Journal of Mathematical Behavior 25 (2006) 57–72

Mathematical Behavior

www.elsevier.com/locate/jmathb

Students' conceptual understanding of a function and its derivative in an experimental calculus course

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Available online 17 November 2005

Abstract

Calculus has been witnessing fundamental changes in its curriculum, with an increased emphasis on visualization. This mode for representing mathematical concepts is gaining more strength due to the advances in computer technology and the development of dynamical mathematical software. This paper focuses on the understanding of the function and its derivative as viewed by students of a reformed Calculus 1 course offered in two experimental sections at the Lebanese American University in Beirut, Lebanon. Results have shown that the general approach adopted in the course proved to be unpopular for a great majority of the students, but rewarding for others. Interviews conducted with some students and a study of their performance on very specific exam questions reveal that for most students, the algebraic representation of a function still dominated their thinking; however, these students showed an almost complete understanding of the derivative, particularly the idea of the instantaneous rate of change and/or the slope of a curve at a given point. Furthermore, very few of these students referred to the mechanical methods for finding derivatives.

Keywords: Experimental Calculus; Functions; Derivatives

Calls for change in calculus instruction have long been initiated in the United States and Europe. Indeed, many steps have been undertaken to ensure that the suggested changes in the calculus curriculum and the way it is taught answer these calls. One fundamental change that calculus witnessed is an increased emphasis on visualization. Zimmermann (1991) considers mathematical visualization as the core of the reform movement; he writes: "Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136). This is, of course, in contrast to what has traditionally been the belief and practice of mathematics teachers and consequently of students. For many teachers and students alike, a visual proof is not a "real" mathematical proof (Eisenberg & Dreyfus, 1991), and for most calculus instructors (and students as well), doing calculus is equivalent to learning the skill of manipulating symbols and numbers (Hughes-Hallett, 1991).

The implementation of the reformed calculus movement has been paralleled by an extensive use of technology, be it graphing calculators, or computer software (particularly, dynamic ones). But the determining role of technology has been an issue of debate; however, it is widely accepted now that the main strength of technology is in its capability of providing greater and easier access to multiple representations of concepts (Fey, 1989; Goldenberg, 1987; Kaput, 1992; Porzio, 1999). Here, multiple representations refer to presentations of a mathematical idea from three different

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doi:10.1016/j.jmathb.2005.11.004

perspectives: numerical, graphical, and symbolic. Also, computer-generated graphs are enabling mathematicians to visualize the content of abstract theorems (Pool, 1992). This is not to say, however, that visualizing concepts has necessarily become a simple task. Some research has shown that graphing calculator students who are given the opportunity to view multiple representations of a concept may not have a better understanding of that concept (Porzio, 1999). Reasons behind this may stem from the fact that some students (even mathematically gifted ones) do not possess visual/pictorial skills, or are not visualizers as defined by Presmeg (1986, p. 298), that is, individuals who prefer to use visual methods when attempting mathematical problems which may be solved by visual and non-visual means.

This paper is about students' understanding of a function and its derivative in a reformed calculus course given at the Lebanese American University (LAU) in Beirut, Lebanon. The course is the first in a sequence of four offered at LAU, but the sections observed were experimental in the sense that concepts were introduced using multiple representations. Attempts in these sections were made to employ graphing calculators, but more often a dynamical calculus software program was used by the instructors for presentation purposes, and by students for homework assignments. This was a pioneering project at the institution, although in other mathematics classes, serious attempts have been made to induce changes in the curriculum in favor of the geometric component. Results of our research have shown that the approach proved to be unpopular and difficult for many students, but very satisfying for others. In fact, the latter students showed an almost complete understanding of the derivative, particularly the idea of the derivative as a rate of change.

1. Students' mathematical background

As mentioned earlier, the course is the first in a sequence of four calculus courses offered at the Lebanese American University. Students are placed in any one of these courses based on a number of parameters: their scores on the entrance exam, their school background, and their intended field of study. A student of Lebanon who has passed the scientific sections of the official Lebanese Baccalaureate enrolls at the university at the sophomore level, and is usually placed in Calculus 3 (but sometimes in Calculus 2). Students with a Humanities background planning to major in the sciences, or students with a high school diploma are usually placed in Calculus 1 (again, sometimes in Calculus 2). The curriculum of Calculus 1 covers the following main concepts: limits of functions, continuity of functions, the derivative of a function, graphs of functions, and other applications to the derivative. The curriculum of Calculus 2 covers topics such as the anti-derivative, the definite integral, the Fundamental Theorem of Calculus, and applications to the definite integral, while Calculus 3 includes topics such as techniques of integration, the calculus of transcendental functions, sequences and series.

Some subjects of our study were Lebanese nationals coming from the Humanities section of the official Lebanese Baccalaureate program. A large number of students were Lebanese and other Arab nationals with a high school diploma, not earned necessarily in Lebanon. The non-Lebanese students had to face another problem, namely that they had never studied mathematics in English prior to their enrolment at the university.

To assess the mathematical knowledge of the students, subjects of this study (89 students in all), a diagnostic exam was administered at the beginning of the semester (the same exam was also given to the students of the non-experimental section). Results showed that a large number of students were in dire need of a pre-calculus course (that the university does not offer), although many did not think so. For instance, in one question students were asked to solve the inequality $3x < 9x + 4$ and to sketch the solution set. Eighty-seven percent of all answers were wrong. When asked to write the equation of the line having slope -2 and passing through the point $(-1, 2)$, 71% of all who responded had an incorrect answer. When asked to define the sine, cosine, tangent and cotangent of an angle, 76% did not have a complete answer. Among other things, students were also asked to complete a table of trigonometric values; the percentage of incorrect answers reached then 87%. These alarming conclusions had a severe effect on the research itself, since the approach adopted in the experimental sections was more demanding than the more traditional one causing a large number of dropouts and failures among the students. Note, however, that this phenomenon has been witnessed also in the traditional sections but to a lesser extent.

2. Methodology

Prior to our experiment, the teaching of Calculus 1 at the Lebanese American University had not benefited from the reform movement in mathematics. The content of the course had not changed in years, and technology had not been introduced in the teaching and learning process. Consequently, instructors of the course aimed simply at teaching techniques for solving drill problems like finding the derivative of a function, or drawing the graph of a function based on the signs of its first and second derivatives. Like many mathematics educators elsewhere, we were dissatisfied with the students' learning outcome; for this reason, we decided to run an experiment whereby fundamental changes are introduced, be it in the curriculum, the teaching process, or the expectations from the students themselves.

The experiment was conducted during two consecutive semesters, the fall of 2001 and the spring of 2002. The two targeted classes consisted of 89 students, but only 56 remained in the course for the entire semester, out of which 13 failed. One of us was the teacher of the course during a whole semester, while the other was an observer. Roles were reversed in the second semester. During the fall semester, a traditional section of Calculus 1 was also offered but by a different instructor. Cooperation was limited, but there was a general agreement on the topics to be covered allowing the final exam to be a common one. There was no traditional section offered in the spring semester. The textbook adopted for the experimental section was *Calculus, from Graphical, Numerical, and Symbolic Points of View*, by Ostebee and Zorn (1999), while the 10th edition of *Thomas' Calculus* by Finney (2001) was used in the traditional section. The approaches adopted in the two books are extensively different. For instance, *Thomas's Calculus* defines the derivative of a function $f(x)$ at a point a analytically by $f'(a) = \lim_{h\to 0}((f(a+h) - f(a))/h)$, proceeds to do examples, and then discusses its geometric meaning as the slope of a line tangent to $f(x)$ at a. Thus, in the traditional setting, students are expected to memorize formulas for derivatives (e.g., the derivative of x^n is $\mathrm{d}/\mathrm{d}x[x^n] = nx^{n-1}$ and $\mathrm{d}/\mathrm{d}x[\sin x] = \cos x \ldots$). They learn later some derivative rules (e.g., the product and quotient rules), and as an application, students are given a set of theorems such as: If the derivative of a function in some interval is positive, then the function increases in that interval. Throughout this discussion, the only "geometry" used is a graphical description of the difference quotient $f(a+h) - f(a)/h$ as a slope of the line through the points $(a, f(a))$ and $(a+h, f(a+h))$ and that the limit of the difference quotient is the slope of the tangent line at the point $(a, f(a))$ (see Fig. 1a and b).

Typical questions for students in a traditional section vary from finding the derivatives of a set of functions to writing the equation of the line tangent to the graph of a function at a given point.

In contrast, the book by Ostebee and Zorn introduces the derivative by first discussing the rate of change of a function at a given point as the limit of an average rate of change, proceeds to relate the result to the slope of a tangent line, to arrive finally at the analytical definition of the derivative. The use of a reformed textbook was accompanied by new methods of teaching and assessment. The traditional lecture method was infrequently used in the classroom; instead and in many situations, activities were designed allowing students to discuss a particular problem or topic at hand. Handouts were additionally prepared to guide the students in these activities. Computers (and to a lesser extent the *TI-89/92* graphing calculators) were frequently used. *Autograph* (Hastell, 2000) was the dynamical software adopted for the course, and it was available for student use in a computer laboratory; however, neither calculators nor computers were used during the exams. In general, technology (particularly its visualization capabilities) was employed in well-chosen problems as a tool assisting in the exploration of problems, allowing students to reflect, analyze, and modify their thinking until an appropriate conclusion was reached. For instance, to introduce derivatives, animation features of *Autograph* were employed to clarify the idea of an average rate of change becoming an instantaneous rate of change (Fig. 2).

In many assignments also, a written component was added forcing students to reflect on their own thinking. Students were required to do a write up detailing the steps used in their analysis and explaining why their conclusion is a valid one. For instance, an essay on car population was given and students were required to work in groups of two in order to analyze the data, represent it graphically, and answer related questions. This assignment had several objectives such as the relevance of mathematics in everyday life, the possibility of defining functions in data form, and reading visually the rate of change of functions. (For further examples, see Appendix A).

To assess the success of this experiment, notes were taken following lecture sessions, copies of exam papers were collected, and most importantly, two sets of semi-structured separate interviews were conducted with students enrolled in the course who volunteered to take part in the study. The interview questions of the first set of interviews revolved around the students' new views of a function, not only as an algebraic formula, but also as graph and a table of values, and their understanding of the differences between the various types of functions presented to them. This discussion was also intended to progress to the idea of the rate of change, and consequently, the derivative of a function. The questions of the second set of interviews focused on the derivative. Interviewees were asked to talk about their understanding of this concept both geometrically and analytically. (See Appendix B for details.)

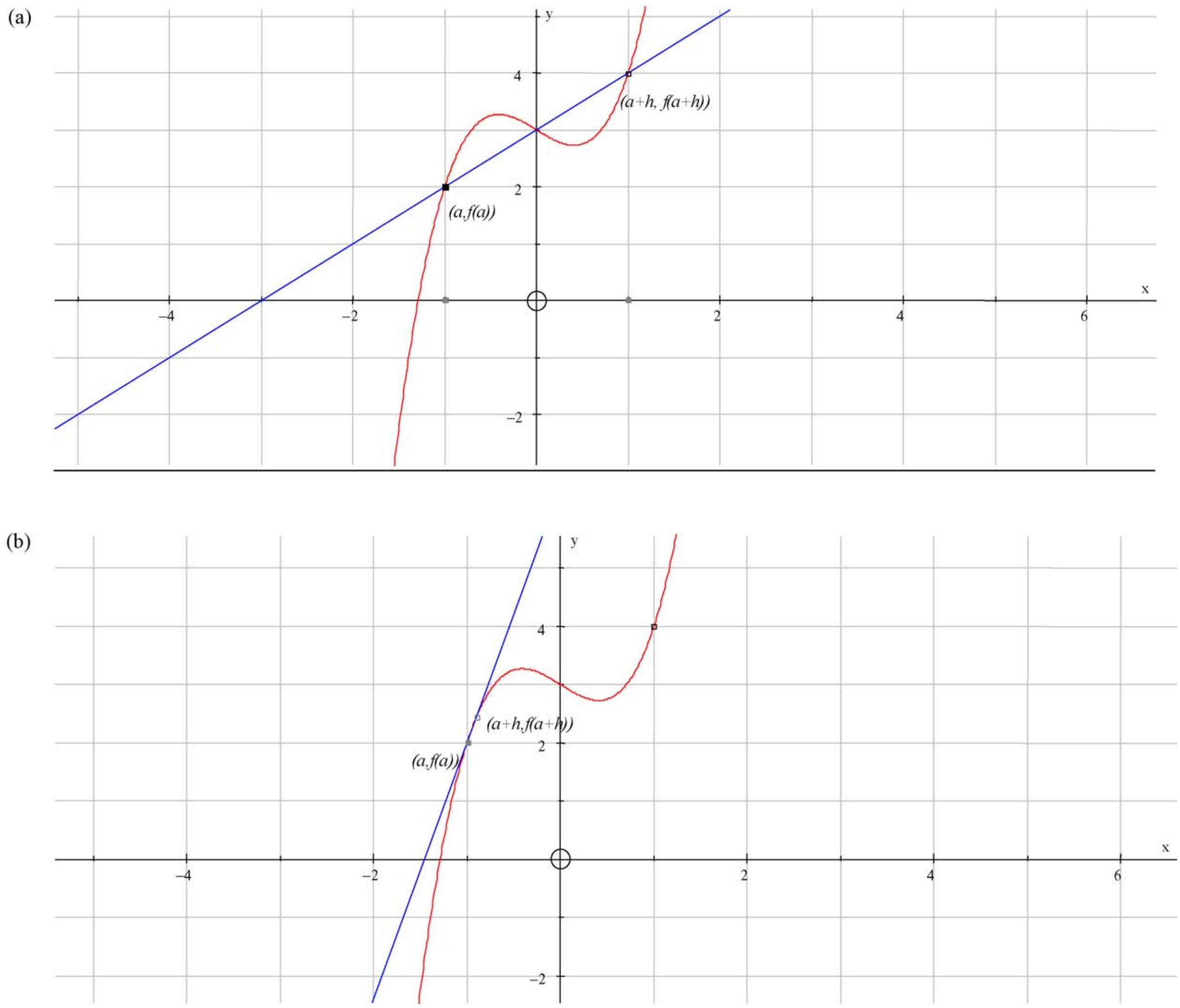


Fig. 1. (a) The line through the points $(a, f(a))$ and $(a+h, f(a+h))$. (b) The line through the points $(a, f(a))$ and $(a+h, f(a+h))$ as $h \to 0$.

3. Results and discussion

In all, 10 students volunteered to take part in the study. It was perhaps unfortunate that most of the volunteer students were better than average students (see Table 1). The interviewees chose the following pseudonyms to identify themselves: Rana, Kareem, Denise, Sarah, Annie, Elie, May, Layla, Biggie T, and Mark. The first set of interviews took place around the middle of the semester following the first mid-term. Interviewees were initially asked to give brief backgrounds of themselves, and to share their opinions on the need for a pre-calculus course to precede Calculus 1.

Table 1
Grade distribution of interviewees

A	B	C
May	Joe	Biggie T
Annie	Layla	
Elie	Denise	
Sarah	Kareem	
Mark		

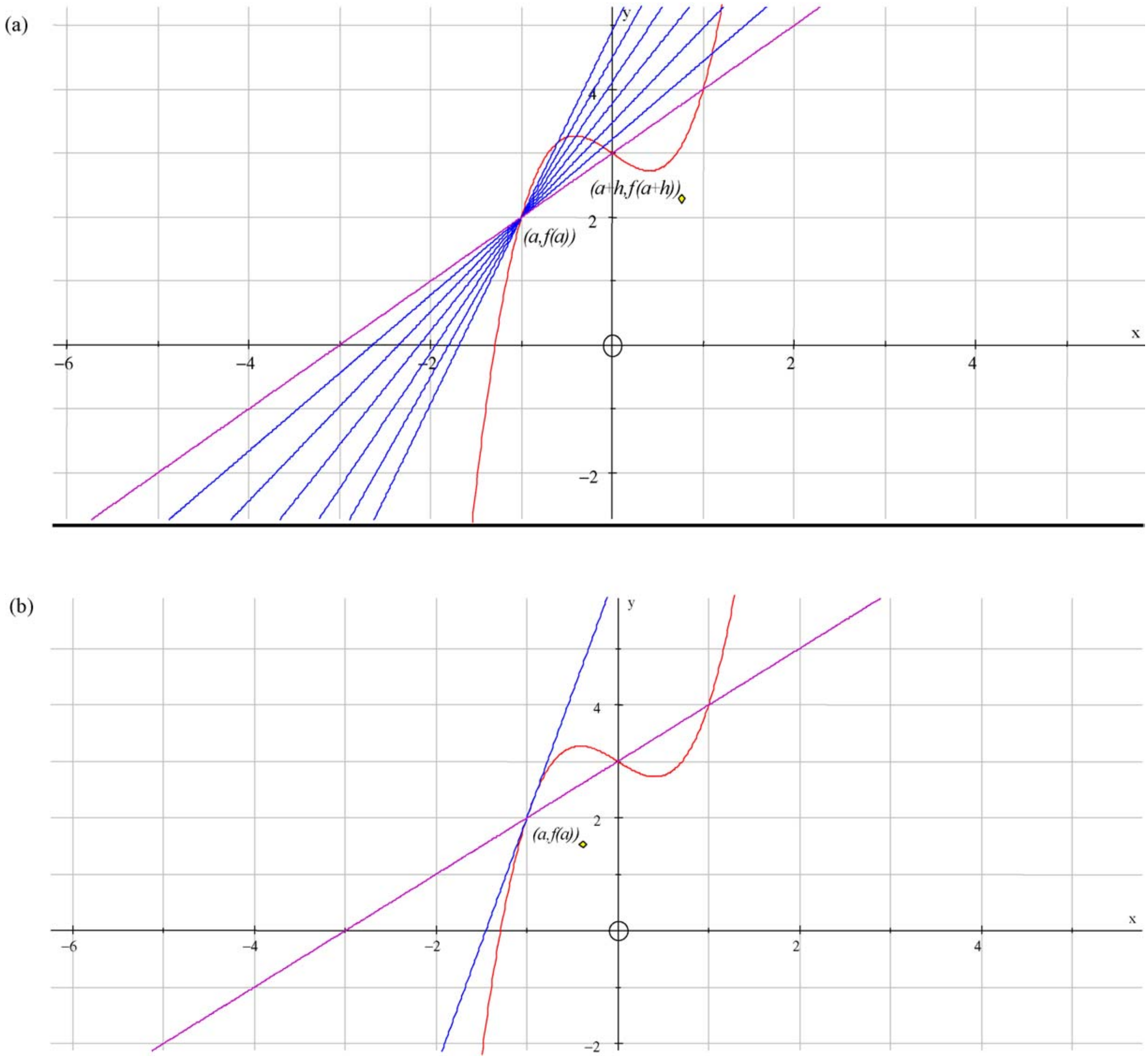


Fig. 2. (a) Average rates of change between points $(a, f(a))$ and $(a+h, f(a+h))$ as h approaches 0. (b) The average rate of change becoming an instantaneous one at the point $(a, f(a))$.

Except for Annie, May, and Kareem, all interviewees had their school education abroad (Arab Gulf countries and Canada). Sarah had taken a pre-calculus course, and except for Layla, none thought that they needed such a course. On the other hand, Elie stood out among all others since he had used *Thomas's Calculus* in studying advanced math in his school in the Gulf.

Following this brief introduction, interviewees were asked if they saw any difference in the way the course Calculus 1 is being conducted at LAU. All agreed that the approach is different. More specifically, most mentioned the visual and analytical thinking required for the learning of the material:

Elie: It is a new way. It depends on analyzing.
Biggie T: Here there is explanation but no exercises . . . but I am not used to visual thinking.
Mark: The basic difference is that you understand why things work the way they do.
Sarah: It is more elaborate . . . it concentrates on geometric issues . . . you have to understand the geometry first, and then comes the algebraic part. This is better.
May: I used to depend more on memory. Now we do it graphically, in a more logical way.

Denise: Each exercise is different. It's not mechanical, so you have to think always.
Rana: The teacher always used to show the graphs first and then talk about the function it represents and the main theoretical points of the lesson.

Sarah also commented that this approach "helps students get ideas from the whole scope", and that she will be able later to maintain the information gained in the course because she is "understanding the content very well". May on the other hand stated that with the traditional approach, one can depend on memory and study at the last minute, but the approach of this course "demands daily work and practice". She also added "the student has to be serious". As for Annie and Layla, they complained that the approach is more difficult, that it is time consuming, that exercises are tough, and that there isn't much time to understand them. Annie, who had taken calculus before as is the case in the baccalaureate program of Lebanon, wondered how students who are taking the material for the first time could cope with its level of difficulty.

Interviewees were then asked to talk about their understanding of a function. Since the course emphasized the geometric approach and to a lesser extent the algebraic one, the definition of a function was not initially given in class explicitly; instead a variety of examples were designed (e.g., car population) by means of which functions were either given as a set of data, or visually as a graph. For this reason, the interviewers hoped that students would speak of the multiple representations of a function: as a formula, as a graph, and as table of numerical values.

Except for Mark and Denise, all interviewees thought of the function in one way or another as a formula, and only three immediately associated it with a graph.

Sarah: If you have an input x, through this function you have an output y . . . It is a rule and a graph.
Annie: A function is a formula that is given. . .we have to put some points and we draw a graph.
Biggie T: It's like a formula or a graph!

Only Denise stated that the graph of a function is what first comes to her mind. She adds:

Denise: I see many things, not only numbers and formulas.
Interviewer: Do you think of values of x and corresponding values of y?
Denise: First I think of graphs.

Mark, on the other hand, thought of functions as objects to be used in physics, biology, or chemistry. He also spoke of the rate of change that a function measures.

Thus, contrary to what we wished, the algebraic representation of a function dominated the thinking of most students. On one hand, the numerical representation of a function was not on any student's mind, perhaps because, in the course, less emphasis was placed on the numerical representation of the function. On the other hand, only one of 10 students thought of the function as a graph, and only three others associated a graphical representation with a function. Studies (e.g., Dreyfus & Eisenberg, 1990; Tall, 1991; Vinner, 1989) have consistently shown that students' understandings are typically algebraic and not visual, that visual information is more difficult for students to learn and is considered less mathematical. Thus, even though the experimental sections, the objects of our study, were geared almost completely geometrically, students' thinking remained algebraic. When asked later if they are able to visualize functions, most interviewees thought that the function had to be linear, or quadratic, to be able to do so. Therefore, for most of them, the algebraic formulation of a function is a prerequisite for visualizing it. In fact, Sarah claimed that no functions exist that cannot be represented algebraically. For some students, the algebraic formulation is thus necessary to answer almost any question related to the function. On the final exam, for instance, students were given the graph of a function (see Fig. 3), and were asked to deduce whether its derivative is increasing or decreasing. Fifteen percent of those who answered (none were interviewees) stated that a formula must be assigned to the graph before making any conclusions. Here is a sample:

> In order to see if the derivative is increasing or decreasing, we must derive the function first. Second, when the function is obtained, we can conclude the derivative, which is $2x$. When $2x$ is plotted, we notice that it is increasing infinitely.

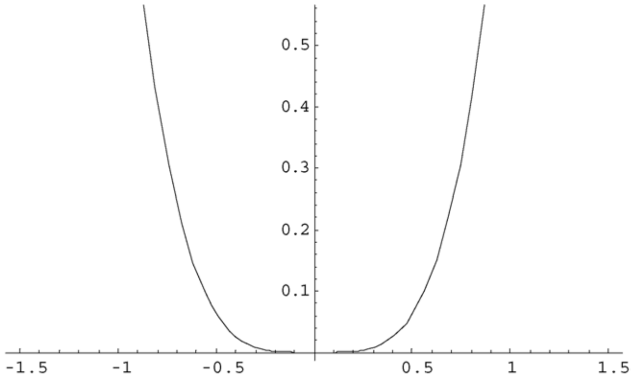


Fig. 3. The graph of a function given on the final exam; students are asked to determine if its derivative is increasing or decreasing.

Thus, for most students (interviewees and non-interviewees alike), the formula comes first, then the graph. Furthermore, most students seem to think only of polynomials as examples of functions while few speak of the trigonometric functions, and even fewer speak of the exponential and logarithmic ones. In *Students' Mental Prototypes for Functions and Graphs* (1991), Bakar and Tall spoke of "prototype examples" that students use for the concept of functions. In the absence of an algebraic definition of a function, students are able to reason only by thinking of functions such as $y = x^2$, or polynomials in general (p. 105). They add: The learner cannot construct the abstract concept of function without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototypes examples having built-in limitations that do not apply to the abstract concept" (p. 111). The students, subjects of this study, have shown that learners cannot construct a visual image of functions without experiencing examples of functions first, and in particular the prototype examples built in their minds (such as $y = x^2$).

3.1. Identifying functions

Questions on the interviews then focused on specific functions. Interviewees were given the graphs of two simple polynomials, $y = x^2$ and $y = x^4$ (see Fig. 4), without their equations. The task was to associate every graph with an equation. The purpose of this exercise was to analyze whether students are able to see the difference between the rates of increase and decrease of these two polynomials specifically over the domains $[-1, 1]$ and $(-\infty, -1] \cup [1, +\infty)$.

In class and prior to the first set of interviews, polynomials had been studied at length. In particular, properties of polynomials of different degrees had been looked at carefully. A lot of emphasis was placed on the effects of transformations (such as $f(x + a)$, $f(x) + a$, $f(ax)$, $af(x)$) on the behavior of functions (polynomials in particular). In this respect, *Autograph* was used to enable the user to see visually the effect of a on the graph of the original function $f(x)$. Thus, polynomials were translated, stretched, and compressed. Other functions had been introduced and studied (to a lesser extent); those were the trigonometric, exponential and logarithmic functions.

A lot of examples of piecewise functions were also given in order to convey to the student the idea that a function is not necessarily synonymous with an algebraic expression. These examples were also important to introduce the concepts of continuity and differentiability over different intervals. On the other hand, the ideas of increase and decrease as well as the rates of increase and decrease (but not the derivative) had been discussed extensively.

Concerning the graph of the parabola $y = x^2$, no one had a problem identifying the equation corresponding to it. However, the answers of two interviewees, Denise and Annie, were more general by suggesting that the equation of the first graph must take the form $y = ax^2 + bx + c$. Denise went on to say: "I think that a is positive since the graph is first decreasing then increasing". Also, Annie spoke of the effect of a on the graph of the equation. When the discussion turned to finding an equation for the second graph, only Elie spoke immediately of $y = x^4$ and was correct about his justification, while the discussion with Kareem, Layla, and Rana lead nowhere. As for Biggie T, May, Denise, Sarah, Mark, and Annie, they all decided that the equation of the second graph must take the form $y = ax^2$. Some of the latter interviewees thought that the second graph was a compression of the first (which seems only true graphically inside the interval $[-1, 1]$) while others thought that the second graph was obtained by stretching the first one (which also seems

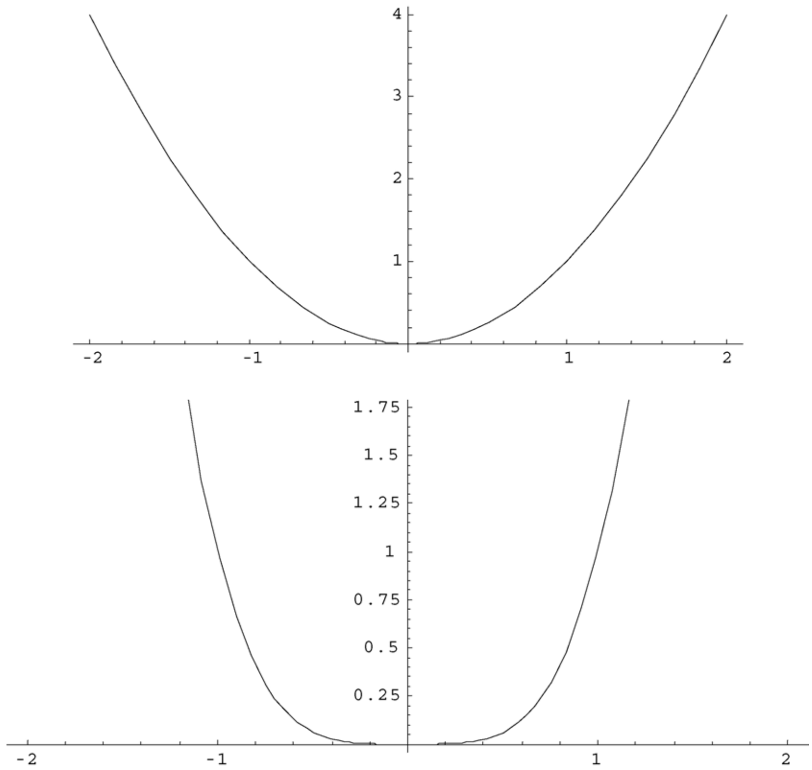


Fig. 4. The graphs of $y = x^2$ (top) and $y = x^4$ (bottom).

true graphically, but inside $(-\infty, -1] \cup [1, +\infty)$). Biggie T, Denise, Sarah, May, and Annie tried to find a constant value that would cause compression inside the interval$[-1, 1]$, and a stretching of the graph inside $(-\infty, -1] \cup [1, +\infty)$, but they soon realized that it was not possible. Mark, on the other hand, observed that $(-1, 1)$ is a common point to both graphs, so one can neither be a compression nor a stretching of the other. The discussion then progressed to finding other possible representations of the second graph. Biggie T suggested products of linear factors, but then eliminated (for an incomprehensible reason) the possibility of it being x^3. He concluded later that the power must be even, suggested x^4 and convinced himself by trying various values of x. May also eliminated the possibility of it being x^3 when she tried to replace x by -2. Then, she concluded that the function "should be an even function like x^4". Mark and Annie made similar remarks about the graph and concluded also that x^4 is a possibility. On the other hand, Denise thought of a piecewise function (different values of a on different intervals), while Sarah could not make any conjecture. Only Denise, Sarah, and Rana spoke of the rate of increase or decrease of functions while comparing the two graphs.

Denise: It [the graph of x^4] looks like a parabola, but it is wider in the bottom. It seems that it is decreasing there at a slower rate.

Sarah: The first one [x^2] is increasing faster between 0 and 1, but when $x > 1$ the second one is increasing faster, hence the rate of change is faster there.

Mark: The second one [x^4] is increasing extremely fast [this she concludes by looking at the slopes at different points].

The ease of identifying the parabola $y = x^2$ is not at all surprising and falls within the "prototype examples" that Tall and Bakar spoke of. The parabola $y = x^2$ is not only an example of a function that students think of immediately, but is also an example that many instructors present in classroom discussions in order to introduce or clarify an idea. The detailed answer of Annie is perhaps a reflection of her educational background. The Lebanese Baccalaureate system extensively covers calculus topics; for instance, sketching graphs of polynomials is done in great detail, using the first derivative to determine increase and decrease, and using the second derivative for the concavity. However, as in many schools, the education system does not seek to develop in the student's mind the ability to analyze; instead, learners are, in most cases, asked to memorize properties such as "when a is positive, the graph is first decreasing then increasing". The difference between the learning approach required for the Lebanese Baccalaureate and the learning approach in this reformed calculus course was reflected in Annie's comment when she wondered how students who are taking the material for the first time could cope with the level of its difficulty.

Concerning the discussion of the function $y = x^4$, six out of the ten interviewees (Biggie T, May, Denise, Sarah, Mark, and Annie) attempted to analyze the graph before assigning an algebraic function to it. It is true that their thinking was geared in one particular direction (compression or dilation, perhaps because a lot of time was spent in class discussing transformations of functions), yet their emphasis on comparing the graphical properties of x^2 and those of x^4 shows that the reformed approach used in teaching the course had its effect on the thinking of some students. The fact that some of them decided eventually that the sketch must be the graph of x^4 stemmed only from an elimination process (since it is neither ax^2 nor a cubic). Elie, on the other hand, was an exceptional student with an exceptionally good background in calculus, so it was not very surprising that his answer was immediate. Only Denise, Sarah, and Rana spoke of the difference between the rates of change of the two sketches, although neither one of them was able to associate an equation to the graph of x^4.

3.2. The derivative concept

The second set of interviews took place almost at the end of the semester (but before the final exams period). At that point, students were exposed to the idea of derivatives in two different contexts: the geometric and the analytic. According to Zandieh (1999), "although knowledge of these processes and the formal definition are certainly needed for a robust understanding of the concept of the derivative . . ., the pseudo structural knowledge of derivative as the steepness of a function at a point, or the speed at an instant in time . . . allow students to solve many problems without the complications of the formal derivative . . .." Thus, more time was spent discussing the meaning of the derivative both as a rate of change and as the slope of a curve at a certain point. This approach had a "negative" effect on the students' conceptualization of the derivative. For instance, when asked on an exam to give the formal definition of the derivative, less than half gave the correct answer. When asked later to identify the function whose derivative at a certain point a is given by $\lim_{h\to 0}((1/(1+h)-1)/h)$, an even smaller percentage of students were successful.

Within that reformed spirit, many of the assignments given to students asked them to match graphs of functions to given equations, or match graphs of functions to the graphs of their derivatives. In other exercises, the graph of a derivative was given and questions on the function itself were posed. Another type of question requested the sketching of the graph of a function given a table of data for the function and its derivative. Some exam questions focused also on the geometric meaning of the derivative (see Appendix A). For instance, on the second and final exams, students were asked to give a geometric definition of the derivative. Results of the second exam show that 42% spoke of the slope of a tangent to the curve at a given point, 10.5% spoke of the instantaneous rate of change (two students also included the tangent line in their definition), while the remaining 47.5% were unsuccessful in their definition. On the final, 50% of those who answered spoke of the derivative as being the slope of the tangent line at a point, 15% spoke of the instantaneous rate of change, and 35% failed to give a comprehensible answer. When interviewees were asked about the meaning of the derivative, 60% (Sarah, Elie, Layla, Biggie T, Mark, and Rana) spoke of the "spontaneous rate of change at a point" and/or "the slope of the tangent at a point" (see Table 2). These results show an improvement in the students' conception of the derivative in favor of its geometric interpretation. The percentage observed in the interviews may be a consequence of the fact that interviewees were better than average students, as mentioned earlier.

Table 2
Defining the derivative

	The derivative as a the slope of a tangent (%)	The derivative as a rate of change (%)
Results of Exam 2	42	10.5
Results of interviews	30	30
Results of final exam	50	15

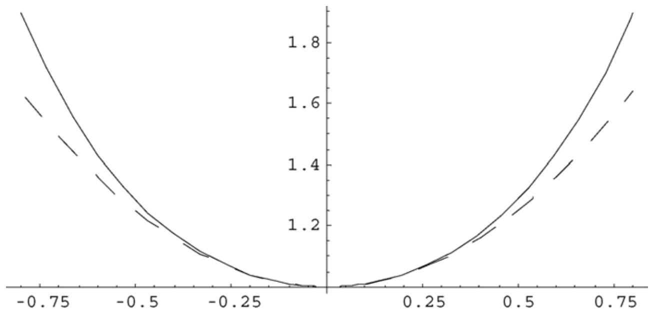


Fig. 5. The graphs of $y = x^2 + 1$ (dashed) and $y = e^{x^2}$.

In some cases, interviewees were asked to speak further about the meaning of the spontaneous rate of change. The answer given by Mark, for instance, showed the effect of the approach on the students' mind.

Mark: At a certain moment, we can know for example if a car is speeding. The cops have radar, so if they want the speed of the car at a certain point, then this is called instantaneous.
I: But what makes it instantaneous?
Mark: h tends to zero.
I: So you mean it is the fact that we are taking the limit. What if we remove the limit?
Biggie T: We get the average rate of change.

A careful look at the conversation with Mark shows that, contrary to what one might expect from a student in a traditional calculus section, Mark attempted to relate the abstract definition of the derivative to its actual meaning. Although Elie was the only student who referred to the abstract definition of the derivative $(f'(a) = \lim_{h\to 0}(f(a+h) - f(a)/h))$, none gave examples for finding the derivative in a mechanical way. On the other hand, Annie, May, Denise and Kareem spoke of the derivative as a tool for determining the increase and decrease of a function. Elie and Biggie T also made a reference to this idea.

Discussions with interviewees then turned to comparing the derivatives of two functions given graphically. Those were the graphs of $y = x^2 + 1$ and $y = e^{x^2}$ (see Fig. 5).

It was clear to all interviewees that the dashed graph is decreasing faster than the non-dashed one for negative values of x, and increasing faster for positive values of x. More importantly, all interviewees were able to make the right conclusions about the derivatives of both functions. Here is a sample of answers.

Biggie T: At each point, there is a tangent line; the tangent lines are steeper for the dashed graph.
Annie: They are both increasing . . . The dashed one is steeper because the slope [at a given point] is greater . . . We draw a tangent line; if the tangent line is increasing the derivative is positive there.
Denise: For every value of x, we have two different values of y [one on each curve] and the slope on the dashed graph is steeper than the one on the non-dashed curve.

Later, interviewees were given a graph that represented the derivative of a function $f(x)$, (see Fig. 6) and they were asked to make a guess for the equation of the function itself.

Disregarding the discontinuity at the origin, students with a good understanding of the derivative should be able to deduce that the function has a constant slope of -2 for $x < 0$, and a constant slope of 4 for $x > 0$. Indeed, all interviewees were thinking in that direction. For some of them, the discussion had started with a question about the derivative of a straight line, which many (if not all) described geometrically as a horizontal line.

I: For a straight line, what is the graph of the derivative?
Denise: It is a horizontal line because the rate of change is constant.
I: If I now give you this sketch [Fig. 6] and tell you that it represents the graph of a derivative of a function, can you guess what the function looks like?
Denise: It is two lines, one is decreasing with slope -2, and the other is increasing with slope 4.

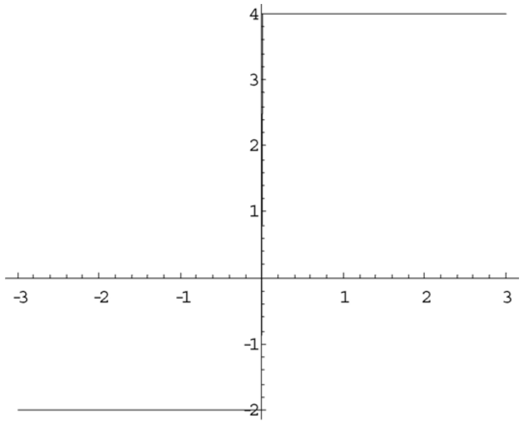


Fig. 6. The graph of $f'(x) = \begin{cases} -2, x < 0 \\ 4, x \geq 0 \end{cases}$.

The same conversation took place with all other interviewees, and surprisingly, they all answered in the same way, revealing a very good understanding of the idea of the derivative. Thus, contrary to what the interviews revealed about the student's conception of a function, the meaning of the derivative, as presented to them, had sunk well in their minds and only in one or two instances did they refer to its abstract definition. In addition, none of them gave examples for finding derivatives in a mechanical way.

Two things can justify the difference in the attitude of the students in the first and second interviews: On one hand, since the approach for explaining concepts was non-traditional, students were showing (unintentionally perhaps) a resistance to the new ideas presented to them. On the other hand, during the first set of interviews, these new ideas were to substitute an earlier knowledge of the same concepts since all interviewees had worked with functions before. By the time the second interview had come, more time had elapsed and students have gotten more acquainted with the new approach. Furthermore, for many of them, this was their first exposure to the derivative concept, thus enabling the instructor to instill in the learner's mind the desired concept definition as well as concept image for the derivative. According to Tall and Vinner (1981), the concept image "includes all the mental pictures and associated properties and processes" of a certain concept; "it is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures". On the other hand, the concept definition "is a form of words used to specify that concept . . .." In the case of this experiment, the concept of the derivative was explained to the students as an instantaneous rate of change, and the concept image students were expected to build was the slope of the curve at a given point, or the decrease/increase at that point. Since for most students, this course was their first exposure to this concept, it was not surprising, therefore, that most if not all interviewees answered in the way they did.

4. Conclusions

In the end, the one thing that is most striking is the large percentage of dropouts (33 students out of 89) and failures (12 of the remaining 56 students) in the observed sections. This could be a consequence of the varied background of the students, but it also means that the approach was very difficult for a great majority of the students. However, the level of understanding of the interviewees and their general attitude show that this approach was rewarding for some students, at least the better ones, specifically by the end of the semester when time contributed to using graphs efficiently. Time, therefore, is a crucial factor in assimilating the idea of thinking visually, but also as Habre (2001) showed, a barrier to visual thinking is the traditional instructional background that most students are still experiencing in their schools.

It has been argued that high-ability students favor looking at functions in a graphical setting. This is not to say that they found the approach easy. For instance on one exam, students were asked the following question: "We have shown geometrically in class that the derivative of $f(x)=x^2$ is $f'(x)=2x$. Consider now the function $g(x)=x^2+1$. Explain *geometrically* why the derivative of $g(x)$ should also be $2x$." In the responses, we expected students to observe that since the graph of $g(x)$ is a vertical translation of that of $f(x)$, then the tangent lines at a given point remain parallel; consequently, the derivatives are equal. Results showed however that only about 13% of the entire student population gave the desired justification, but none were interviewees. Here is what two students wrote on their exam papers:

> When drawing $g(x)$, we see that it has the same graph as $f(x)$ but it has been taken up one unit. The slopes of the tangent at any point in $g(x)$ and $f(x)$ are the same.
>
> It will be $2x$ because the tangent of $f(x)$ and $g(x)$ are parallel to each other.

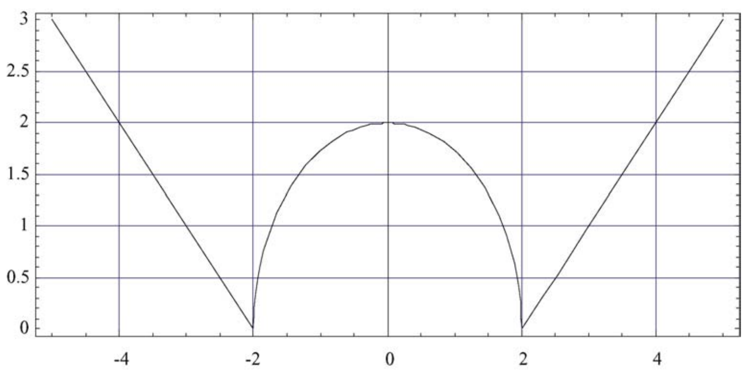
On the other hand, about 26% attempted to do the problem mechanically noticing that derivative of a constant is zero, but three of them (one of which is the interviewee Kareem) added a geometrical flavor to their reasoning:

> $g(x)$ differs from $f(x)$ by adding +1 to $g(x)$[he means $f(x)$] and by shifting $g(x)$[he means $f(x)$] one unit upwards. We know that the derivative of a constant is zero. (Kareem)

All other students (61%) reasoned incorrectly both geometrically and analytically. This particular exam question requires a certain mathematical maturity that few students can acquire from a one-semester course in which a new approach has been adopted. For that reason, these results are not very surprising. In general, however, interviewees had a better capability of handling the difficulties associated with the geometrical approach, and many of them seem to have assimilated the idea of thinking visually. Thus, although most interviewees thought of functions only in the analytic setting, yet when attempting to identify the graphs of x^2 and x^4, their discussion was almost purely geometric. In fact, only 3 out of the 10 interviewees even spoke of the rate of change of these functions, a concept emphasized later in the semester. Concerning the derivative concept, interviewees showed an almost complete geometric understanding of this topic. The one thing that many students (interviewees and non interviewees alike) failed to do properly was to define the derivative geometrically (47.5% on second exam, 35% on final, and 40% of interviewees). This is perhaps a consequence of the fact that mathematical definitions are traditionally analytical, creating an obstacle in the minds of the students. It should be noted here that the interviewees enrolled in the fall semester had taken a common final exam with the traditional calculus section; consequently, the exam emphasized mechanical techniques. Their performance, however, was as good as their performance on the midterms. Also, brief observations of those who took the later calculus courses (offered using *Thomas's Calculus*) showed that interviewees are able to do mathematics in traditional as well as non-traditional settings (without necessarily liking both approaches). Finally, even though comments like "... you understand why things work the way they do", and "this approach helps students get ideas from the whole scope" are very encouraging, yet it is the large percentage of dropouts and failures that discourages us from repeating this experiment.

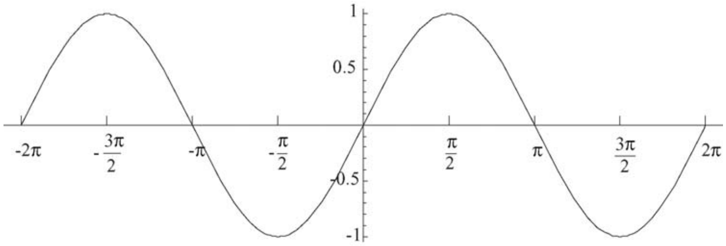
Appendix A

(1) A. The graph below represents a piecewise function consisting of two lines and a semicircle. Write a formula (in fact three formulas) for this function. Justify your answer.

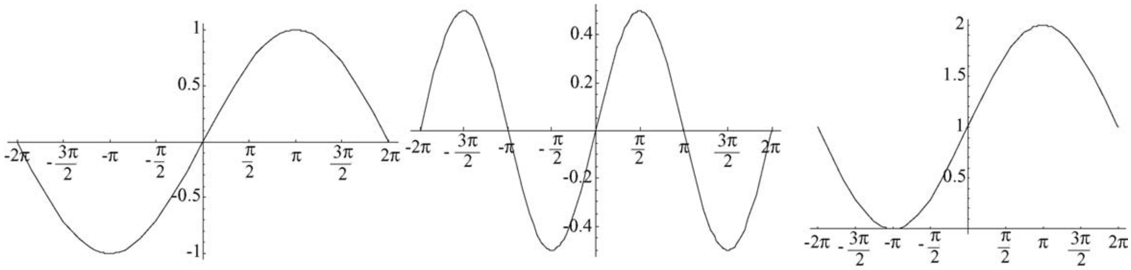


B. Using the graph in A, determine if this function is even, odd, or neither. Justify your answer.
C. Compute $f(-3), f(3), f(-1), f(1)$.
D. Are your answers in C consistent with your answer in B? Justify.

(2) Here is the graph of *Sin* (x) between -2π and 2π:

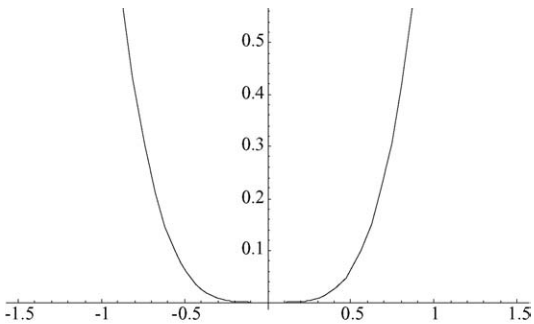


A. What is the period of this function?
B. Below are three more graphs. Associate each with one of the following function:
$$\frac{1}{2}\text{Sin}(x);\quad \text{Sin}\left(\frac{x}{2}\right);\quad \text{Sin}\left(\frac{x}{2}\right)+1;\quad \frac{1}{2}\text{Sin}\left(\frac{x}{2}+1\right)$$
Justify your answers. Also, determine the period of each function.



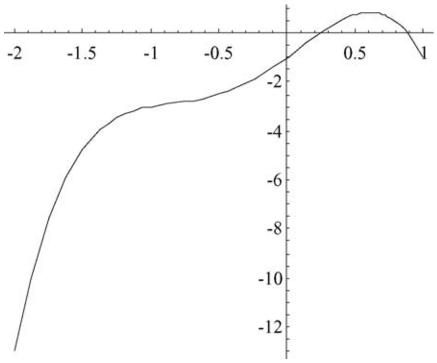
(3) Below is the graph of the function $f(x) = 3\,(x-2)\,(x-1)\,(2x+1)\,(x-3)$.
A. Use the TI-89 to produce polynomials of degree 4 having the following graphs.
B. Describe the process by means of which you were able to obtain your answer, including those that did not yield the desired graph.

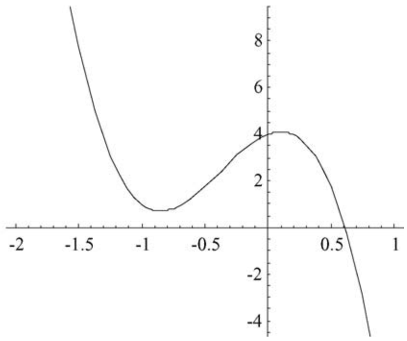
(4) Consider the following function $f(x)$:



Is the derivative of this function increasing or decreasing or both? Justify.

(5) The first sketch represents the graph of a function. Explain why the second sketch represents that of its derivative. (In particular, discuss not only the increase and decrease of f but also the rate of increase and decrease.)

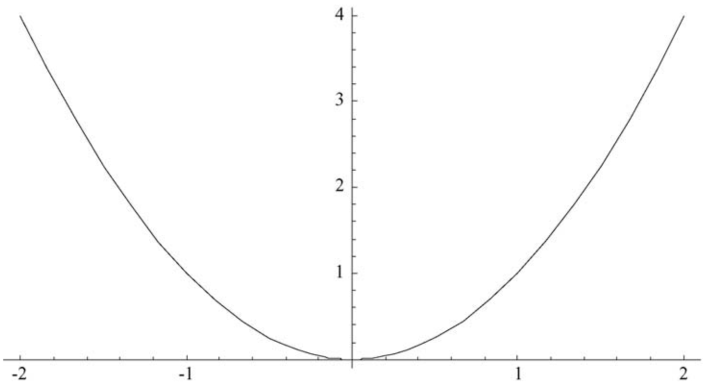


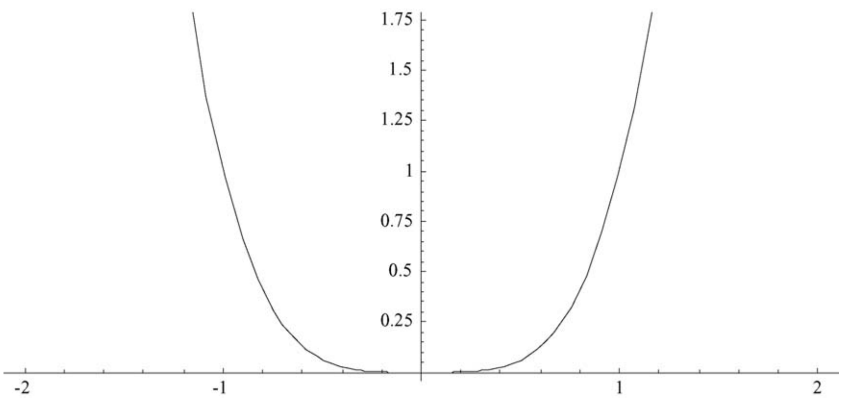


Appendix B

B.1. Interview 1: (Interview took place following the first mid-term)

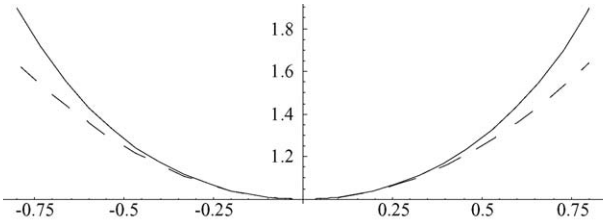
1. Give a pseudonym for yourself
2. Describe your educational background; more specifically tell me whether you think you should have taken a pre-calculus course prior to Calculus 1.
3. Identify any differences if any in the way Calculus 1 is being taught in comparison to other mathematics classes you have taken.
4. What is your understanding of a function?
5. Can you visualize functions?
6. Here are two graphs of functions. Associate to each an algebraic formula.



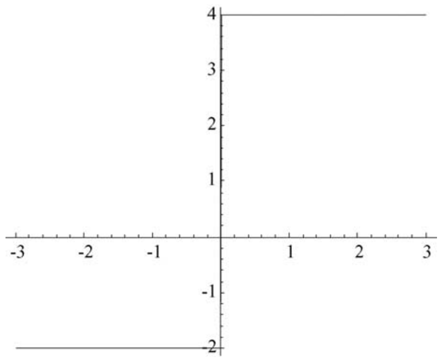


B.2. Interview 2: (Interview took place at the end of the semester, following the second mid-term)

1. What is your understanding of the derivative of a function?
2. Compare the derivatives of the functions below:



3. The graph below represents the derivative of some function $f(x)$. Can you guess the equation for $f(x)$?



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