T^{2k})(I + \bar{q}_1^{2k} R^{2k})(I + \bar{q}_2^{2k} R^{2k})
= A_{k+1} T^{2k+1} - B_{k+1} T^{2k} + C_{k+1} I - \bar{B}_{k+1} R^{2k} + \bar{A}_{k+1} R^{2k+1}.
$$
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Here the coefficients $A_k, B_k$ and $C_k$, can be defined either by recurrence or explicitly in terms of the roots $q_1$ and $q_2$ as follows:

\begin{align}
A_{k+1} &= \rho^{2k}(q_1^{2k} + q_2^{2k}) = A_k^2; \\
B_{k+1} &= -\rho^{2k}(q_1^{2k} + q_2^{2k} + q_1^{2k}q_2^{2k} + q_2^{2k}q_1^{2k}) = 2A_kC_k - B_k^2; \quad \text{and} \\
C_{k+1} &= \rho^{2k}(1 + q_1|q_2|^{2k+1} + q_2|q_1|^{2k+1} + q_1^{2k}q_2^{2k} + q_2^{2k}q_1^{2k} + |q_1q_2|^{2k+1}) \\
&= 2|A_k|^2 - 2|B_k|^2 + C_k^2.
\end{align}

The point we are trying to make here is that as $k$ goes to infinity, the operators $\rho^{-2k}S_{k+1} : L^p(C) \to L^p(C)$ converge to the identity in the operator norm for any $p \in (1, \infty)$.

Now, we are in a position to examine the operator $L$ with variable coefficients. For this we notice that the calculation done above remains valid modulo terms involving the commutators $\mu T - T\mu$ and $\mu R - R\mu$ where $\mu$ is one of the functions $A_k, B_k$, or $C_k$. It is of importance to observe that the recurrence relations for $A_k, B_k$ and $C_k$ (see (5.6) and (5.9)) are free of the roots $q_1(z)$ and $q_2(z)$. Therefore $A_k, B_k, \text{and } C_k$ are VMO-functions.

We refer now to a result of Uchiyama [9] which states that the commutator $\mu T - T\mu$ of a Calderón Zygmund type operator $T: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and the multiplication by a function $\mu \in \text{VMO}(\mathbb{R}^n)$ is compact for all $1 < p < \infty$. In other words, the calculations we have done for constant coefficients are valid modulo compact operators.

The leading coefficient $\rho(z)$ is a measurable function defined by (5.5). We will show that $\rho(z)$ is separated from zero by a positive constant.

Let $\epsilon = \inf\{|1 + \alpha(z)\xi + \beta(z)\bar{\xi}|^2 - |\gamma(z)\xi + \delta(z)\bar{\xi}|^2; (\xi, z) \in S^1 \times \mathbb{C}\}$. By uniform ellipticity, $\epsilon$ is a positive constant. On the other hand, the following real-valued harmonic polynomial

$$(\beta\bar{\alpha} - \delta\bar{\gamma})\xi^2 + (\alpha + \bar{\beta})(\xi + (1 + |\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2) + (\beta\alpha - \delta\gamma)\xi^2$$

coincides with $|1 + \alpha\xi + \beta\bar{\xi}|^2 - |\gamma\xi + \delta\bar{\xi}|^2$ for $\xi \in S^1$. Therefore, applying the mean value theorem we find that $1 + |\alpha(z)|^2 + |\beta(z)|^2 - |\gamma(z)|^2 - |\delta(z)|^2 \geq \epsilon > 0$. Thus $\rho(z) \geq \frac{\epsilon}{6}$, as desired.
Again, by uniform ellipticity, there exists a positive number \( r < 1 \) such that \( |q_1(z)| \leq r \) and \( |q_2(z)| \leq r \) for almost every \( z \in \mathbb{C} \). The estimates below now follow immediately:

\[
\rho^{-2^k} |A_{k+1}(z)| \leq r^{2^k}; \quad \rho^{-2^k} |B_{k+1}(z)| \leq 4r^{2^k}; \quad \left| \rho^{-2^k} C_{k+1}(z) - 1 \right| \leq 5r^{2^k}.
\]

On another note, the definition of \( S_{k+1} \) yields:

\[
\rho^{-2^k} (S_{k+1} - J_{k+1}) - I = \rho^{-2^k} (A_{k+1}T^{2^k+1} + B_{k+1}T^{2^k} + (C_{k+1} - \rho^{2^k})I
+ \overline{B}_{k+1}R^{2^k} + \overline{A}_{k+1}R^{2^k+1}),
\]

where \( J_{k+1} : L^p(\mathbb{C}) \to L^p(\mathbb{C}) \) is a compact operator by the theorem of Uchiyama. Hence by (5.10),

\[
\|\rho^{-2^k} (S_{k+1} - J_{k+1}) - I\|_p \leq r^{2^k} \left( \|T^{2^k+1}\|_p + 4\|T^{2^k}\|_p + 5\|I\|_p + 4\|R^{2^k}\|_p + \|R^{2^k+1}\|_p \right) \leq 15C_p r^{2^k} 2^{k+1},
\]

where \( C_p \) is the constant determined by Theorem 2.1.

This estimate shows that \( \lim_{k \to \infty} \rho^{-2^k} (S_{k+1} - J_{k+1}) = I \) in the operator norm. In particular, for every \( p \in (1, \infty) \), we can always find an integer \( k \) and a compact operator \( J_{k+1} : L^p(\mathbb{C}) \to L^p(\mathbb{C}) \) such that \( S_{k+1} - J_{k+1} : L^p(\mathbb{C}) \to L^p(\mathbb{C}) \) is invertible. This implies that \( S_{k+1} \) is a Fredholm operator.

Returning to the recurrence relations of \( S_{k+1} \) we obtain the following factorization modulo compact operators: \( S_{k+1} = P_k P_{k-1} \cdots \hat{L} L = KL = LK \), where \( K \) is a bounded operator in \( L^p(\mathbb{C}) \) given by:

\[
K = P_k P_{k-1} \cdots P_1 \hat{L} : L^p(\mathbb{C}) \to L^p(\mathbb{C}).
\]

Since \( S_{k+1} \) is Fredholm, the general Fredholm theory (see [7]) implies that \( L \) is also a Fredholm operator.

What remains is to show that the Fredholm index of \( L \) is equal to zero. To this end, we consider a one-parameter family of operators:

\[
L_t = I + \alpha(z)T + \beta(z)R + t\gamma(z)\overline{T} + t\delta(z)\overline{R}, \quad 0 \leq t \leq 1.
\]
Thus $L = L_1, L_t$ corresponds to a differential operator $S_t \in \mathcal{E}(\Delta)$ (see (0.7)), and the coefficients of $L_t$ remain in VMO ($\mathbb{C}$) for $0 \leq t \leq 1$. Therefore $L_t$ are Fredholm and index $L = \text{index } L_0$. 

That the index of $L_0$ is zero follows from the simple homotopy:

$$\alpha_t = \frac{ta}{1 + t^2 ab}, \beta_t = \frac{tb}{1 + t^2 ab}; \ 0 \leq t \leq 1.$$  

connecting $L_0$ with the identity operator. Since $a(z)$ and $b(z)$ are VMO-functions, this homotopy keeps the coefficients $\alpha_t$ and $\beta_t$ in the VMO-class for all $0 \leq t \leq 1$. Hence index $L_0 = 0$. $\Box$

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