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Students' Conceptual Understanding of Derivatives in Freshmen Calculus

By

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Dedication Page

To my loving parents

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Students' Conceptual Understanding of Derivatives in Freshmen Calculus

Hana Ghassan Shatila

ABSTRACT

The freshmen calculus curriculum has witnessed many changes in the past few years due to the development of computer technology and dynamical mathematical software. This study aims to examine the students' conceptual understanding of derivative in a calculus I course. Fifty- two students participated in this study consisting of 27 males and 25 females. All students attended a calculus I course at a private Lebanese University. Two groups, each of 26 students, are considered: a control group (taught by instructor X) learning derivatives using Book 1 that emphasizes the symbolic approach of the concept, and an experimental group (taught by instructor Y) learning derivatives using Book 2, which emphasizes the multiple- representations approach of the concept. In the experimental group, cooperative learning, technology (*Autograph*), and a series of activities incorporating the *APOS* (action- process, object- schema) levels were integrated in the teaching and learning of derivative. Data were collected using qualitative and quantitative methods such as the content analysis of two calculus books (Book 1 and Book 2), observations, and questionnaires on derivatives administered to all students before and after the implementation of the unit on derivatives. Moreover, data was collected using a test, consisting of five conceptual-understanding based problems on derivatives, and clinical interviews conducted with twelve students. Results show that students in the experimental group have better conceptual understanding of the derivative concept than those in the control group. Many students in the experimental group seem to have an *object* conception and almost a comprehensive understanding of the derivative particularly concerning the slope of a tangent line at a point, the instantaneous rate of change, and the relation between a function and its derivative. However, many students in the control group had deficiencies in their understanding, showing an action/ process conceptions of the derivative. Findings in this study are parallel to the findings of several studies. In addition, the quantitative analysis of both the questionnaires and the tests reveal significant statistical differences in the mean scores between the two groups in favor of the experimental group. Finally, the observations reveal that many students in the experimental group were more interested and motivated to learn mathematics. On the other hand, some experimental students resisted the approach used and found it difficult and demanding**.** This study highlights the need for all stakeholders to work collaboratively to integrate technology in teaching calculus and to consider changes in the calculus curriculum and the books adopted to encourage the use of multiple representations.

Keywords: Calculus, Derivatives, Conceptual Understanding, Multiple Representation- Visual Approach, Formal Symbolic Approach, APOS theory

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CHAPTER ONE

INTRODUCTION

1.1 Overview

Poor math achievement and negative attitudes toward mathematics are two subjects of utmost importance and interest in the field of mathematics education. Research has shown that many students face difficulties in learning mathematics, find it abstract and boring, and perform poorly in it. Calculus, as one of mathematics disciplines, is not an exception (Ferrini- Mundy & Graham, 1994; Orton, 1983). Calculus is a rich subject that deals with functions, limits, derivatives and their applications, integrals and their applications, sequences and series and others.

Calculus is the study of change, yet for decades the teaching of calculus has been based on rote memorization of formulas and procedures, algebraic manipulation and solving drill problems. As a result of this, students' drop-out and failure rates in calculus have always been high compared to other courses. Due to this crisis, the movement towards reform calculus began in the late of 1980's, and calls for change in calculus instruction have come from different sources (Douglas, 1986; Steen, 1987; Tall, Smith & Piez, 2008; Vinner, 1989). Many documents, books and conferences have emerged as a part of a reform movement such as the *"Curriculum and Evaluation Standards for School Mathematics"* (NCTM, 1989), the conference "*Toward a Lean and Lively Calculus*" (Douglas 1986), the colloquium "*Calculus for a New Century: A Pump, not a*

Filter" (Steen 1987) and others. The National Council of Teachers of Mathematics (NCTM) (1989) requested a mathematics curriculum that "emphasizes conceptual understanding, multiple representations and connections, mathematical modeling, and mathematical problem solving" (p. 125). Therefore, many steps have been carried out to ensure changes in the calculus curriculum such as changes in textbooks and delivery style, having group work discussion, students- teacher interactions and emphasizing conceptual understanding and visualization and others. Zimmermann (1991) emphasized the importance of visualization in mathematics ; he argued that " the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136).

The development of computers, graphic calculators and dynamic mathematic software played an important role in the reform movement in mathematics. The role of technology in the teaching and learning process has provoked debates among mathematicians and educators. Research (Fey 1989; Porzio, 1999) has shown that there is a consensus that technology provides students with an easier access to the multiple representations of the math concepts. Representation is a process used to express mathematical concepts, thoughts and relationships. Different forms of representation (numerical, graphical, and symbolic, and others) can be used to convey the same concept, and each mode of representation has advantages that make it better than other representations (Kaput, 1987).

1.2 Statement of the Problem

The concept of derivative is one of the key ideas in calculus. It is a concept that is built in connection to other concepts such as functions and limits. Derivative can be approached using different modes of representation such as numerical, graphical, and symbolic. Making connections and translations among and within these representations is important for understanding derivatives (Ferrini- Mundy & Graham, 1994; Orton, 1983; Zandieh, 1998). Generally, a good conceptual understanding of derivative includes the following ideas: the idea of a differentiable function at a point, the idea that the derivative of a function is itself a function, the instantaneous rate of change, the slope of a tangent line, the formal definition of derivative, and finally the ability to relate and connect all together (Ellison, 1993). Research on understanding derivative has shown that many students have deficiencies in their understanding of derivatives. Some students face difficulties in understanding derivatives as a rate of change (Bezuidenhout, 1998), understanding the graphical representation of derivatives as the slope of a tangent line (Asiala, Cottrill, Dubinsky, &Schwingendorf, 1997; Ferrini- Mundy & Graham, 1994; Orton, 1983; Vinner 1982), and understanding the formal definition of derivatives (Zandieh, 1998).

Based on existing studies that are discussed further in this study and on the researcher's own experience, many students exhibit a complete dependence on algebraic formulas and rules when dealing with derivatives and have little conceptual understanding of the concept. Part of this may be attributed to the traditional instructional methods, which place a strong emphasis on formulas, equations, memorizing rules and manipulating symbols. Therefore, a visual, dynamical and a multiple-representation approach might enhance and deepen students' conceptual understanding of derivatives.

1.3 Purpose of the Study

This study aims to examine how the concept of derivative is presented and developed in two different calculus textbooks that adopt two different approaches: a formal symbolic approach and a multiple-representation visual approach. The second purpose is to investigate the types of difficulties that students face when learning the various notions related to the derivative concept. Finally, it aims to compare the impact of two different instructional methods, one using a multiple- representation visual approach and the other using a formal symbolic approach, on students' conceptual understanding of derivatives and their attitudes toward math.

1.4 Research questions

This study addresses the following questions:

1. How is the concept of derivative presented and developed in two different calculus textbooks? (formal symbolic approach versus multiple-representation visual approach)
2. What types of difficulties do first- year (Freshmen) students face when learning and applying the various notions of derivatives?
3. What are the differential effects of the two approaches (formal symbolic approach and multiple-representation visual approach) on students' conceptual understanding of derivatives?
4. Does the use a multiple-representation visual approach improve students' motivation and attitudes toward math?

1.5 Definition of Key terms

The terms conceptual understanding, multiple representation approach, formal symbolic approach are defined as follows:

- **Conceptual understanding:** It is rich in relationships, ideas, connections and patterns. It cannot be based only on memorization, but it needs a reflective and thoughtful thinking. While the procedural knowledge refers to memorized facts, rules, procedures, methods and formulas, Arslan (2010) defined conceptual understanding as follows: "Learning that involves understanding and interpreting concepts and the relations between concepts" (p. 94).
- **Multiple- representation approach:** An instructional approach involves many types of representations of the concept. In other words, expressing the same concept using different external representations: graphs, tables, numbers, symbols, etc. and connections among these representations. This kind of approach caters for students' needs and different learning styles.
- **Formal symbolic approach:** An approach places emphasis on symbols, rules, equations, and algebraic expressions. In addition, it is characterized by 'chalk and talk', repetition, drill and practice, teacher centeredness and memorization.

1.6 Significance/ Usefulness of the Study

The concept of derivative is an interesting math topic that allows measuring the steepness of the graph of a function, finding the rate of change of the output relative to the input, calculating the slope of tangent lines, and finding the critical points of a graph, and others. Also, derivative can be used as a tool to model the behavior of changing quantities such as: rising prices, growing population, decaying radioactive materials, finding velocity and acceleration of moving objects and others. In other words, the applications of derivative are essential in other disciplines such as physics, chemistry, economics, medicine, engineering, etc. Therefore, having a solid conceptual understanding of derivative is important.

Moreover, there is a vast literature on students' understanding of derivatives, but few research studies have been conducted in Lebanon and the Arab region on that topic. Therefore, this study will help in creating a base line for conducting further research on the topic in Lebanon.

Furthermore, this study will draw teachers' and curriculum developers' attention to the following:

- Difficulties that students face when learning the different notions of derivatives
- Rethinking the curriculum and the textbook they are using in order to deepen and increase students' conceptual understanding.
- The importance of translating among different representations of derivatives and connecting them
- The importance of using a multiple- representation approach and integrating technology in calculus courses in order to maximize students' understanding, address their different learning styles and needs, and motivate them to learn.

CHAPTER TWO

LITERATURE REVIEW

Derivative is a multifaceted concept that is built on the function and limit concepts (Zandieh, 1998). Therefore, the analysis of literature will briefly focus on students' understanding of functions and limits, extensively focus on students' understanding of derivatives, and then discuss the role of technology and visualization in teaching and learning of mathematics. Multiple representations, concept image and concept definition, and APOS *(Action-Process-Object-Schema)* theory are used as theoretical frameworks in describing students' understanding of functions, limits and derivatives.

2.1 Multiple Representations

Many researchers (Janvier, 1987; Kaput 1987, Lesh, Post & Behr, 1987) have investigated the usage of multiple representations in math education. According to Kaput (1987) representation is a tool that is used to express mathematical concepts, thoughts or communication. He distinguishes two types of representations: internal and external. Internal representations refer to mental images or structures constructed by a student, while external representations refer to observable pictures, equations, tables, symbols and other physical configurations. Lesh et al. (1987) have suggested another approach to multiple representations. He identified five types of external representations: pictures, written symbols, spoken language, manipulative, and relevant situations.

According to Lesh et al. (1987), it is important for student not only to understand the different types of external representations, but they should have the ability to translate between and within modes of representations. Figure 1 presents the representations model developed by Lesh and his colleagues.

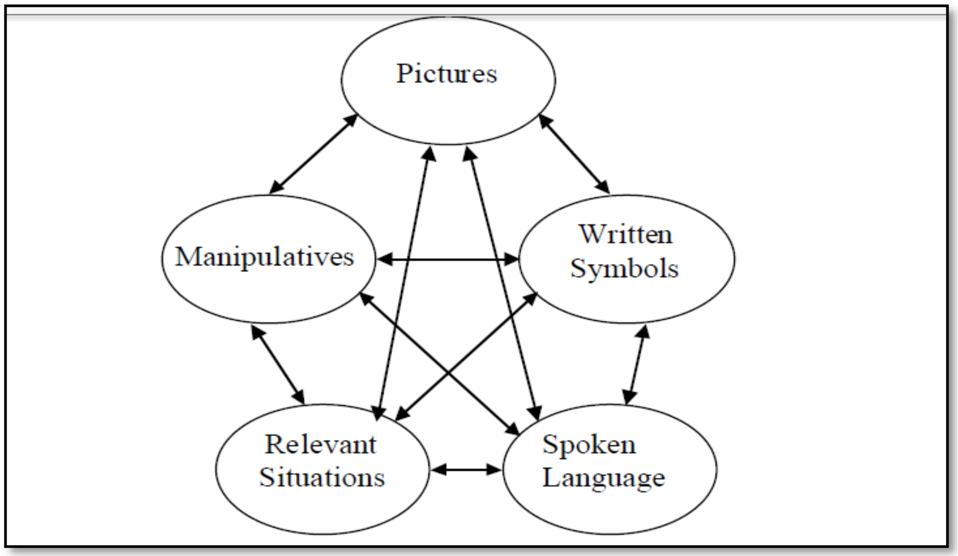


Figure 1. Representation Model Adopted from Lesh, Post, & Behr

From: "Representations and Translations among Representations in Mathematics Learning and Problem Solving," In C. Janvier, (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics*, pp.33–40. Hillsdale, NJ: Lawrence Erlbaum Associates, 1987.

Janvier (1987) refers to *translation* as "the psychological processes involved in going from one mode of representation to another" (p.27). Table 1, presents in details Janvier's model, showing translations among different types of representations. Janvier called the cells, in which translation takes place within the same mode of representation such as from graphs to graphs or from formulas to formulas, as transposition. Moreover, he classified translations into two types: direct and indirect. Direct occurs when translation takes place from one type of representation to the other one without using any kind of other representational mode in this translation process; for example, translation from a formula to table. On the other hand, Janvier (1987) refers to the indirect translation, as the process of translation from one representational mode to another by passing through another mode; for instance, the translation from a formula to a table and then from table to graph.

Table 1

Translations among Different Representations

To / **From**	Verbal Description	Tables	Graphs	Formulas
Verbal description	–	Measuring	Sketching	Modelling
Tables	Reading	–	Plotting	Fitting
Graphs	Interpretation	Reading off	–	Curve fitting
Formulas	Parameter recognition	Computing	Sketching	–

***Notes*:** From "Translation Processes In Mathematics Education", *Problems of Representations in the Teaching and Learning of Mathematics*, (p. 28), by C. Janvier , 1987, Hillsdale, NJ: Lawrence Erlbaum Associates

Moreover, representation is one of the standards of the *Principles and Standards for School Mathematics* (NCTM, 2000). NCTM recommends that students should "create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena" (p. 67).

2.2 APOS Theory

Dubinsky and his colleague developed APOS, which is similar to the constructivist theory by Piaget. APOS states that one has to build certain mental structures in order to understand a given mathematical concept. APOS theory has been used significantly in many publications, in the design of textbooks, in the teaching practice, and in explaining the difficulties students face when learning concepts such as functions, derivatives, linear and abstract algebra, discrete math and others.

APOS is used to categorize students' thinking about mathematical concepts and not categorize the concepts themselves. That is one student may have an action conception of derivative and another have an object conception. In APOS-based research, the terms conception and concept have different meanings. McDonald et al. (2000) describe the distinction as follows: "We distinguish between conception and concept as the first is intrapersonal (i.e., the individual's idea or understanding) and the latter is communal (i.e., a concept as agreed upon by mathematicians)" (p. 78).

2.2.1 Elements of APOS theory

APOS is an acronym that stands for Action, Process, Object, and Schema. These four stages represent different levels of abstraction. According to Dubinsky and McDonald (2001), the elements of the APOS theory can be described as follows:

1. <u>Action</u>: It occurs when the individual reacts to external stimuli using algorithms and step-by-step procedure. An action remains, for the individual, as externally driven.

2. Process: It occurs when the actions become interiorized, and thus the individual can repeat, reflect on, describe or reverse the steps without using step-by-step procedure.
3. Object: It occurs when the mental process encapsulated to become a *total entity* or formalized object.
4. Schema: Finally, schema are collections of processes, objects and previously constructed schema which can be themselves encapsulated into objects. They allow an individual to make sense of a given situation.

2.2.2 APOS cycle

Dubinsky and his colleagues (1997) proposed a cycle according to which APOS theory can be applied, tested, and refined. The cycle consists of three components: *theoretical analysis, instructional treatment, observations and assessment of student learning* (See Figure 2).

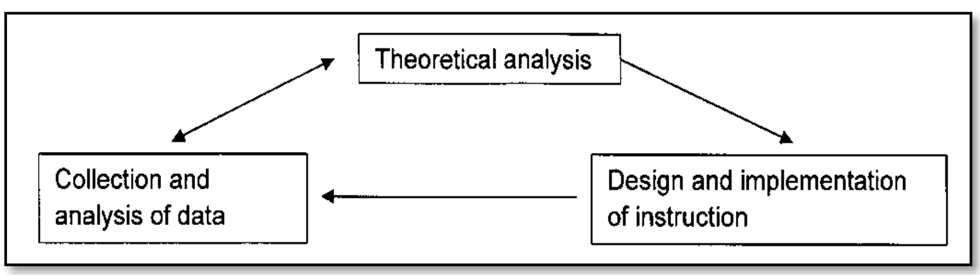


Figure 2. Cycle for APOS theory

From Asiala,M., Cottrill,J., Dubinsky, E., & Schwingendorf, K. (1997). The development of students' graphical understanding of derivative. *Journal of Mathematical Behaviour, 16*(4), 339-431

- The theoretical analysis: It is based on the researcher’s knowledge of the concept, APOS theory and on literature.
- The instructional treatment: It is based on the ACE teaching cycle (Activities, classroom discussions, and exercises). Thus, the instructor has to design activities/ handouts that give students opportunities to explore and construct their knowledge and think critically while working in groups followed by class discussions. After that, the students have to solve exercises and problems in order to reinforce the materials learned.
- Collection and analysis of data: It includes observations and assessment of students learning. In addition, it consists of interviews and written exams. The data is then gathered and analyzed. Based on the results , the data either concur with the theoretical analysis, or lead to some modifications and changes.

2.3 Concept Image and Concept Definition

Tall and Vinner (1981) defined concept image as "the cognitive structure that is associated with the concept which includes all mental pictures, properties and processes related and linked to the concept" (p. 152). It changes as the individual matures, and meets new experiences. It is worth noting that new conceptions, difficulties and misconceptions may come across whenever change occurs on existing concepts. Fischbein (1987) used 'intuitive' knowledge, which is similar to the concept image. According to Fischbein (1987), an intuitive knowledge includes ideas, beliefs, and mental pictures that are associated with the concept. It is characterized by being immediate, obvious, acceptable, and develops from experiences.

Concept definition refers to the words used by the individual to specify and define the concept (Tall & Vinner, 1981). It could be a personal definition or formal, which is the concept definition accepted by a mathematical community. Fischbein (1987) used the term 'formal' knowledge, which refers to the knowledge of the definitions, proofs, and axioms of the mathematical concepts. For example, the formal concept definition of derivative of a function f at a point $x = a$ is taken as" $f\,'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists". However, the concept image of the derivative of a function at a point may include other aspects, such as the idea of derivative as the instantaneous rate of change, the slope of the tangent line, and the slope of a curve. In addition, it may include the steepness of the graph of a function, the increase and decrease of a function, the relation between the graphs of a function f and its derivative $f\,'$, and the rules of differentiation (power rule, sum, difference, product and quotient rules). What exists in an individual's concept image, are none, some or all aspects of the derivative.

Reseach (Ferrini- Mundy & Graham, 1994; Habre & Abboud, 2006; Mahraji, 2013; Orton 1983) has shown that many students' concept images of functions and derivatives were dominated by the computational/symbolic representation (using rules and formulas). In other words, students' knowledge of functions and derivatives was procedural than conceptual.

The following sections present and discuss several studies on students' understanding of functions, limits, derivatives, and the role of technology and visualization in mathematics. Finally, a summary is presented.

2.4 Functions and Limits

Functions lie at the heart of calculus and are a central part of pre-calculus and calculus curriculum (Tall, 1997). They are also the building blocks for other important concepts such as limits, derivatives and integrals. Even though the function concept is a very important concept in mathematics, many students hold superficial understandings and serious misconceptions (Tall & Vinner, 1981).

Functions describe the relationships among varying quantities. They are used to represent and model real life situations and their applications are essential to many disciplines: physics, engineering, chemistry, medicine etc. For example, the area of a circle depends on the radius, the voltage depends on current and resistance, the speed depends on the distance travelled and time taken, and the amount of money you make depends on the number of days or hours you work and others. A function can be represented by an equation, a graph, a table or in words. A good conceptual understanding of functions requires making connections between different representational environments (Zandieh, 1998).

2.4.1 Studies on students' understanding of functions and their difficulties

There is a vast literature on students' understanding of functions and their difficulties, which has been documented in various studies (Breidenbach, Dubinsky, Hawks & Nicolas, 1992; Orton, 1983; Tall, 2011; Tall & Vinner 1981 and others).

Research has shown that many students have weak *concept image* of function, excluding some aspects of the function concept. Also, it has been reported that many students face difficulties when dealing with functions that are presented in a graphical form and exhibit a complete dependence on the use of equations or their algebraic expressions (Asiala et al., 1997). Tall and Vinner (1981) attribute part of the difficulties students encounter when working with the graphical form of functions to the traditional instructional methods. Moreover, Leinhardt (as cited in Asiala et al., 1997) added that the little time and practice spent on constructing or converting functions from graphs to tables or algebraic expressions may lead to the development of a partial concept image of function.

Vinner (1983), and Vinner and Dreyfus (1989) discussed understanding of functions in terms of concept image. In their study, Vinner and Dreyfus conducted a study to examine the concept image of functions among college students and high school teachers through seven questions administered to them through questionnaire. Students were asked first to define function and to determine if the given relations or mathematical correspondences were functions or not. The findings of the study revealed that the formulas come first to students' minds, and that they rely on algebraic formulas when dealing with functions. Moreover, researchers like Tall (2011) and Dubinsky and Harel (1992) used the terms *process* and *object* to describe the different levels of understanding of functions among students. The process, second stage of the APOS theory, is a mental process that occurs completely in the mind. Then, it becomes an object when the individual can perform actions on it. Sfard (1992) used the terms *operational* and *structural* and others used *static* and *dynamic* in the same way.

2.4.2 Studies on students' understanding of limits and their difficulties

The concept of "limit of a function" plays an essential role in grasping basic calculus notions such as: derivative, integral, and continuity. Students' understanding of limits can be also discussed using the constructs of *concept image* and *concept definition* (Tall and Vinner, 1981), multiple representations, and process or dynamic conception of limits.

Several studies showed that students face difficulties when dealing with limits. Some of the obstacles are due to representational means (verbal, visual, and symbolic) that students use when thinking of limits (Tall, 1992), prior knowledge, teachers' pedagogical style (Barbe, Bosch, Espinoza & Cascon, 2005), and others are related to the notion of infinity and continuity (Tall, 1992). In their study, Tall and Vinner (1981) reported that even when a student can give a correct static or dynamic definition of limit, his or her concept image of limit is not necessarily clear and may include some contrasting elements. Another study conducted by Ferrini - Mundy and Graham (1989) showed that students were able to evaluate the limit of a function $f(x)$ as x approaches a, but failed to give a geometric interpretation. This means that for many students the graphic and algebraic representations of a mathematical concept are not related and are seen as separate worlds.

2.5 Understanding Derivatives

The concept of derivative is one of the fundamental and important concepts in calculus. It is a concept built from other concepts such as functions and limits (Zandieh 1998).

Research on understanding derivative has shown that students have deficiencies in their conceptions of derivatives. More specifically, they face difficulties in understanding the derivative as a rate of change (Bezuidenhout, 1998; Orton, 1983), and understanding the graphical representation of derivative as the slope of the tangent line (Orton, 1983, Ferrini- Mundy and Graham 1994; Asiala et al., 1997; Vinner 1982). Moreover, students face difficulties in relating the derivative function with the original function (Orhun, 2012; Ubuz, 2007) and understanding the formal definition of derivative (Zandieh, 1998). Research has shown (Orton, 1983; Tall, 2011) that students' difficulties in learning derivative are due to their lack of the conceptual understanding of the concept.

Orton (1983) interviewed 110 college and precollege students to investigate their understanding of derivatives. Students were asked to perform some routine problems (finding derivative of functions using some derivatives rules) and do some conceptual tasks that include interpreting graphs, finding slopes graphically, graphical interpretation of both average and instantaneous rate of change, etc. Orton found that almost all students did well on the routine differentiation items such as finding the derivative of polynomial functions. However, he noticed that many students were not able to relate the derivative of a function at a point with the slope of the tangent line, nor to the limit of a set of secant lines. Other areas of difficulties are related to the ideas of instantaneous rate of change versus average rate of change. Orton also found that about 20% of students got confused with the derivative at a point and the y-value of the point of tangency.

Ferrini- Mundy and Graham (1994) conducted a study to investigate college students understanding of different calculus topics including derivatives. All of the students were interviewed and asked to think aloud while completing a set of tasks some of which were presented graphically and others algebraically. Graham and Mundy discussed in details one student's attempts to find the equation of a function presented graphically before sketching the graph of the derivative. Although this student was able to find the derivative of a function given its equation and sketch its curve, yet she had difficulties in other areas. She was unable to relate the derivative of a function to the tangent line, and she had no geometric meaning of differentiability. The researchers concluded, "Graphical contexts and algebraic contexts may function for students as separate worlds" (p.42).

Asiala et al. (1997) conducted a study to investigate the graphical understanding of a function and its derivative by university students. Forty -one students participated in the study where 17 students took a reformed calculus course and were taught using cooperative learning and computers, while the remaining took a traditional calculus course. Clinical interview conducted with each student consisted of eleven questions. Students were asked to justify their answers and reasoning while solving the questions. In one question, students were given only the graph of a function and a tangent line on some particular point, and they were asked to find the derivative at that point. Another question asked students to sketch the graph of a function based on some information given in a table form. Students' responses were analyzed according to APOS levels. The results showed that students who received the instructional treatment given in the reformed calculus course showed better understanding of the function and derivative concepts (at the process and object levels) than students in the control group.

In his study, Huang (2011) examined engineering students' conceptual understanding of the derivative concept. The sample of the study consisted of 35 students. They were exposed to the derivative concept using different modes of representation: symbolic (rules of derivatives), graphical and numerical (tables and rate of change). After the implementation of the unit on derivatives, a test composed of two test problems was given to students who were required to explain their thinking and problem- solving processes. Moreover, semi- structured interviews were conducted with some students. The results showed that almost 80 % of students' conception approached the object level. Another study that examined students' conceptual understanding of derivative, based on *APOS*, was that conducted by Maharaj (2013). A multiple-choice test composed of six questions was administered to 857 university students. Students were tested on applying the rules of derivatives and their applications (e.g. rate of change, interpretation the graph of a derivative and an optimization problem).The findings of the study revealed that students had troubles when applying the chain rule, solving the rate of change problem and interpreting the derivative of a function given graphically. The researcher concluded that the majority of students do not have adequate mental structures at the *process, object* and *schema* levels. It was suggested that more time should be devoted to instruction focusing on the numerical (rate of change) and graphical approaches of the derivative concept.

Concerning the formal definition of derivative, which includes the knowledge of the limit of the quotient difference or ratio, Zandieh (1998) conducted a study with nine high school students to explore their abilities to apply the definition in different representations. All students were interviewed and asked different questions on the concept of derivative. As to students’ responses, they were classified into three categories. Some students showed a good understanding of the formal definition of derivative, three students had memorized the definition with no understanding and they were not able to relate it to the notions of limit, ratio and functions, while two of them did not memorize the definition nor did they understand it.

2.6 Role of Visualization and Technology in Calculus and Derivative

Many researchers (Zimmermann 1991; Arcavi, 2003; Zarzycki, 2004, Rolka & Rosken, 2006; Presmeg, 2006; Ubuz 2007; & Natsheh &Karsenty; 2014) have investigated the role of visualization in mathematics as well as the strengths and difficulties associated with it. Arcavi (2003) defined visualization as follows:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about, developing previously unknown ideas, and advancing understandings. (p. 217).

This definition emphasizes the powerful role that visualization has in teaching and learning mathematics. Visualization is not just seeing pictures, graphs or any other visual tool, but it gives meaning, depth understanding, helps in problem solving and encourages discoveries. Giaquinto (2007) argues that visualization's role goes beyond just being a supportive role (demonstrating cases or providing examples for a definition etc), but it provides a mean for discovery, understanding and can be seen as a proof itself.

2.6.1 Difficulties around visualization

It has been said that "a picture worth of thousands of words." This proverb is true only if one is able to use it effectively. A student may be able to plot a graph or a draw a diagram, but he may not be able to extract information or use it to solve the problem. Interpreting graphs, tables, diagrams and other visual tools involves analytic and synthetic skills, intuition and sense of judgment. The difficulties around visualization and students' reluctance to use visual images have been widely investigated (Alcock and Simpson 2004; Arcavi, 2003; Guzman, 2002). Reasons behind this may be because many instructional and curriculum materials emphasize the symbolic approach over the visual one, and visual thinking demands higher cognitive thinking than the algorithmic one (Dreyfus and Eisenberg, 1991). In addition, many students and even teachers believe that a visual proof is not accepted or not considered a "real" proof. Another reason is that many students have weak visualization skills, or are not visualizers as defined by Presmeg (1986), "individuals who use visual methods in solving mathematical problems when there is a choice "(p. 298).

Since visualizing a mathematical concept is not an easy task, teachers should provide students with enough preparation, training and time. Students should be taught how to read, interpret any visual tool (graphs, tables, etc), extract information and translate it into other forms of representation (verbal using words, symbolic using equations etc). A student should have a repertoire of images, pictures, properties, relations, and different forms of representation associated with the concept as a musician who has a repertoire of melodies.

2.6.2 Technology and dynamic software

The calculus reform movement since 1980's has encouraged the integration of technology in the teaching and learning mathematics. Graphic calculators, computer programs and dynamic mathematical software (*GeoGebra, Maple, Cabri, Geometer's Sketchpad, Autograph and others)* have been used for teaching and learning many math concepts such as: polygons, triangles, quadratic equations, functions, limits, derivatives, integrals and others. The dynamic math programs allow users to create objects (points, lines, graphs etc), make measurements (angles, areas, slopes of lines etc), do transformations (rotation, reflections, symmetry, enlargement etc) and perform other manipulation of the selected or constructed object. In addition, the 'dragging' and 'animation' features of most dynamic programs provide students with an environment for discovery, experimentation, seeing patterns, generating and testing conjectures and visualizing mathematical objects (Gonzalez & Rodriguez, 2011; and Herceg, 2010). Moreover, most of these dynamic programs allow the user to visualize the concept in many representations (algebraic, numeric, and graphical). For example, a student can create a table of values, enter an equation and draw the graph of any function. Dynamic representations of mathematical objects allow learners to visualize mathematical problems or processes in ways that are not possible using paper and pencil (Sacristán et al., 2010). For example, the use of both the 'slope function' and 'slow plot' buttons, in *Autograph,* allow the user to plot gradually the derivative of the selected graph or equation showing the moving tangent on the original function.

2.6.3 Studies on the effects of technology on students' learning

Several studies have been conducted on the impact of technology (internet, simulation ion of the games, graphic calculators and dynamic programs etc) on students' math achievement, understanding and attitudes.

In their study, Simonsen and Dick (1997) found that graphing calculators enhanced students' conceptual understanding, and allowed them to be active learners and autonomous who construct their knowledge. Students were able to visualize the problems using multiple representations. In contrast, Porzio (1999) has shown that students who used graphic calculators did not show a better understanding of the concept. Heid (1984) conducted a study on two groups of college calculus students to investigate their understanding of functions, limits, integrals and derivatives. One group used dynamic computer software in the course while the other group did not. Heid collected her data based on the interviews she conducted with students in addition to their class work, assignments, quizzes and tests. Heid asked students to explain the meaning of derivative and interpret real world problems that include derivatives, and others. In general, the results showed that students in the experimental section held rich conceptual understanding, while students in the traditional section showed little and superficial understanding.

Habre (2006) investigated university students' conceptual understanding of a function and its derivative in an experimental calculus course. Students in the course were given the opportunities to use both the graphic calculator and *Autograph*, a dynamic mathematical software. At the end of the study, Habre noticed that some students struggled and faced difficulties in the course, while many enjoyed the course and showed a good understanding of derivative particularly the ideas of derivative as the slope of the tangent line and as the rate of change. However, Habre noted that even with the instructor emphasis on the graphical/visual approach of derivative, several students preferred the symbolic approach (equations, formulas).

Further studies have been conducted on the effects of dynamic software on students' achievement. Naidoo (2007) developed an interactive and dynamic module for teaching derivative. A group of 33 engineering students was taught using an interactive software, while 30 students were taught using the traditional lecture method. Students were tested on the ideas of average rate of change, instantaneous rate of change, limit of sequence and some rules of differentiation. Some students were clinically interviewed while solving the math tasks. Students' scores on the test were significantly different in favor of the experimental group. The findings showed that students in the experimental exhibited deep understanding of the concepts while the control group had superficial understanding and exhibited more structural errors compared to the experimental group. In their study, Zulnaid and Zakaria (2012) examined the effects of using dynamic software, *GeoGebra*, on the procedural and conceptual knowledge of functions. A total of 124 high school students participated in the study where 60 students were in the experimental group and 64 were in the control group. The difference in the mean scores between the two groups was significantly different at $p < 0.05$ in favor of the experimental group. Students who took lessons on functions using *GeoGebra*, showed better conceptual and procedural knowledge of the function concept, compared to the control group.

2.7 Summary of the literature analysis

Students' understanding of functions, limits and derivatives in particular, has been extensively discussed in many studies as cited in the literature analysis. It was shown that many students' conceptual understanding of calculus topics such as derivative is superficial and limited. Moreover, many of the difficulties that students face in learning calculus and derivative in particular are related to the emphasis on the use of symbolic approach, which places emphasis on rote memorization of formulas, rules and algebraic manipulation, on the expense of other modes of representations. Difficulties are also related to the difficulty in translations and connections among different types of representations of the derivative.

CHAPTER THREE

METHODOLOGY

The following section explains the design of the study and includes the tools and instruments that are used to collect data. After that, the theoretical framework of the study, based on *APOS* and multiple representations, is presented and discussed.

This study aims to examine how the concept of derivative is presented and developed in two different calculus books (Book 1, which emphasizes the formal symbolic approach of the concept, and Book 2, emphasizing the multiple- representation approach of the concept).The second purpose is to investigate the types of difficulties students face when learning derivative. Finally, it aims to compare the students' conceptual understanding of derivative in two Calculus I groups: a control group learning the derivative concept in a computer- free environment and using Book 1, and an experimental group learning the derivative concept using Book 2 and activities (paper- pencil and *Autograph*- based).

The triangulation through the use of different sources and instruments for data collection (observation, questionnaires, tests, interviews) increase the validity of the study.

3.1 Research Context and Calculus Courses

This study is conducted at a private Lebanese university adopting an American program. The university is considered among the top universities in Lebanon. This university offers both undergraduate and graduate programs in Arts, Sciences, Business, Pharmacy, Nursing, Engineering, and Medicine.

The university offers four calculus courses: Calculus I, Calculus II, Calculus III, and Calculus IV. Several factors determine which calculus course is required from students such as students' scores in their entrance exams (SAT), school background, and their future field of study. Students who have passed the Humanity/ Economic sections of the official Lebanese Baccalaureate, or students who are holders of a high school diploma have to take calculus I course. In calculus I, students take the following topics: functions, limits, continuity of functions, the derivatives of functions and their applications.

The topic chosen in this study is the "Derivative Concept" which is taught in Calculus I course.

3.2 Participants

The study is conducted during two consecutive semesters, the fall of 2013 and the spring of 2014. In fall, the researcher observed two sections of Calculus I offered by instructor X. In spring, only one section of Calculus I was offered which was taught by a different instructor (hereafter referred as instructor Y). Instructor X is an assistant professor of Mathematics at the university, while instructor Y is a holder of an MA degree in mathematics education who has experience in using a visual, technology-based multiple-representation approach to teaching calculus.

3.2.1. Sampling method

The total number of students is 52 whose age ranges between 17 and 20 years old. Two groups of students are considered: a control group and an experimental group. The two groups have equal sizes. Twenty-six students, including 13 males and 13 females, are selected from the two calculus I sections offered at fall to form the control group. These students are taught by instructor X (fall 2013).The same number (26), including 14 males and 12 females, is selected to form the experimental group, which is taught by instructor Y (spring 2014). It is worth noting that the selection of students took into consideration those who completed all the tests and questionnaires given in the study.

Most of the participants are Lebanese except for six students. There are four students from Syria, one student from Al Bahrain, and one from Saudi Arabia.

Concerning the ethical considerations of the study, the researcher gets an official permission from the university (Appendix). She provides the participants with a written letter explaining the purpose of the study and its importance ensuring confidentiality and anonymity of the participants.

3.2.2 Students' mathematical background

Assessing students’ prior knowledge is important to check whether the two groups are comparable. It is worth noting that the researcher was able to get students' scores on the SAT from the registrar office of the university. Students' scores on the SAT of the math section are approximately normally distributed (whose distribution forms a normal bell curve). The mean score on the SAT for the control group is M =

525.7 with a standard deviation SD = 55. On the other side, the mean score for the experimental group is lower with M = 509.2 and a standard deviation SD = 57.8 indicating that students varied widely in their achievements. However, the results of the independent t-test showed that difference in achievement between the two groups is not statistically significant at $\alpha = 0.05$ level of significance.

Moreover, a diagnostic test (Appendix A) was administered to all students at the beginning of each of the semesters (fall 2013 and spring 2014) to assess their previous knowledge. The results show that both groups (control and experimental) are approximately of the same level of achievement.

For instance, when students were asked to find the domain of definition of two functions ($f(x) = \frac{1}{x^2+x-6}$ and $f(x) = \sqrt{-2x+8}$), 50 % of the answers in the control group, compared to 54 % in the experimental group, were incorrect. Also, when asked to find the rate of change of a function ($f(x) = x^2$) over a given interval, the percentage of incorrect answers reached 62 % in the control group, compared to 66 % in the experimental group. Moreover, when asked to find the cosine and tangent of an angle given its sine (*sin x= 3/5)*, 66% of the answers in the control group, compared to 68 % in the experimental group, were wrong. In one question, which requires finding the equation of the line tangent to the curve of $f(x) = x^2 - 5x + 6$ at x = 5/2, only four students in the control group, compared to two students in the experimental group, had correct answers. This question, which tests students' understanding that the derivative of a function at a given point is equal to the slope of the tangent line at that point, indicates that the majority of students have poor graphical understanding of the derivative concept. Finally, when students were asked to find the derivative of three different functions given their equations, it is noticed that 62 % and 66 % of the answers were incorrect in the control and experimental group respectively. This question tests students' procedural knowledge of using some rules of differentiation (constant rules, power rule, and sum / difference and product rules).

In general, the results of the diagnostic test suggest that many students are weak in math and in need of a pre-calculus course.

3.3 Procedures

Several procedures were followed to accomplish this research.

3.3.1 Diagnostic Test

A diagnostic test (Appendix A) was administered to both groups to assess their mathematical background and record their weaknesses and gaps. The test includes nine questions that cover the following topics: quadratic equations, domain of definition of functions, rate of change, trigonometry, lines, parabola, limits, and derivative. The test took place during the lab time of the course (once per week), and students were given 65 minutes to complete it. The researcher was present to make sure that everything was clear and to answer any technical questions not related to the solutions.

3.3.2 Two calculus books (Book 1 and Book 2)

Book 1, which emphasizes the symbolic approach of the derivative concept, was used in the control group (fall 2013). In the experimental group (spring 2014), Book 2 was used which, emphasizes the multiple- representation of the concept. A general comparison of the development of the derivative concept in the two books (Book 1 and

Book 2) is provided. Then, the content and the structure of each of the four selected sections on derivative, in the two books, are discussed. Finally, the modes of representation (symbolic, graphical, and numerical) used in Book1 are discussed and compared to those used in Book 2. Section 4.1 provides detailed analysis of the books.

3.3.3 Derivative Questionnaire

A questionnaire on *"Students' perceptions of the notion of derivatives"* (Appendix B) was administered to all students before the implementation of the unit on derivatives. The questionnaire was developed by the researcher, and it consists of three parts (I, II & III). The first part (I) is an open-ended question that asks students to freely write what they know about derivative. Parts (II) and (III) are a multiple – choice items that ask students to choose the correct statements. Students took 15 minutes to complete the questionnaire.

The questionnaire aims to record students' dominant image of the concept of derivative. In addition, it aims to examine students' conceptions related to derivative. The questionnaire was administered again, immediately after the implementation of the unit on derivative to check the development and the progress of students' conceptions of derivative. Data are analyzed both qualitatively, and then quantitatively analyzed using the test. The null hypothesis claims that there is no significant difference in the means between the two groups.

3.3.4 Implementation of the derivative unit

In the control group (fall 2013), four sections on derivative are selected from Book 1, which address the concept derivative using different modes of representation.

The three modes of representation include symbolic (formal definition of derivative using the limit of quotient of differences and denoted by, $f'(x) = \lim_{h \to 0}(f(x + h) - f(x))/h$), geometrical (as the slope of a curve or the slope of a tangent line), and numerical (as the instantaneous rate of change). The emphasis, however, is on the symbolic approach (using algebraic expressions, equations and the formal definition of derivative).The implementation of the unit was carried out over five sessions of work (50 minutes each). Homework was usually assigned at the end of each teaching session. The homework exercises and problems are selected from Book 1. It is worth noting that there is no group work nor is there any use of technology. The lecture method was frequently used; the teacher explains the lessons, students take notes and answer questions when needed. Section 4.2.1 describes the flow and the content of sessions addressed by instructor X during the teaching of derivative.

On the other side, four sections on derivative are selected from Book 2, which also address the derivative concept using different modes of representations. The three modes of representations are equitably used, with more emphasis on the graphical representation. The four selected sections in Book 2 cover almost the same objectives as those sections selected in Book 1. The implementation of the unit was carried out over 6 sessions and a half of work (50 minutes each) because group work and technology were used, which required more time for technical support. The researcher with the cooperation of instructor Y designed activities (Appendix F) that encourage the exploration of the derivative through different representations (tables, graphs, formulas, word problems). Moreover, the activities were designed to guide students to promote the formation of the mental structures (*Action- Process- Object- Schema*) as described in the genetic decomposition of the derivative concept (see section 3.3.7). It is worth noting that the teaching sessions were interactive where students felt free to ask questions, listen to others and express their ideas. During the activity part, the teacher encouraged discussions among group members and sharing results before having class discussion and presentation. The activities were then followed by homework exercises that were similar to the activities. Section 4.2.1 describes the flow and content of sessions addressed by instructor Y during the teaching of derivative. It is worth mentioning that the pedagogy used in the experimental group (cooperative learning, technology and activities) was only done in the derivative chapter. The approach was traditional in the earlier chapters (functions, limits, continuity).

It is important to note that the lack of activities in the control group is due to several factors. First, Book 1 seems to follow a traditional approach where the definitions of the concepts are directly stated at the beginning of the *Text* without giving students the opportunities to discover and be active learners. Moreover, in Book 1, the dominant mode used in the given part of the exercises is the symbolic representation (using algebraic equations and formulas) which are used as tools to find the equation of the derivative function, slope of a curve at a given point or slope of a tangent line at some point and others. However, the graphical and numerical representations are used mostly as an end and not as teaching tools.

3.3.5 Observations

During the implementation of the unit on derivatives, the researcher is present in both groups (control and experimental) to observe the instructional method/s and strategies (lecture, group work, technology, individual work) used in the classroom as well as students' participation and work, using an observation work log (Appendix C). In addition, a major attention is directed toward the different representations of the concept of derivative that are emphasized or de-emphasized in the two groups. The researcher took detailed notes about how each lesson was conducted. Moreover, at the end of each teaching session, notes were taken about students' opinions and attitudes toward the teaching approach used.

3.3.6 Derivative Test

After the implementation of the unit on derivatives, a common written test (Appendix D) was adminstered to all students. It includes five conceptual – understanding based problems on derivatives, constructed keeping in mind the *APOS* levels. Four questions out of five are developed by the researcher, and one question (question I) was adopted from a study by Asiala et al. (1997). In this test, students were asked to justify their answers and reasoning and were given 65 minutes to complete it. The main characteristic of the test is that only graphs and table of values of the functions are given without their algebraic expressions. The purpose of the test is to identify the types of difficulties that students face in relation to the notion of derivative. In addition, it aims to assess students' ability to translate among the different representations of derivatives. The researcher was present to make sure that everything was clear and to answer any technical questions not related to the solutions of the questions. Students' responses on the test are qualitatively analyzed based the apriori analysis provided in section 4.4.1. Moreover, a quantitative analysis using descriptive and t-test is applied to check whether the difference in the mean scores between the two groups is significant.

The null hypothesis claims that there is no significant difference in the performance between the two groups.

3.3.7 Interviews

Interviews are conducted with 12 students while solving the derivative test. The purpose of the interview is to obtain a clear, explicit and better picture of students' conceptual understanding of derivative. Students were asked to think aloud while solving the test. Moreover, the interviewees were asked about their educational background, their opinions toward math, their opinions toward the teaching approaches used in their classes, and finally about the meaning of the derivative. The interviewed students include five males and seven females who have different levels of achievements. Even though the researcher explained to the interviewees the purpose of the interview, and assured them both confidentiality and anonymity, she could not audiotape nor videotape the interviews upon their request.

3.4 Theoretical Framework

The framework that is used in this study is based on two main components: Multiple representations and APOS Theory. These two components allow the researcher to study the derivative concept from two different but complementary viewpoints. A discussion of the two ideas is presented with reference to the works that influenced my ideas.

3.4.1 Representations

Types of representations. This study distinguishes three types of representations of derivative: numerical, graphical, and symbolic. The concept of derivative can be viewed numerically using the rate of change, which refers to the instantaneous rate of change of a function at a point, which is equal to the average rate of change over small interval of time. In such a case, data is best represented in a table form or in ordered pairs (e.g. (1, 3) and (-4, 2)). Second, the graphical representation of derivative can be viewed as the slope of the curve of a function at a particular point, or as the slope of the line tangent to the curve at the particular point. Finally, the concept of derivative can be viewed symbolically using the formal definition of derivative ($f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$). In such a case, the equation of the function is given and the derivative is calculated by applying the formal definition.

Representations and translations. The representation of derivative is recognized in two different ways. First, it is identified by the language, words, symbol or phrase used. For example, the phrase "slope of tangent line" indicates the graphical representation of derivative. Similarly, the phrase "instantaneous rate of change" designates the numerical representation of derivative and the word "derivative" or the symbol $f'(x)$ in a question denotes the symbolic representation. Second, the representation of derivative is identified by the procedures or steps carried out. For example, finding the slope of a curve at a particular point leads to the graphical representation of derivative, while finding the rate of change through the calculation of the difference quotient ($\frac{f(x+h)-f(x)}{h}$) of two close points results in a numerical representation of derivative.

Connections and translations among and within these representations are an integral impart in understanding derivatives (Ferrini- Mundy and Graham, 1994; Orton, 1983; Zandieh, 1998). Translation can take place from one mode of representation to another, or it occurs within the same mode of representation (Janvier, 1987). In this study, the researcher is interested in investigating whether students are able to translate from one mode of representation to another and/or to work within the same mode of representation.

For example, consider the following question from the derivative test (Appendix D): Given $F\ (x)$; Let $G(x)\ = F(x) + C$, where C is any constant. *Clearly $G' = F'$ since $\frac{d}{dx}[C] =\ 0$ (derivative of a constant = 0). Now, explain geometrically why the two derivatives are equal.*

In this question, it is proved symbolically that the derivative of the two functions are equal because the derivative of a constant is zero. It is required to graphically interpret this fact. One way to prove it graphically (geometrically) is to recognize that the function $G(x)$ is obtained by shifting the graph of F vertically C units (either upwards or downwards depending on the sign of C). In addition, the slopes of the tangent at any point a on the graph of F and G are the same since the graphs are identical and just translated along the vertical direction, thus the tangent lines are parallel. Therefore, answering this question requires relating the derivative with the slope of the tangent line or slope of the curve. Therefore, this question involves translation from symbolic to geometric representation (S → G).

3.4.2 Genetic decomposition of the *Function* Concept

The concept of function can be represented in different forms: symbolically (equations, algebraic expressions), numerically (table form) and graphically (graph) and others. Since the concept of *derivative* is built on the concept of *function*, then it is reasonable to provide a brief genetic decomposition of the function concept based on *Action- Process- Object* levels of *APOS*. For constructing this genetic decomposition, the researcher referred to the work done by Breidenbach *et al* (1992) and Asiala et al. (1997). Students' inability to grasp the process and object perspectives of functions make it difficult for them to understand the derivative concept (Asiala et al, 1997).

***Action*:** An individual is restricted to an *action* conception of the *function* concept if he / she is unable to interpret a situation as a function unless he/ she has a formula/ algebraic expression of the function; for such students, having a table or a graph that represent a function without an explicit equation is meaningless. Students who have an action conception of function think of it as an expression that evaluates something when numbers are plugged into the equation.

***Process*:** An individual who has a *process* conception of a function, thinks of a function as a machine that maps an input to a corresponding output. In other words, whether the function is given graphically, symbolically or numerically students understand the $f(x)$ notation as linking x and y values: for every x input, the function has a corresponding y value. Also, students who have a *process* conception of functions can identify if a function is one-to-one function, or onto functions etc. A function is called one- to- one if every element in the range of the function corresponds to exactly one element of the domain. A function f from a set B to set C is onto if every element c in C has a corresponding b in B.

Object**:** An *object* conception of function occurs when an individual can think of a function and any of its representations as entities. For example, students should interpret a table of values or graph as representing functions. Moreover, students can perform actions or processes on the object and indicate, for example, the interval where the function is increasing, decreasing, constant, etc. Students with *object* conception can deal with composite functions, piece- wise functions, etc.

3.4.3 Theoretical Analysis (Genetic decomposition) of Derivative

In this section, the researcher presents a genetic decomposition, in the sense of APOS theory, for the concept of derivative. Asiala et al. (1997) have used two paths, graphical and analytical, in their study of students' graphical understanding of the derivative. The graphical path is related to the geometric meaning of derivative, while the analytical path includes both the symbolic and numerical representations of derivative. Therefore, the researcher will adopt these two paths (Asiala et al., 1997, p. 7), but some changes and modifications are included.

1a: Graphical-*Action*: Joining two points on a curve to form a line (secant line) and calculating the slope of the secant line passing through the points.

1b: Analytical- *Action*: Calculating the average rate of change of a function over an interval .In such a case, the data is given in table form or through the equation.

2a: Graphical- *Process*: Forming the process, as the two points on the curve get "closer and closer" together.

2b: Analytical- *Process*: Forming the process as the size of the interval gets "smaller and smaller".

3a: Graphical- *Object*: Encapsulation of the process in point **2a** to produce:

- Tangent line as limit position of the secant lines
- Slope of the tangent line to the curve at a point as the limit of slope of secant lines as the points get closer

3b: Analytical- *Object* : Encapsulation of the process in point **2b** to produce :

- the instantaneous rate of change at a point as the limit of the average rate of change over a very small interval.

By number 3, the object level is attained. The coming numbers 4, 5, and 6 are also considered at the object levels for several reasons:

- Point 4 links and connects the three representations of derivative; the paths are coordinated.

- Point 5 uses the derivative of a function at a point a for approximating the function linearly near that point. (*Linearization*)

- Points 6a and 6b extend the definition of derivative from a single point to the derivative function.

4. Object: Coordinate several interpretations of $f'(a)$. Students should be able to relate the formal derivative of a function at a point ($f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$)with the slope of the tangent line and the instantaneous rate of change at that point. Students should be able to explain the relationship between the three representations. Also, students should be able to move between these interpretations. If students just stated the relationship as a result of a remembered fact without showing any understanding, or if they demonstrated no ability to de- encapsulate it back to the process from which it came, then they don't have an *object* conception of derivative. Instead, they tend to have *action/ process* conception.

5. **Object:** Encapsulation of the processes in points **2a** and **2b** to find the value of the function $f(x)$ at a point that is very near to $x = a$ by approximating the function $f(x)$ using the linear equation *L(x)*. The function is approximated to a linear function L(x) as following : $f(x) \approx L(x) = f(a) + f'(a)(x-a)$.

6a: Graphical- *Object*: producing the derivative function f' by:

- **6a.1:** Producing the derivative at a point. That is seeing the derivative as the correspondence: $a \to$ slope of tangent line at the point $(a, f(a))$
- **6a.2:** Encapsulation of the process in point **6a.1** to extend the concept of derivative from a single point to the derivative function. That is seeing derivative as a function itself.

6b: Analytical - *Object*: producing the derivative function f' by:

- **6b.1:** For any input a, the derivative function $\boldsymbol{f}\,'$ produces the output $f'(a)$. That is seeing the derivative as the correspondence $a \rightarrow$ instantaneous rate of change at the point $(a, f(a))$
- **6b.2:** Encapsulation of the process in point **6b.1** to extend the concept of derivative from a single point to the derivative function (students can use the formal definition of derivative).

7a: Graphical-*Schema*: Using the concept derivative and the constructions mentioned before, students should be able to:

- Interpret the graphs of $\boldsymbol{f}$ and $\boldsymbol{f}\,'$ and explain how they are related. In other words, use the relations between properties of functions and derivatives; discussing monotonicity of the function and sign of the derivative. If students mentioned the relationship between the variation of a function over an internal with the sign of its derivative based on a memorized fact without being able to explain the relationship using, for example, steepness of a function, rate of change or slope of tangent line, etc., then they are not considered to have a good schema. They tend to have an *action/ process* conception.
- Operate on the derivative graph as representing a function.
- Use the idea of steepness of the tangent or rate of change to indicate whether the derivative is increasing or decreasing.
- Relate the roots of $\boldsymbol{f}\,'$ with the critical points of $\boldsymbol{f}$ (maximum / minimum)

- Determine conditions for differentiability and cases where a function fails to have a derivative at a point (*corner, cusp, vertical tangent, discontinuity*)

8b: Analytical -*Schema*: Using the *derivative* concept and the constructions mentioned before, students should be able to:

- Relate between the sense of variation of a function $\boldsymbol{f}$ (increasing, decreasing) over an interval with the sign of the derivative function $\boldsymbol{f}'$ (positive, negative**)** given symbolically (as equations) or through table format. There should be an indication of understanding the relationship between $\boldsymbol{f}$ and $\boldsymbol{f}'$ and not just based on memorization.
- Find and discuss the nature of critical points of $\boldsymbol{f}$ **(**maximum/ minimum), not just based on a memorized fact. Students should be able to explain their reasoning based on the rate of change of $f(x)$ or something else.
- Interpret, analyze and solve problems that involve rate of change (velocity, acceleration, population change, volume change, etc.). (Asiala et al., p. 7).

CHAPTER FOUR

FINDINGS

This chapter consists of six parts ($4.1 \rightarrow 4.6$) which represent the findings of the study. The first part includes an analysis of the chapters' sections on derivative as presented in two calculus books: Book 1 emphasizing the symbolic approach and Book 2 emphasizing the multiple-representation approach. The second part presents an analysis of the exercises, activities and observations of the sessions on derivative that occurred during fall 2013 (control group) and spring 2014 (experimental group). The third part includes an analysis of the data collected from the derivative questionnaire. A comparison of students' responses on the derivative test in the two groups is discussed in part four. Part five includes an analysis of the data collected through the clinical interviews of 12 students (six students in each group). The last part presents the summary of this chapter.

4.1- Content Analysis of the Calculus Books

The textbook used in the control group is referred as Book 1, while the book adopted for the experimental group is referred as Book 2. First, a general comparison of the development of the derivative concept, in the two books, is presented. Then, detailed comparisons of the sections on derivative, based on specific criteria, are discussed.

4.1.1 General comparison of the approaches in the two books.

The approaches adopted in the two books are different. Book 1 follows the sequence: formal definition of derivative → slope of curve or slope of tangent line → rules of differentiation such as $(x^n)' = nx^{n-1}$, product and quotient rules etc → rate of change as an application of the rules. However, Book 2 follows the sequence: derivative as a rate of change → slope of curve or slope of tangent line → formal definition of derivatives → rules of differentiation.

4.1.1.1 Global development of the derivative concept in Book 1.

In general, Book 1 emphasizes the symbolic approach of the concept (using algebraic formulas and equations), and includes to a lesser extent other forms of representations (graphical and numerical). In this book, the derivative concept is introduced in chapter 3 (*DIFFERENTIATION*). Chapter 3 includes nine sections (3.1 →3.9) which are as follow:

- 3.1: Tangents and the Derivative at a Point (Formal definition)
- 3.2: The Derivative as a Function
- 3.3: Differentiation Rules
- 3.4: The Derivative as a Rate of Change
- 3.5: Derivatives of Trigonometric Functions
- 3.6: The Chain Rule
- 3.7: Implicit Differentiation
- 3.8: Related Rates

- 3.9: Linearization

However, in this book, only four sections are considered. The selected sections are:

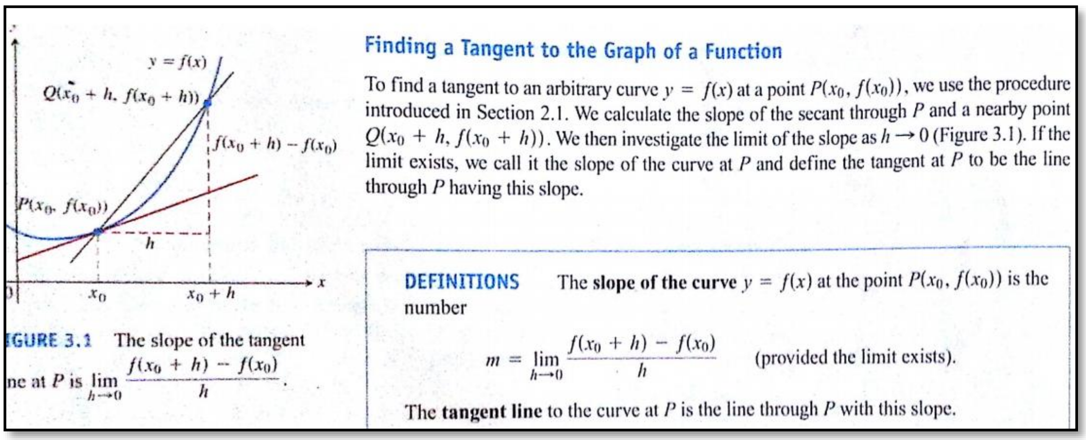
- 3.1: Tangents and the Derivative at a Point (Formal definition)
- 3.2: The Derivative as a Function
- 3.4: The Derivative as a Rate of Change
- 3.9: Linearization

Choice of sections in Book 1. The researcher is interested in the selected four sections because they address the *concept of derivative* using different modes of representation: symbolic (formal definition of derivative using the limit of quotient of differences), geometrical (as the slope of the curve or the slope of a tangent line) and numerical (as a rate of change). The remaining sections include "rules of differentiation which are not part of this study. On the other hand, the researcher is interested in investigating students' conceptual understanding of the derivative concept more than their procedural knowledge of the rules and formulas (e.g., $(x^n)' = nx^{n-1}$, $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}[f(x).g(x)] = f'(x)g(x) + g'(x)f(x)$ and others) which are used for finding the derivative of a given function (constant function, power function, polynomial, rational function, trigonometric function and a combination of them).

Development of the derivative concept in Book 1 (overview of all sections even those not included in the study). Section 3.1, introduces derivative symbolically by defining the derivative of a function $f(x)$ at a point *a* by $f'(a) = \lim_{h\to 0} \frac{((f(a+h) - f(a))}{h}$ provided the limit exists, progresses to do examples on the definition, and then discusses its geometric (graphical) meaning as the slope of tangent line to the graph of f at *a*

(Figure 3). In section 3.2, the concept of derivative is then extended from a single point to the derivative function using mainly the formal definition

($f\,'(x) = \lim_{h\to 0} \frac{((f\,(x+h) - f\,(x))}{h}$, provided the limit exists). Section 3.3 introduces some rules of differentiation (constant rule, power rule, sum /difference rules, and product and quotient rules). Most of the rules are followed by their proofs using the



IGURE 3.1 The slope of the tangent ne at P is $\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

formal definition of derivative.

Figure 3. Formal definition and slope of tangent line, from Book 1

Later, in section 3.4, as an application of the derivative rules, the derivative is interpreted as the instantaneous rate of change of a function. For example, consider *Example 4* from section 3.4 in Book 1 (Figure 4).

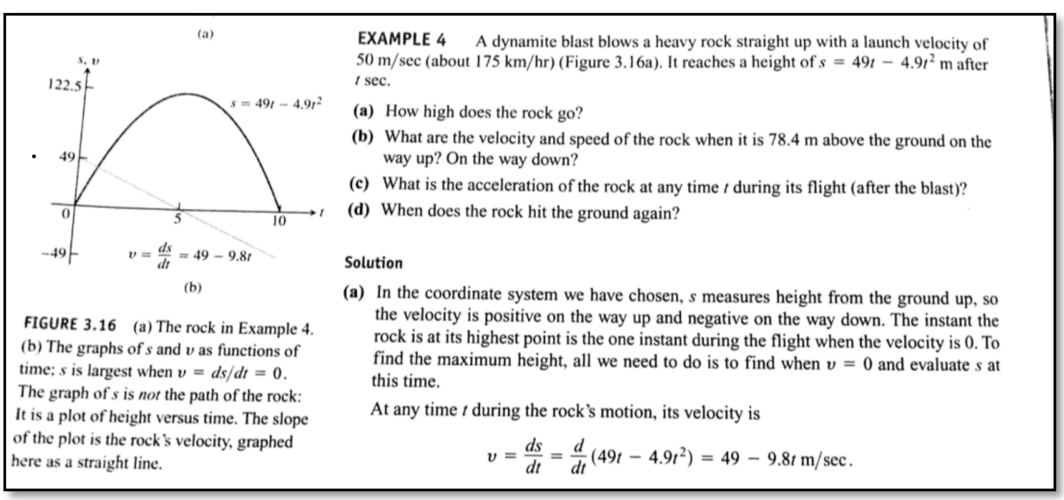


FIGURE 3.16 (a) The rock in Example 4. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

EXAMPLE 4 A dynamite blast blows a heavy rock straight up with a launch velocity of 50 m/sec (about 175 km/hr) (Figure 3.16a). It reaches a height of $s = 49t - 4.9t^2$ m after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution

(a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t during the rock's motion, its velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(49t - 4.9t^2) = 49 - 9.8t \text{ m/sec}.$$

Figure 4. Rock's height, velocity and acceleration, from Book 1

As shown in Figure 4, the equation of the rock's height is given. The question requires finding the rock's velocity, speed and acceleration with respect to time. The velocity is obtained by differentiating the function that represents the rock's height in terms of time, and acceleration is obtained by differentiating the velocity (or double differentiating the height's function).

After that, through sections 3.5 to 3.8, other rules of differentiation are given (chain rule, derivative of trigonometric functions). Finally, in section 3.9, the *linearization* concept is introduced, where any differentiable function, say f, at point a is approximated using a linear function (For example, $L(x) \approx f(x) = f(a) + f'(a)(x - a)$). Later, as an application of the derivative concept (chapter 4 in the book), students are given sets of theorems such as "If the derivative of a function in some interval is negative, then the function decreases in that interval."

4.1.1.2 Global development of the derivative concept in Book 2**.**

The approach used in Book 2 is different. This book includes numerical, graphical and symbolic representations of derivative and emphasizes the translations among and within these modes. The title of this book, *"from Graphical, Numerical, and Symbolic Points of View"*, reflects the multiple- representation approach adopted. In this book, derivative is explained in chapter 1 and chapter 2. Chapter 1 introduces the derivative concept informally using two approaches, which are rate of change and a graphical approach, in terms of slopes. Then chapter 2 presents the formal definition of derivative and the symbolic techniques for calculating derivatives. Chapter 1 includes seven sections (1.1→ 1.7) and chapter 2 includes seven sections (2.1→ 2.7) which are the following:

CHAPTER I: FUNCTIONS AND DERIVATIVES: THE GRAPHICAL VIEW

- 1.1: Functions, Calculus Style
- 1.2: Graphs
- 1.3: A Field Guide to Elementary Functions
- 1.4: Amount Functions and Rate Functions: The idea of the Derivative
- 1.5: Estimating Derivatives: A closer Look
- 1.6: The Geometry of Derivatives
- 1.7: The Geometry of Higher- Order Derivatives

CHAPTER II: FUNCTIONS AND DERIVATIVES: THE SYMBOLIC VIEW

- 2.1: Defining the Derivative (formal definition)
- 2.2: Derivatives of Power Functions and Polynomials
- 2.3: Limits
- 2.4: Using Derivative and Antiderivative Formulas
- 2.5: Differential Equations; Modeling Motion
- 2.6: Derivatives of Exponential and Logarithmic Functions; Modeling Growth
- 2.7: Derivatives of Trigonometric Functions; Modeling Oscillations

However, in this book, only four sections are considered. The selected sections are:

- 1.4: Amount Functions and Rate Functions: The idea of the Derivative
- 1.5: Estimating Derivatives: A closer Look
- 1.6: The Geometry of Derivatives

- 2.1: Defining the Derivative (formal definition)

Choice of sections in Book 2. The four selected sections in Book 2 cover almost the same objectives as those sections selected in Book 1. In these sections, the *concept of derivative* is approached using different modes of representation: numerical, geometrical and symbolic. The other sections include rules for differentiation, which are not of the researcher's interest.

Development of the derivative concept in Book 2. Section 1.4 introduces the idea of derivative informally as a "rate function", as opposed to the "amount function" for which it is the derivative. It first discusses the rate of change of a function using a real life example involving distance, velocity and acceleration of a moving car. Figure 5 presents the definition of derivative, informally as a rate function.

DEFINITION (The derivative, informally, as a rate function) Let f be a function. The new function f', called the **derivative** of f, is defined by the rule

$$f'(x) = \text{instantaneous rate of change of } f \text{ at } x.$$

We'll sometimes use "derivative" and "rate function" interchangeably to describe f'. When we do, we'll call f an **original function** or an **amount function**. We'll revisit and improve this definition in later sections. We'll explain, for instance, just what "instantaneous" means, and we'll consider whether a given function f *has* a derivative f'. But first we'll take derivatives out for a drive.

Rates, amounts, and cars: derivatives on the road

At noon on October 1, 2000, a car enters Interstate Highway 94 from the State Capitol on-ramp in Bismarck, North Dakota. For several hours, the car moves—sometimes east, sometimes west—at varying speeds.

Functions, derivatives, and the prime notation offer a powerful language for describing the situation. First, we define some symbols:

- t: the elapsed time, measured in hours since noon;
- $P(t)$: the car's position at time t, measured in miles *east* of the State Capitol on-ramp;
- $V(t)$: the car's *eastward* velocity at time t, measured in miles per hour (mph).

Figure 5. Derivative as rate of change, from Book 2

Two things are noticed from Figure 5. First, the fact that from the very beginning the, derivative is defined as another function, and second the developmental aspect; the book addresses the students as mathematicians and partners in developing the definition. Then, the relationship between amount functions and rate functions (the derivative of amount functions) are discussed through other examples (rate of change of balloon's height, bank deposit, volume of water in tank etc.). Then, the instantaneous rate of change of a function at a point is related to the slope of a curve at that point, which is equal to the slope of the tangent line. The slope is defined as a rate (ratio of two changes; rise / run). In other words, it is a measure of how fast one variable (y) changes with respect to another (x). It is noticed that two examples are used to explain how derivative is related to the slope of the curve. The first one is related to the derivatives of linear functions, and the second is related the derivative of a curved function. The second example demonstrates how to calculate a curved graph's slope at a point by drawing a tangent line at that point, and then finding its slope using any two points on the line.

Section 1.5 uses the strategy of "Zoom in" graphically and numerically at any point to estimate the derivative of a function at that point. *Example 2* of section 1.5 requires estimating the derivative of the function $f(x) = x^2 \ at \ x = 1$ by zooming in on the graph near the point (1, 1) and calculating $f\,'(-1.5)$ by zooming in numerically (Figure 6).

Then, the concept of derivative is extended from a single point to the derivative function; that is seeing derivative as the function: $x \rightarrow$ slope of tangent line at the point $(x, f(x))$. Section 1.6 describes how the graphs of f and its derivative function $f\,'$ are related, and what each tells about the other. For example, the sign of the graph of the derivative function f' (positive/ negative) tells whether the graph of f is increasing or decreasing. Also, if f has a maximum or a minimum point at x = c and f' is defined, then $f'(c) = 0$.

Solution Zooming in successively on the f-graph near (1, 1) gives the results shown in Figures 2(a) and 2(b).

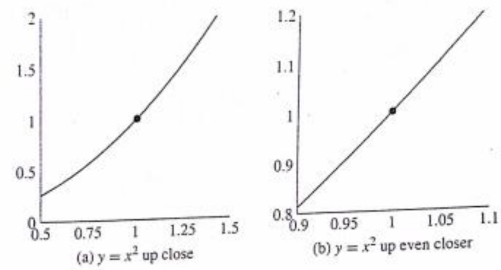


FIGURE 2
Closer and closer looks

In the right-hand picture the graph looks virtually straight; its slope seems to be about 2. Thus, the (approximate) tangent line has slope 2 and passes through (1, 1); its point–slope equation is therefore $y = 2(x-1)+1$.

To "zoom in numerically" near $x = -1.5$, we can look at a table of closely-spaced values of f:

Values of $f(x) = x^2$ near $x = -1.5$							
x	−1.8	−1.7	−1.6	−1.5	−1.4	−1.3	−1.2
$f(x)$	3.24	2.89	2.56	2.25	1.96	1.69	1.44

Figure 6. Example of " zooming in" graphically and numerically, from Book 2

Finally, in section 2.1 the derivative of a function f at $x = a$ is defined symbolically, using the limit of difference quotient, by $f'(a) = \lim_{h\to 0} \frac{((f(a+h) - f(a))}{h}$. This definition is deduced and approached using two problems; the -rate- of- change problem (Figure 7) and the tangent - line problem (Figure 8). In the 'rate problem', the question requires interpreting the given table to estimate Professor X's speed at t = 2:00 pm. The average speed over the interval [2.00, 2.05] which includes t = 2 should be close to the instantaneous speed at t = 2. Therefore, $D'(2) = \frac{D\,(2.05) - D(2)}{0.05} = \frac{3.54}{0.05} =$ 70.8.

The rate-of-change problem

EXAMPLE 1 At exactly noon, Professor X → began a 2.5-hour trip in her trusty but aging automobile. The clock and trip mileage meter still work, but the speedometer gave out long ago. Worried about speeding (the limit is 70 mph), the compulsive professor recorded her travel time t (in hours) and her distance traveled $D(t)$ (in miles) every 3 minutes during the trip. Here are some of her data:

Professor X's time and distance data

t (hours)	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20
$D(t)$ (miles)	66.60	69.84	73.15	76.54	80.00	83.54	87.15	90.84	94.60

Figure 7. Professor's X trip, distance traveled with respect to time, from Book 2

In the' tangent- line problem', the graph of the function $f(x) = x^2 + x$ is given with its equation. The question requires calculating $f\,'(2)$. To solve this question, one is expected to calculate the slope of the line tangent to the graph at x = 2 graphically using two points, for example P (2, 6) given on the line (L) and (3, 12).

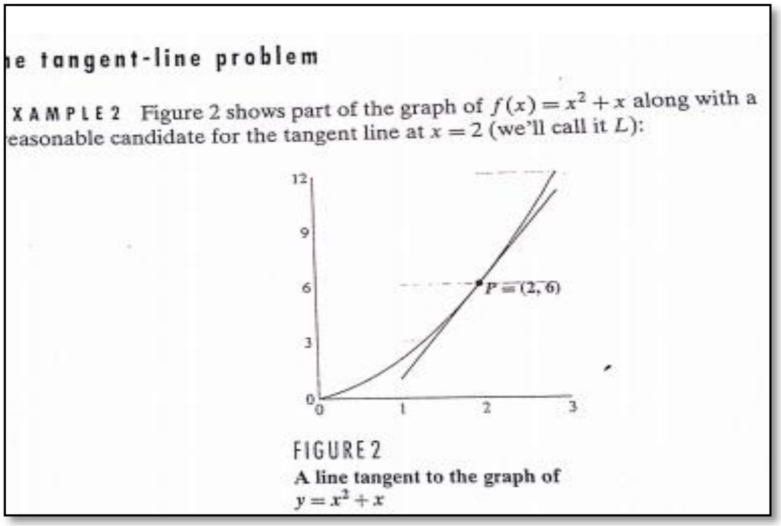


Figure 8. Graphs of a function f and its tangent at x= 2, from Book 2

4.1.2. Detailed analysis of the sections on derivative in the two books.

The analysis of the parts on derivative considers the following aspects: structure of the chapters (sections), modes of representation used in the 'Text' section, modes of representation used in the 'Exercises' section, and finally the translations among representations.

4.1.2.1 Structure of the sections on derivative (Book 1 and Book 2).

Each section on derivative, in both books, includes two main parts: the *Text* part and *Exercises* part.

Structure of Book 1. The *Text* part directly states definitions, properties and theorems to be learned. Then, it proceeds to prove them and provide examples. The number of examples varies from one section to another, depending on the objectives of each section. In addition, in the *Text* section most of the graphs are marked as figures and are placed on the left margin of the page, giving the impression that they are marginal, and don't constitute an integral part of the lesson. The *Exercises* section consists of three parts. The first part includes drill exercises that are mostly direct applications of the definitions or formulas learned in each section. The exercises are divided into sub-sections. Each sub-section holds a title for the concept to be used in solving the exercises, thus inducing the solution. The second part of the *Exercise* section includes exercises that are more challenging; these exercises are marked “Applications and Theory”. The last part includes exercises that require the use of technology such as Computer Algebra Systems (CAS); these are marked with the letter ***T*** or labeled as *Computer Explorations*.

Structure of Book 2. All sections in Book 2 have the same structure. Each section builds on the previous section. In addition, each section consists of two parts: the *Text* part and the *Exercises* part. The *Text* part starts with examples that aim to introduce the concept and set the stage for new ideas and definitions. It is noticed that 'remarks' are placed at the margin of each page and graphs are included everywhere, not only in the very margin. The exercises are of two types: “Basic” and “Further”. Exercises that fall under the “Basic” part are straightforward and focus on one idea. Exercises that fall under the “Further” part are more challenging; they require synthesis of several ideas and translations among graphical, numerical and symbolic points of view. Moreover, many exercises require the use of technology, but they are not labeled as “technology exercises”, thus reflecting the book's authors belief that technology is not an add- on to be applied at the end, but provides an environment for discovery, exploration, experimentation, gaining insights, generating and testing conjectures, and checking answers for plausibility and others.

In the following part of this analysis, since counting will be used in calculating the percentages of the types of representation (symbolic, graphical, and numerical) especially in the *Exercise* section, it important to refer to the definitions of the following terms: item, given representations, requested representation, and translation among representations.

- Item: An item is considered to be any form of statement or a question that calls for a response from the student. In the case where the sentence or the question includes many components, then we have more than one item. For example," Find $f\,'(1)$, $f\,'(2)$ and $f\,'(-1)$ where $f(x) = x^2$" is counted for three items, because it stands for " find $f\,'(1)$, $find\ f'(2)$ and find $f\,'(-1)$. " Since each selected section in the books has different total numbers of items, the data was converted to percentages to unify the basis of comparison.
- Given representation: Given representations are those that appear in the given information of the exercises. For instance, the given representation is considered numerical if the data is given in a table form or in ordered pairs (e.g. (2, -3), (1,6), etc.), and it is considered graphical if the graph of a function is given or slope of curve or tangent lines are given at some points. If an exercise includes both the graph and the equation of a function, then the given representation is considered both symbolic and graphical and so on.
- Requested representation: Requested representations are those that appear in the questions or orders of the items. For example, the requested representation is considered numerical if the question requires finding or explaining the rate of change (average rate of change or instantaneous

rate of change). If the exercise requires finding the slope of tangent line or the slope of curve or graphing the derivative function, then the requested representation is considered graphical and so forth.

- <u>Translation among representation:</u> The translation will occur from the given representation to the requested representation. Translation can take place from one mode of representation to another, or it may occur within the same mode of representation (Janvier, 1987).

4.1.2.2 Modes of representation in the 'Text' Section.

Based on the definitions of an item and given representation, the researcher counted the number of functions, in the *Examples* of the *Text Sections*, that are expressed through algebraic expressions alone, tables alone, graphs alone, equations and graphs, equations and tables, graphs and tables, and through the three modes together. Then, the data were converted to percentages.

Book 1. In the text of the selected sections on derivatives, the symbolic representation is the most dominant mode used (65 %) where the functions are expressed through equations. The questions differ from finding the derivatives of a set of functions using the formal definition of derivative (limit of quotient of differences), checking the differentiability of a function at a given point, writing the equation of the line tangent to the graph of a function at a given point, and finding the linearization of a function at some point. The other modes that appeared in the text of the four sections with a lesser occurrence than the symbolic mode are the graphical mode (14%) and the graphical and symbolic representations linked together (21%). It is noticed that most of the graphs are placed on the margins of the pages of the four sections, but they are referred to in the texts. Finally, the percentage of using numerical representations in the four sections is zero (0 %). Functions, expressed in a table form, and phrases like "*rate of change*" are not used in the given statements of the *Examples* of the four sections. However, in section 3.4, even though all the functions are represented through equations, they are related to the idea of derivative as a rate of change where they require finding the velocity and acceleration of a moving object, calculating the marginal cost of manufacturing some product, etc.

Book 2. In text of the four selected sections, the three representations (symbolic, graphical, and numerical) are equitably emphasized where 24 % of the functions are expressed through graphs alone, 16 % are expressed through equations alone, and 16 % are represented in a table form. The three modes of representation are used as teaching tools that help students finding the derivative of a function at some point either by calculating the average rate of change using two points, or drawing a tangent line to the graph and calculating its slope, or using the formal definition ($(f'(a) = \lim_{h\to 0} \frac{((f(a+h) - f(a))}{h}$). Moreover, 16 % of the functions are represented through graphs and equations together, 16 % include graphs and tables, 8 % include tables and equations, and finally 4 % include the three modes connected together. For instance, the equation, the graph and a table of values of the function $f(x) = e^x$ are used in one exercise.

4.1.2.3 Modes of representation in the 'Exercises' Section.

Book 1 (Given and requested Representations). In the exercises of the four sections in Book 1, the symbolic and graphical modes of representation are only used in the given statement of the exercises, whereas the numerical representation is not used at all.

Based on the definition of the given representation, the researcher counted 173 items. The details concerning the number and types of representation used in the given part of the exercises of each section are presented in Table 2 and Figure 9.

Table 2.

Number and Types of Given Representations in the Exercises of Book 1

	Symbolic	**Graphical**	**Numerical**	**Total**
Section 3.1	46	6	0	52
Section 3.2	48	20	0	68
Section 3.4	27	7	0	34
Section 3.9	24	0	0	24
Total	145	33	0	178
Percentage	81.46 %	18.53 %	0 %	100%

According to Table 2, it is clear that the dominant mode used in the given part of the exercises is the symbolic algebraic representation (145 out of 178). In the four sections, 81.46 % of the given representations are symbolic and 18.53 % are graphical while the numerical representation is inexistent. Although the second mode used is graphical, the difference between the symbolic and graphical percentages is 62.93 %, which is a significantly high difference.

In addition, it is interesting to interpret the differences among sections. For example, the graphical mode is not equally used in all the sections. It is used mostly in section 3.2, and never used in section 3.9. These facts reflect the objectives of sections 3.2 and 3.9. In section 3.2, students are expected to learn how to (1) make a reasonable plot of the derivative of $y = f(x)$ by finding the slopes on the graph of f; and (2) determine the cases when a derivative fails to exit at a point using both the limit definition and graphically (existence of a corner, a cusp, a vertical tangent, and a discontinuity).

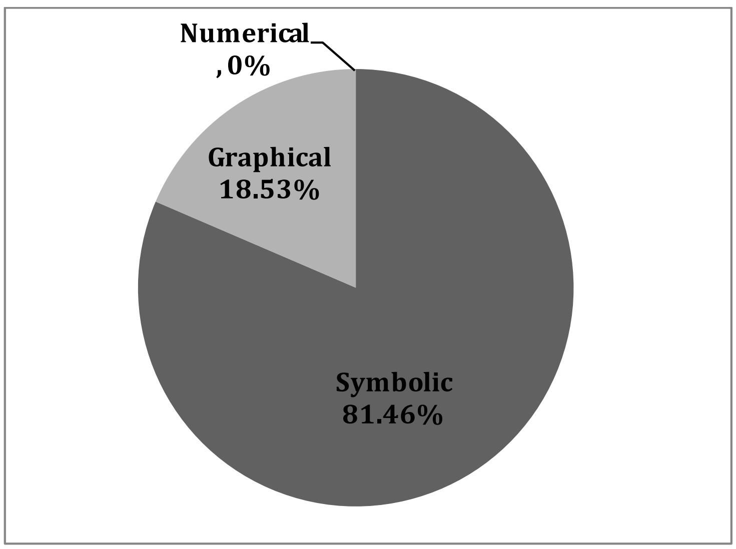


Figure 9. Percentages of the three types of representation used in the given part of all the exercises in Book 1

On the other hand, section 3.9 aims to find the linearization (based on tangent lines) of a differentiable function, given its algebraic expression, at a point. Since the rate of change is used as an application of derivatives and not as a teaching tool, the numerical representation is totally ignored in all the sections. This is explained by the absence of table of values or phrases like "average rate of change" or "instantaneous rate of change" in the given part of the exercises.

Based on the definitions of an item and requested representations, the researcher counted 316 items as requested representations. Concerning the requested representations, the dominant mode of representation in Book 1 is graphical (147 out of 316) which represents 46.5 %. Moreover, it is noticed that 32. 6 % (103 out of 316) of the requested representations are symbolic, while 20.9 % (66 out of 316) are numerical. Figure 10 is a pie chart that provides a visual distribution of the requested representations in Book 1.

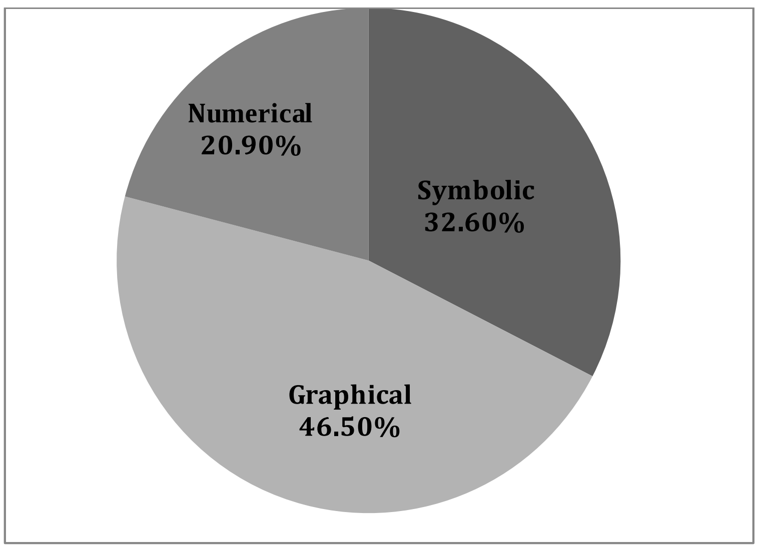


Figure 10. Percentages of requested representations in the exercises of Book 1

Moreover, Table 3 provides in details the number and the three types of requested representations in the exercises of each section in Book 1.

Table 3.

Number and Types of Requested Representations in the Exercises of Book 1

	Symbolic	**Graphical**	**Numerical**	**Total**
Section 3.1	4	62	4	70
Section 3.2	63	50	9	122
Section 3.4	4	28	53	85
Section 3.9	32	7	0	39
Total (Percentage)	103 (32.6 %)	147 (46.5 %)	66 (20.9 %)	316 (100%)

Moreover, several things can be noticed from Table 3. In section 3.1, the requested graphical representation is high (88.57 %) while the symbolic and numerical representations are low. This is justified by the fact that in section 3.1, the formal definition of derivative is used to find the slope of a curve and the slope of a tangent line at some point. This is opposite to section 3.9, symbolic is high and graphical is low, which aims to find the linearization of a function. Moreover, the numerical representation is not equally requested in all the sections. It is frequently requested in section 3.4 (62.35%) than in the other sections. This reflects the objective of section 3.4 where derivative is used to study rate problems, mainly motion of objects.

Book 2 (Given and Requested Representations). Table 4 and Figure 11 presents in details the number and types of representations used in the given statements of the exercises of each section.

Table 4

Number and Types of Given Representations in the Exercises of Book 2

	Symbolic	**Graphical**	**Numerical**	**Total**
Section 1.4	11	12	18	41
Section 1.5	18	27	8	53
Section 1.6	10	32	6	48
Section 2.1	23	11	13	47
Total (percentage)	62 (32.81 %)	82 (43.39 %)	45 (23.8 %)	189 (100%)

In the exercises of the four sections in Book 2, the three types of representations (symbolic, graphical, and numerical) are used in the given part of the exercises. Based on the definitions of the given representation, the researcher counted 189 items. According to Table 4, the representation that gets the highest percentage (43.39%) is the graphical mode (82 out of 189). Next is the symbolic representation (32.8 %), then the numerical representation (23.81 %).

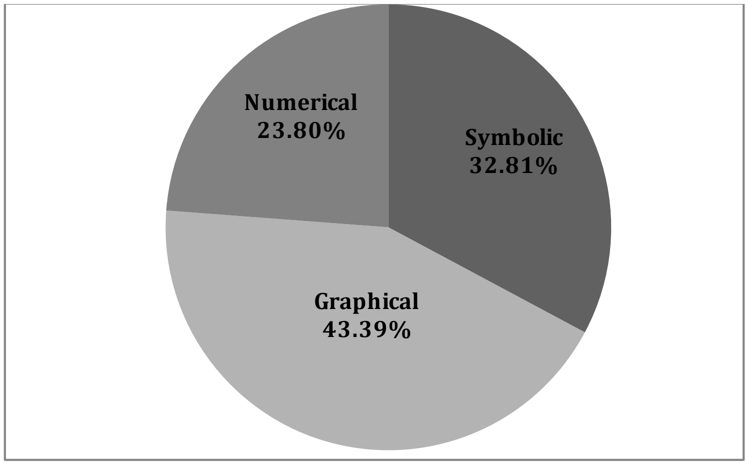


Figure 11. Percentages of the three types of representation used in the given part of all the exercises in Book 2

Thus, as shown in Figure 11, the distribution is rather balanced, with more emphasis on graphical representation. This reflects the book's authors belief that any mathematical concept can be represented in several ways (such as symbolically, graphically and numerically), and thus understanding each representation contributes to a deeper understanding of the concept as a whole.

Based on the definitions of an item and requested representation, the researcher counted 434 items as requested representations. The representation that gets the highest percentage (41.48 %) is the symbolic mode (180 out of 434). Next is the graphical representation (39.4 %), then numerical representation (19.12 %). Table 5 and Figure 12 provide details per section.

Table 5

Number and Types of Requested Representations in the Exercises of Book 2

	Symbolic	Graphical	Numerical	Total
Section 1.4	45	44	32	121
Section 1.5	73	45	20	138
Section 1.6	16	64	5	85
Section 2.1	46	18	26	90
Total (percentage)	180 (41.48 %)	171 (39.4 %)	83 (19.12%)	434 (100%)

The symbolic and graphical representations are close to each other (41.48% and 39.40 % respectively). Next is the numerical representation, which represents 19.12 % in all the sections. It is important to note that more applications of derivative as rate of change are provided in sections 2.4, 2.5 and 2.6 of Book 2, which are not part of this study.

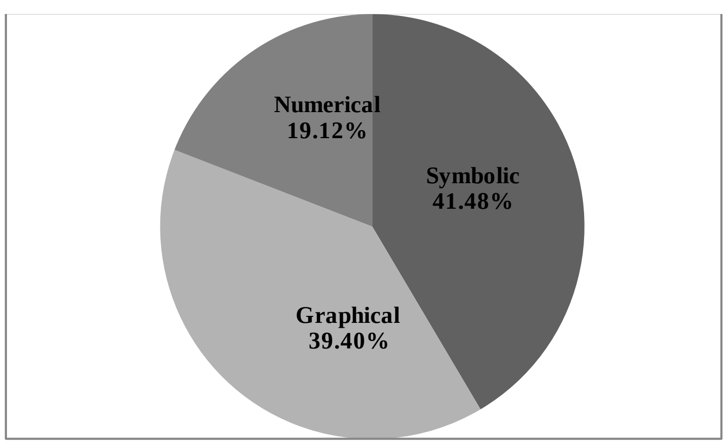


Figure 12. Percentages of requested representations in the exercises of Book 2

4.1.2.4 Translations among representations

Book 1. The translation between representations in the exercises takes place in most of the cases from the symbolic representation to other forms mostly to the graphical form. Table 6 presents in details the number and the percentages of translations among representations in the *Exercises* of Book 1.

Table 6

Translations among Representations in the Exercises of Book 1

	TO			
FROM ↓	Symbolic	Graphical	Numerical	**Total**
Symbolic	99 (31.33%)	109 (34.49 %)	54 (17.1%)	**262** (82.92%)
Graphical	4 (1.27 %)	38 (12.03%)	12 (3.77%)	**54** (17.07 %)
Numerical	0 (0%)	`0 (0%)	0 (0%)	**0**
Total number of items ***(Percentage)***	**103** (32.6%)	**147** (46.5%)	**66** (20.9%)	**316** 100 %

According to Table 6, the translations from symbolic to other forms of representation get the highest percentage, 82.91% (262 out of 316). Next is the translation from graphical to other forms of representation, which gets 17.07 %. It is noticed, from Table 6, that none of the translations occurred from the numerical representations to other forms simply because none of the functions in the given part of the exercises are presented in a table form. Moreover, ordered pairs (e.g., (1, 3), (-2, 4)) or phrases like" rate of change"," instantaneous rate of change", or "average rate of change" are not used in the given of the exercises. Putting such an emphasis on the direction of translations from symbolic to other forms increases students' inability to solve problems related to derivative whenever functions are not presented in the symbolic form. Also, less use of visualization.

On the other side, the translations to the graphical representations get the highest percentage (46.5 %) which is followed by symbolic representations (32.6 %) then numerical representations (20. 9%).

In general, the previous statistics reflect the approach used in Book 1, which emphasizes the symbolic approach of the concept. In general, most of the functions are expressed through their algebraic equations (82.92 %) which are used at tools to find the slope of a curve at a point, find the slope or the equation of a tangent line at a point, plot the graph of a function, graph the derivative of the function and find the linearization of a function. Thus, as teaching tools, the graphical and numerical representations do not play an efficient role in the teaching of derivative. The translations from graphical representations to other forms of representations are rare while the translations from numerical representations to other forms do not exit. Instead, they are deduced from the algebraic expressions of the functions, where 17.1 % of the translations take place from symbolic to numerical representations and 34.49 % from symbolic to graphical representations.

Book 2. It is noticed, from table 7, that the translations from graphical to other forms of representations get the highest percentage, 50 % (217 out of 434). Next is the translation from symbolic to other forms of representations (26.49%), then come the translations from numerical to other forms of representations, 23.5 % (102 out of 434).

Table 7

Translations among Representations in the Exercises of Book 2

	TO			
FROM ↓	Symbolic	Graphical	Numerical	**Total**
Symbolic	62 (14.28 %)	25 (5.77%)	26 (6%)	**113** (26.03%)
Graphical	70 (16.13%)	124 (28.57 %)	23 (5.3%)	**217** (50 %)
Numerical	48 (11.06%)	22 (5.1%)	34 (7.83%)	**104** (23.97 %)
Total (Percentage)	**180** (41.47 %)	**171** (39.41%)	**83** (19.12 %)	**434** 100 %

On the other side, the translations to the symbolic representations get the highest percentage (41.47%) which is followed by graphical representations (39.41 %) then numerical representations (19.12 %).

In general, in this book (Book 2) translations take place among and within each mode of representation. This improves students' visualization skills and increases students' flexibility to move among representations. Moreover, the above percentages, in

Table 7, reflect the goal of the book, which is achieving the conceptual understanding using the multiple representations of the concept (numerical, graphical, and symbolic). The three modes of representations are used both as teaching tools and as an end. Thus, Book 2 provides students with more opportunities than Book 1 to make connections and translations among and within these representations. Unlike Book 1, when students are given a table of values or a graph for a function, they are expected to be flexible and comfortable in handling these representations and use them to answer problems related to derivative.

4.1.3 Conclusion

In conclusion, the approach followed by Book 2 for developing the derivative concept, and represented by: [rate of change (numerical) → slope of curve or slope of tangent (graphical) → formal definition of derivative (symbolical)], is better than the approach adopted by Book 1, and represented by: [formal definition → graphical → rate of change (numerical)], in terms of the representations used and the translations among them.

Explaining derivative starting from real life examples using rate of change will create a need for students to study derivative. In Book 1, the idea of derivative as rate of change is only used as an application of the derivative rules. However, in Book 2, the idea of derivative is introduced first by discussing real life problems involving rate of change such as falling objects, growing populations, increasing bank deposits, decaying radioactive materials etc.

Book 1 seems to follow the traditional approach where the definitions of the concepts are directly stated at the beginning of the 'Text' without giving students the opportunities to foster their curiosity and critical thinking abilities and be active learners. Book 2 follows the constructivist approach whereby students are independent learners who construct their knowledge. In other words, Book 2 follows the learning cycle where the ' Examples' included in the 'Text section' serve for exploration and set the stage for new concepts, ideas and definitions, the ' Narrative Part' serves for development and finally ' Exercises' serve for applications.

In Book 1, the dominant representation used is the symbolic representations of the functions, which are used as tools for finding the equation of the tangent, finding the slope of a curve at a given point, finding the rate of change at a given point, or sketching graphs. This strong emphasis on the symbolic representation might restrict students' thinking and causes learning difficulties whenever functions are represented in other modes (graphs or tables). Therefore, it is very important to use other representations such as graphical and numerical representations as teaching tools and not just as an end. However, Book 2 focuses on the three modes of representation (individually and linked). According to Arcavi (2003), the use of visuals such as graphs, tables and diagrams enhances the learning process. Varying the modes of representation will allow students to use these representations as tools in order to organize, communicate, connect and interpret mathematical ideas, and hence increase their understanding (NCTM, 1989).

Another difference between the two books is the use of verbs in the items of the exercises. In general, most of the items in Book 1 are of types "find", "calculate", "sketch", all of which require the use of known steps, procedures and calculations. However, in Book 2 in addition to these verbs, other verbs have been used like

"Explain", "Justify", "Interpret", "verify", all of which require critical thinking. Such examples include questions such as " Explain how the average and instantaneous rates of change are related?" "What does the difference quotient $\frac{f(a+h)-f(a)}{h}$ represent graphically?", or " Explain how the graph of f' can be used to determine the maximum or minimum points of f "and" Justify in the language of rate your result", and others.

4.2- Analysis of Exercises, Activities and Observations

During the implementation of the unit on derivatives, the researcher was present in both groups (control and experimental) to observe the instructional method (s) and strategies used in the classroom as well as students' work, using an observation work log (Appendix C).

A total number of five sessions (50 minutes each) were observed in fall (control group), while 6 sessions and a half (50 minutes each) (experimental group) were observed in spring. The difference in the number of the sessions is because group work and technology were used in the experimental class, which required more time for technical assistance and guidance. Also, instructor Y (in the experimental group) spent almost 25 minutes explaining to students about the new approach and the software, *Autograph,* that she is going to use for teaching derivative. In addition, students were given reasonable time to work and reflect on each activity. In the following, four major sections are presented:

- Section (4.2.1) presents a comparison of the flow and content of sessions addressed by the two instructors during the teaching of derivative

- Section (4.2.2) provides a detailed analysis of the *Exercises* solved in the control group based on the different types of given representations used. In addition, it presents an a priori analysis of the *Activities* conducted in the experimental group based on the different types of representation and the genetic decomposition of the derivative concept (*APOS framework*). Also, it focuses on the objective(s) behind each activity and explains the logic behind assigning such an activity.

- Section (4.2.3) presents a picture of the distribution of time spent on representations (individual or connected) in class during the teaching of derivative in the two groups, based on the analysis of the observation logs.

- Section (4.2.4) describes the teaching methods (lecture, group work, technology, etc) used by the two instructors, based on the analysis of the observation logs.

4.2.1. Comparison of the flow and content of sessions addressed by the instructors during the teaching of derivative

Control Group. Instructor X followed the same approach used in Book 1, which is represented by: [formal definition of derivative, $f'(a) = \lim_{h\to 0} \frac{((f(a+h) - f(a))}{h}$ → graphical (slope of curve / slope of tangent line) → rate of change (numerical)]. However, the emphasis was on the symbolic representations of derivative (using equations and formal definition of derivative). In general, her approach was teacher-centered where her role was to deliver the course materials directly to students without having them involved in the discovery of the new knowledge. Instructor X spent nine sessions for explaining chapter 3, *Differentiation*, from Book 1. It is worth noting that only five sessions, which are sessions 1, 2, 3, 6 and 9 of the control group, are part of this study because the other sessions are related to the rules of differentiation.

In session 1, Instructor X defined the derivative of a function f at a point a using the formal definition, $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, and solved several examples. Then, she discussed the geometric meaning of derivative as the slope of the line tangent to the graph of f at a, and then solved some exercises. In session 2, she defined the derivative as a function using again the formal definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, presented examples, and then discussed the cases where the function is not differentiable at a point using both equations and graphs. For example, a function fails to have derivative at a point if right- hand derivative is different from the left- hand derivative, or the graph has a corner or discontinuity at point. In session 3, she solved one exercise on derivative presented graphically, explained the relationship between the sign of $\boldsymbol{f'}$ (positive, negative) and the variation of ***f (increasing, decreasing)***, and the roots of $\boldsymbol{f}$ ' with the critical points of $\boldsymbol{f}$. In the remaining time, she explained some rules of differentiation *(constant function, sum/ difference and power rules*). In sessions 4 and 5, she continued explaining the rules for derivative (*product and quotient rules, chain rule, and derivatives of trigonometric functions*). In session 6, she explained the idea of derivative as rate of change and presented examples involving rate of change such as distance, velocity, acceleration. In sessions 7 and 8, she explained *implicit differentiation*, that is the use of the chain rule to differentiate implicitly defined functions such as $x^3 + y^3 =$

18 xy; where y is a function of x. Finally, in session 9, she explained the *linearization concept* using equations. Appendix- G includes an outline of the sessions on derivative with detailed procedures, conducted in the control group

Experimental Group. Instructor Y followed a different path. She followed the approach used in Book 2, which is represented by: [rate of change (numerical) → slope of curve or slope of tangent (graphical) → formal definition of derivative (symbolical)]. Instructor Y spent 6 sessions and a half for teaching the derivative concept. In general, the classroom environment was cooperative, interactive and supportive. Students were actively involved in their learning; they completed a set of activities (*Activity 1* → *Activity 20*/ Appendix F) mostly working in groups. *Autograph,* dynamic software, was used by the teacher in all sessions. It is worth to mention that the activities (paper-pencil and *Autograph based*), which form the first step of the ACE cycle according to the APOS framework, were designed to guide students to form the mental structures as described in the genetic decomposition of the derivative concept. As a reminder, the genetic decomposition of the derivative concept was based on the researcher's knowledge of the derivative concept, APOS theory and the findings obtained in the control group (fall 2013). Section 3.4.3 provides all the details on the genetic decomposition.

In the first session of the derivative unit, the teacher addressed the idea of derivative as a rate of change. In this session, students solved *Activity 1*, about the rate of change of a candy bar thrown by an astronaut, as an exploratory activity where they deduced the definition of derivative as an instantaneous rate of change. In session 2, students performed several activities (*Activity 2* → *Activity 4*) to deduce the geometric meaning of derivative as the slope of the tangent line. In session 3, students performed several activities (*Activity 5* → *Activity 7*) to deduce the meaning of derivative as the slope of the curve and discuss the cases where the function is not differentiable. In session 4, students solved several activities (*Activity 8* → *Activity 12*) to deduce that the derivative of a function is a function itself. In session 5, students worked with several activities (*Activity 13* → *Activity 16)* designed to explain the relationship between the sign of the derivative f' (positive, negative) and the variation of the function f (increasing, deceasing) and between the roots of the derivative function f' with the critical points of the function f**.** In session 6, the instructor introduced the formal definition of derivative using *Autograph,* and then students applied the definition to solve exercises (*Activity 17* →*Activity 19*) related to slope of tangent line and instantaneous rate of change. Finally, in part of the session 7, Instructor Y explained the *concept of linearization* using *Autograph*, and students solved several exercises (*Activity 20*). In the remaining time, she started with the rules of differentiation (*power rule, product and quotient rules, chain rule etc*). Appendix -H includes an outline of the sessions on derivative with detailed procedures, conducted in the experimental group.

4.2.2. Analysis of the Exercises and Activities conducted in the two Groups

The researcher kept track of the content taught in the two groups by taking notes from the board and photocopying two students' copybooks (one from control group and one from the experimental group). In addition, the researcher kept a record of detailed notes about how each lesson was conducted and what kind of exercises or activities were solved in each group.

4.2.2.1 Analysis of the exercises solved in the control group.

The exercises solved in the control group are shown in Appendix F. Most of the functions solved in class are represented through equations, four functions are represented graphically and none of the functions is represented in a table form. The exercises varied from finding the derivatives of a set of functions using the formal definition (Figure 13), to checking whether the function is differentiable at a given point, and finding the slope of the curve or the slope of the line tangent to the curve, using formal definition. In addition, one exercise requires sketching the graph of the derivative function (Figure 14), and others require finding the velocity, speed and acceleration of a moving object to finding the linearization of a set of functions near some point.

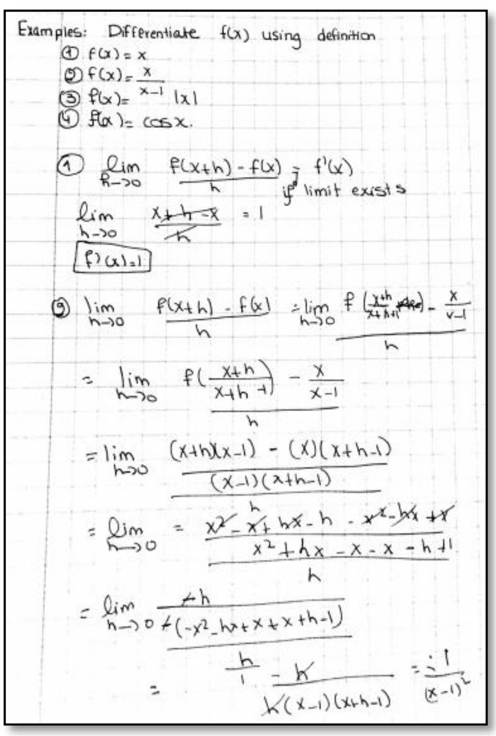


Figure 13. Sample of a student's work, in the control group, in applying the formal definition

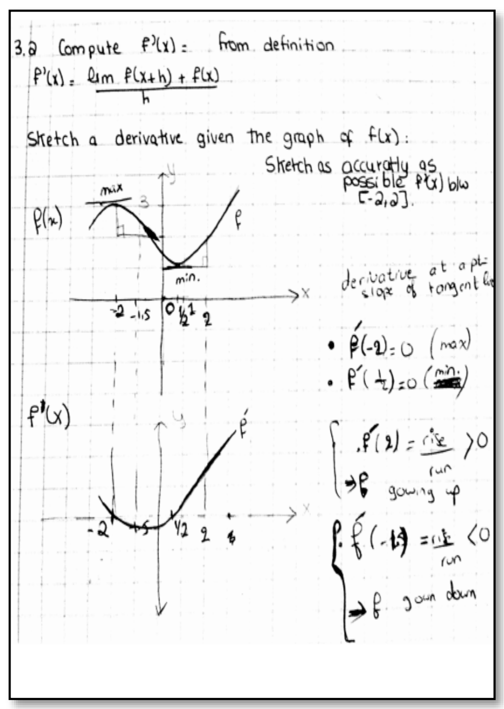


Figure 14. A 'exercise' where the function presented graphically, was solved in the control group

Representations in the Given Statement of the Exercises. The analysis technique, based on representations, conducted for the books is used for the Exercises. It is clear that the dominant mode used in the given statement of the exercises is the symbolic representation (Figure 15). The researcher noticed the following:

- 81 % of the representations are symbolic
- 19 % of the representations of the functions are graphical
- There was no use of numerical representations

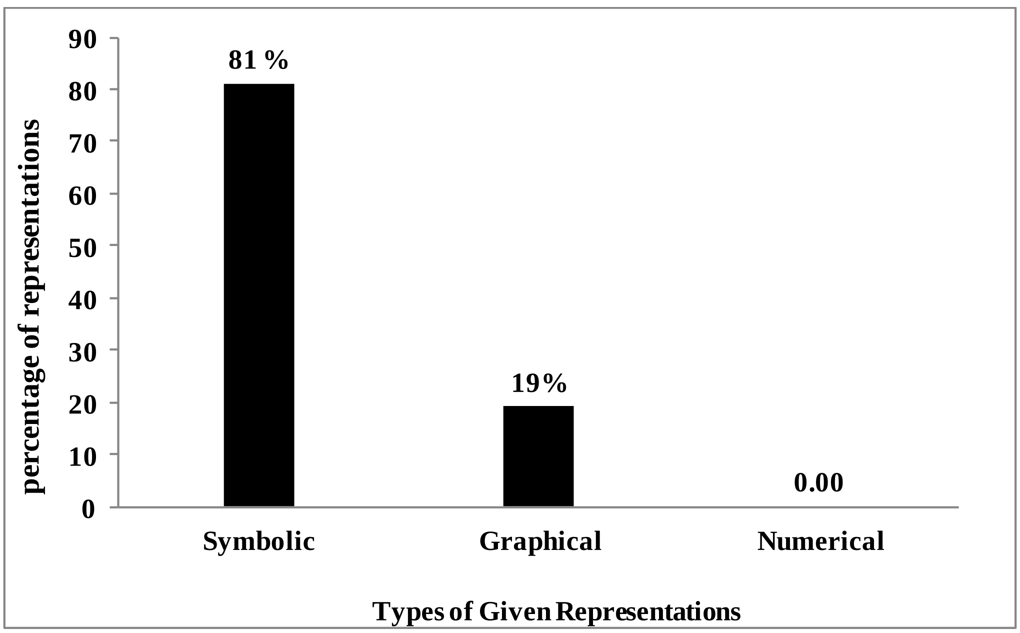


Figure 15. Percentage of different types of representations used in the given statements of the Exercises on derivative in the control group

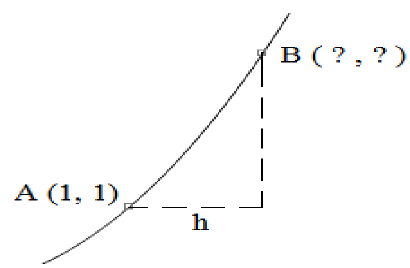
4.2.2.2 A priori Analysis of the activities (Activities 1 - 19) in the experimental group

The researcher with the cooperation of instructor Y designed activities (see Appendix F) that encourage the exploration of the derivative concept through different representations (tables, graphs, formulas, word problems). In addition, *Autograph* was applied in well-chosen problems assisting students in the exploration of the problems. It is worth mentioning that the teaching sessions were interactive where students felt free to ask questions, listen to others and express their ideas. During the activity part, the teacher encouraged discussions among group members and sharing results before having class discussion and presentation. Most of the time, the teacher used the results and the solutions of the activities to present formally the new concept. The activities were then followed by homework exercises that were similar to the activities.

Activity: This activity is designed to help students achieve the levels **1b and 2b** in the genetic decomposition of the derivative concept. Students will work on this activity in groups of three. This activity is designed to make students build connections between the symbolic and numerical representations of a function since the data is given through an equation and through a table. This activity is important since it allows students to compare between the average rate of change and the instantaneous rate of change. In part a) of this activity, students are asked to calculate the average rate of change of the height of a candy bar over time using different intervals. In part b), students are asked to observe what is happening to the rate of change as the time interval gets smaller and smaller. Finally, in part c), they are asked to estimate the rate of change at a particular time; students need to choose the best interval for their estimate.

Activity 2: Students will work on this activity individually, and then class discussion will take place. This activity is important because it allows students to give a geometric interpretation of the average rate of change as the slope of the secant line. In this activity, the equation and the graph of the function are given. Part a), which requires calculating the average rate of change of $\boldsymbol{f(x) = x^2}$ between x =1 and x = 3, is used to achieve the level **1b** in the genetic decomposition. Part b), which requires calculating the slope of the secant line, is designed to achieve level **1a** in the genetic decomposition. Finally, the purpose of part c) is to help students recognize that the slope of the secant line between two points on a curve is equal to the average rate of change between those points.

Activity 3: Students will work on this activity in groups of two. In this activity, the graph of the function $\boldsymbol{y = x^2}$ near the point A (1, 1) is given in the diagram below. The point B is at a horizontal distance h from A.



In this activity, students need to calculate the slope of the line (AB) in terms of h, and then estimate the slope of the tangent at point A. Therefore, this activity aims to show how the tangent line can be obtained from the secant line and how the slope of the secant line approaches the slope of the tangent line. This activity is important because it gradually guides students to deduce the formula for the difference quotient in terms of h (i.e. $\frac{f(a+h)-f(a)}{h}$). This activity is designed to achieve level **2a** in the genetic decomposition.

Activity 4: The instructor will use *Autograph* to conduct this activity which is similar to Activity 3, but it addresses a more complex polynomial function ($y = x^3 - 2x + 2$). Having in mind Activity 3, this activity aims to confirm the formulated conjecture: slope of tangent is the limit of the slope of secant line. This is achieved through the 'animation' feature of *Autograph*, which will allow students to observe how the secant line moves gradually, and coincides with the tangent line. Students can observe the equations of the tangent and secant lines at the bottom of the screen and compare their slopes. This activity is designed to achieve levels **2a** and **3a** in the genetic decomposition.

Activity 5: This activity is to be performed as pair work. In this activity, the graph of a function is given (without its algebraic equation). Part a) stresses the connection between the average rate of change and the slope of secant line, while part b) stresses the connection between instantaneous rate of change and the slope of the tangent line. This activity is important because students are expected to compare the slopes of the lines using the idea of steepness of the line. Part b) achieves level **4** in the genetic decomposition.

Activity 6: This activity is to be conducted by the instructor using *Autograph*, however all students will be asked to participate. The purpose of this activity is to help students understand how to measure the slope of a curve at a point. The instructor will plot the graph of y = x^2 and the tangent line at the point A (1, 1). Then, she will zoom in repeatedly at the point A (1, 1) to get a close up view of the curve. It is noticed that the scales will automatically adjust to each zoom ratio. Students are then asked to make observations. They are expected to see that the more the graph of the function is magnified near the point A, the flatter the graph becomes and the more it resembles the tangent line .Therefore, the slope of the curve at a certain point is equal to the slope of the tangent line which represents the derivative. In addition, students can see that if the instructor zooms in at any point, the graph will be almost straight. This activity achieves level **5** in the genetic decomposition. A snapshot the activity is shown in Figure 16.

Figure 16. Close look at the graph of f and its tangent line coinciding at the point (1, 1)

Activity 7. This activity is to be conducted by the instructor using *Autograph*, however all students will be asked to participate. The idea that the function is differentiable at a point if it is locally straight is very important. This activity builds on Activity 6. It is very important because it introduces the left and right derivatives and examines the case where the function is not differentiable. The instructor will plot the graph of y = $|x|$ and zoom in at the point (0, 0). Students are then asked to make observations and are expected to see that at the point (0, 0) the function is not locally straight and hence not differentiable. Another example, she will plot y = $x\ |sinx|$ and zoom in at the points (0, 0) and $(\pi, 0)$. Students are asked to make observations. They are expected to see that at the point (0, 0) the function is locally straight, while at the point $(\pi, 0)$, the function is not locally straight (corner) and thus not differentiable. This activity achieves level **7a** in the genetic decomposition.

Activities 8 and 9. These activities are to be performed as class work. The teacher will write the questions on the board and ask students to participate. These activities aim to analysis the idea that the derivative of a function is equal to the slope of the curve. In Activity 8, the equation of the function $f\ (x) = 2x + 1$ is given. Students are asked to explain, geometrically and in terms of rate, why the derivative is the constant function $f\ '(x) = 2$. In Activity 9, the graph of the derivative function is given, and students are asked to guess a formula for $f(x)$. This activity is important because it tests students' ability to use the information from the graph and come up with the equation of $f(x)$. that is it stresses the translation from graphical to symbolic representation. This activity achieves level **5** in the genetic decomposition.

Activity 10. This activity is to be performed in groups of two then corrected using *Autograph*. It is very important because it introduces derivative as a function. In this activity, the graph of a function is given without is algebraic expression. In part a), students are asked to match the points labeled on the curve with the slopes of the curve, all of which are given in a table. To solve this part, students are expected to relate the slope of the curve at a point with the slope of the tangent line at that point, and then use the idea of steepness of the tangent lines. The steepness of the line is measured by the absolute value of the slope. Therefore, the line with the greater slope (in absolute value) indicates a steeper line and vice versa. As a hint, students will be asked to place their pen and move it along the graph keeping it tangent to the curve and observe the steepness of the tangent lines (or their pen). In part b), students are asked to sketch the graph of the derivative function. Students are expected to join the points obtained in part (a) with a smooth curve. That is seeing the derivative function as the correspondence: $a \rightarrow$ slope of tangent line at the point $(a, f(a))$. This activity achieves level **7a** in the genetic decomposition.

Activity 11. In this activity, some values of the derivative of the function $f(x) = 2x^2$ are given in a table. Part a) asks students to plot the graph of $f\,'(x)$, and part b) asks students to guess a formula for the $f'(x)$. This activity is important because it involves translation from numerical to graphical and then to the symbolic representation of derivative. Moreover, students can see that the derivative of a function is a function itself. This activity achieves levels **6a and 6b** in the genetic decomposition.

Activity 12. This activity is to be performed in pairs using *Autograph* followed by a whole class discussion. The graph of a function f and its derivative $f\,'$ are given (without equations). This activity aims to help students construct the relation between the sign of the derivative f' and the increase or decrease of the function f. Moreover, it examines the relation between the minimum point of f with the root of is derivative. The use of *Autograph* is interesting because of the 'slow plot' and 'slope function' buttons that are located on the top horizontal ' main ' tool bar. When the 'slow plot' is pressed, the derivative function will be plotted slowly showing a moving tangent on the original function and pausing at minimum point of $\boldsymbol{f}$. Thus, when the plot is paused, students are asked to make connections between $\boldsymbol{f}$ and its derivative $\boldsymbol{f\,'}$**.** They are expected to see that when the function is increasing on an interval, its derivative is positive. Also, when the function is decreasing on an interval, its derivative is negative. In addition, when the function has a minimum point, its derivative is zero because of the horizontal tangent. Another interesting feature of *Autograp*h is that it allows the user to construct a table of values for the selected functions. Thus, students can test their conjectures through observing and comparing the table of values for f and $f\,'$. This activity achieves levels **7a and 7b** in the genetic decomposition.

Activity 13. This activity is to be performed in pairs using *Autograph* followed by a whole class discussion. It is similar to the previous activity, but it addresses a more complex polynomial function. The graphs of a function (solid line) and its derivative (dotted line) are given, and students are asked to explain how the two graphs are related (Figure 17). This activity aims to validate and emphasize the previous formulated relationship. This activity achieves level **7a** in the genetic decomposition.

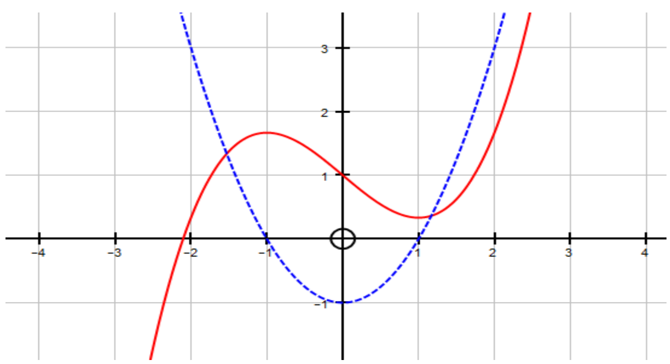


Figure 17. A function (solid curve), and its derivative (dotted curve)

Activities 14 and 15. As an application of the relation between a function and its derivative, students will perform Activity 14 and Activity 15. These activities are important because they include numerical (table form) and symbolic representations (equations) of derivative. In Activity 14, it is given that a function f has a positive derivative on some closed interval and students are asked to select among three tables the one that represents the function $f(x)$. In Activity 15, the equation of the derivative function $f\,'(x)$ is given as a piece- wise function. Students are asked to determine the intervals where the function is increasing or decreasing. Activities 14 and 15 achieve level **7b** in the genetic decomposition.

<u>Activity 16.</u> In this activity, the graph of the derivative function $f\,'(x)$ is given. Students are asked to use this graph to determine the interval where f is increasing or decreasing. This activity is very important because it is opposite to what a typical (traditional) question would be. A typical (traditional) question would give the graph of a function and require determining the interval (s) where the derivative is positive or negative. Solving this question correctly means that students have constructed a schema of the derivative concept.

<u>Activity 17</u>. The purpose of this activity is to introduce the formal definition of the derivative function as $\boldsymbol{f\,'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}}$. This activity is to be conducted by the instructor using *Autograph*, however all students will be asked to participate. The use of *Autograph* is useful to introduce the formal definition. In this activity, the instructor will plot three functions: $f(x) = x^2 - 4x - 3$; its derivative $f'(x) = 2x - 4$; and $g(x) = \frac{f(x+h)-f(x)}{h}$Then, students are expected to know that the quotient $\frac{f(x+h)-f(x)}{h}$ represents a set of slopes of secant lines or average rate of change. Using the 'constant controller' button, the instructor will increase or decrease the value of h. *Autograph* automatically uses the default value of 1 for h , but the user can enter any

value of h .Then, students are asked to make observations and deduce what happen when *h* approaches zero. This activity achieves levels **6a** and **6b** in the genetic decomposition. A snapshot of the activity is shown in Figure 18.

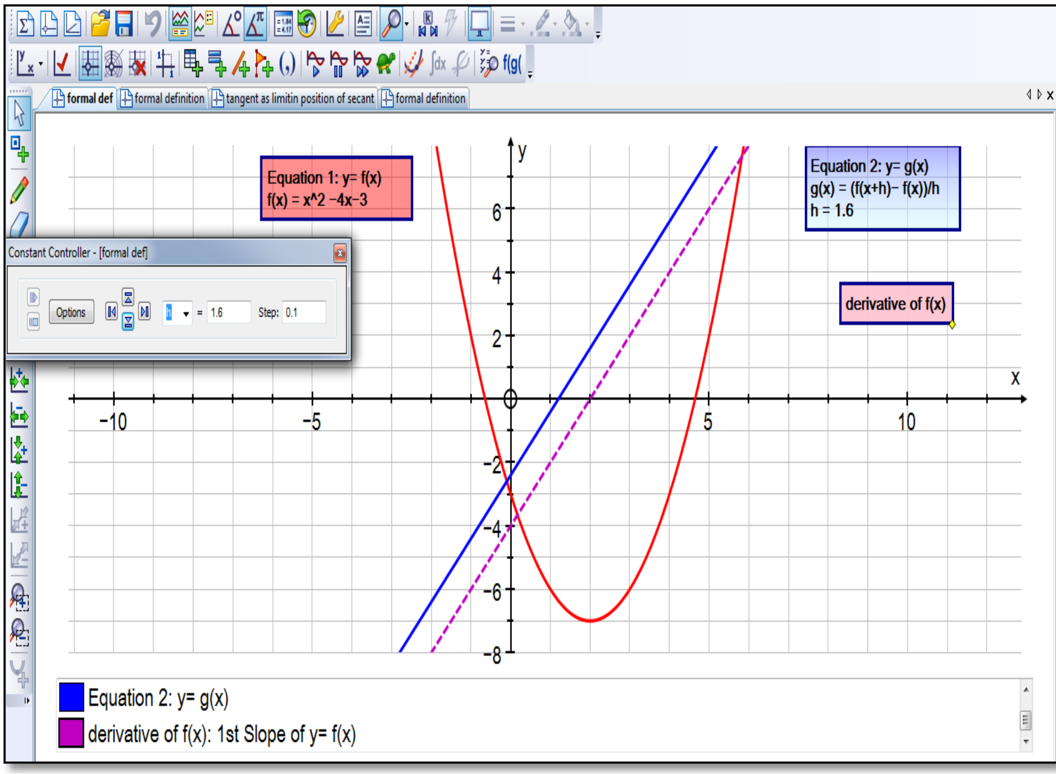


Figure 18. Graphs of *f(x)*, its derivative *f '(x)* (dotted line) and *g(x)*, solid line. As h approaches zero, *g(x)* approaches *f '(x)*

<u>Activity 18.</u> This activity is an application of the formal definition of derivative. It is important because it links different meanings of derivatives (slope of tangent line, instantaneous rate of change and symbolic definition using the limit). In Part I, the equations of two functions are given and students are asked to find their derivative functions. Part I achieves level **6b** in the genetic decomposition. In part II, students will be asked to use the formal definition of derivative to find the instantaneous rate of change at a point. Part III asks students to use the definition to find the slope of tangent line at a point. Parts II and III achieve level **4** in the genetic decomposition.

Activity 19. This activity introduces the concept of linearization. It is to be conducted by the instructor using *Autograph* where all students will be asked to participate. In this activity, the graph of the function $f(x) = \sqrt{x}$ and its tangent at $x = 1$ will be plotted. Students are asked to approximate $\sqrt{1.1}$ without using a calculator. Through the use of the 'zoom in' button near x =1, students are expected to observe that the more the graph is magnified around x = 1, the flatter it becomes which resembles the tangent line. Hence, the function $f(x) = \sqrt{x}$ can be approximated using the tangent line $y = \frac{1}{2}x + \frac{1}{2}$. A similar exercise is to be solved for the function $f(x) = \frac{x}{x+1}$ near x = 1. This activity achieves level **5** in the genetic decomposition.

Representations in the Given Statements of the Activities. In the experimental group, the three types of representation are used in the given statements of the activities (see Figure 19):

- the most dominant representation used in the given of the exercises is the graphical mode (39%)
- 16 % of the exercises are expressed through tables of values alone (numerically)
- 26 % of the exercises are given through equations alone (Symbolic representation)
- 16 % of the exercises are expressed using both graphs and equations
- 3 % are expressed using both equations and tables (symbolic and numerical representation)

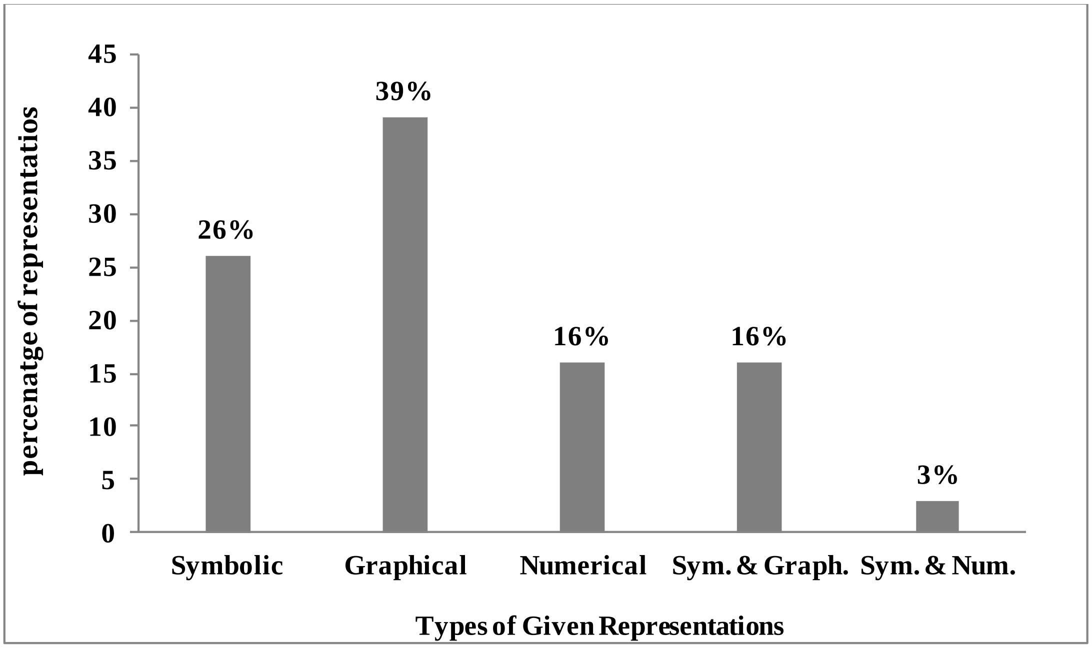


Figure 19. Percentage of different types of representations used in the given statements of the Activities, in the experimental group

Note. Sym. & Graph. = symbolic and graphical; Sym. & Num. = symbolic and numerical

4.2.3. Time spent on representations in the two groups.

Table 8 presents the distribution of time spent on each representation (individual or connected) for teaching derivative as well as the teaching methods used during one of instructor X ' lessons, in the control group. In addition, it includes researcher's notes about the session. Table 8 consists of three big columns that have the three types of representation as headings (Symbolic /S, Graphical/ G, and Numerical /N). Under each of these headings, there are 5 columns that represent the different teaching or learning methods denoted as L ,GW, IW, Q, and T which stand for lecture, group work, individual work, questioning, and the use of technology respectively (Check Appendix D for more details concerning the definition of each teaching /learning method).

Table 8

Types of Representations & Teaching Methods used for teaching derivative in the first session of the control group

TIME	**Symbolic (S)**					**Graphical (G)**					**Numerical (N)**				
	L	GW	IW	Q	T	L	GW	IW	Q	T	L	GW	IW	Q	T
2:00 - 2:07															
2:07 - 2:12															
2:12 - 2:20															
2:20 - 2:35															
2:35 - 2:50															

Notes:

Name of instructor = X; Class duration = 50 minutes; Session 1: Formal definition and tangent line

L= Lecture; GW = group work; IW = Individual work; Q = asking questions about previously learned materials, asking questions for clarifying some idea and answer questions related to homework ; T = use of technology

Class notes*:*

First 7 mins., teacher asked questions about concepts learned previously: existence of limits, continuity of function at a point. In the next 5 min, instructor X wrote on the board the three different interpretations of derivative of a function at a point (instantaneous rate of change, slope of the tangent line to *f* at a point, then the formal definition). Students took notes.

- For 8 mins., the teacher explained step by step how to calculate $\boldsymbol{f'(x)}$ using definition with an example

- For 15 mins., students solved individually four exercises on derivative using the formal definition:

(3 *questions related to tangent lines; 1 requires finding derivative at a point*.)

- Some students participated; others just copied the solutions from the board.

- Remaining 15 minutes, the teacher provided students with the solutions of the exercises (students took notes)

Each row represents a time interval of the teaching session. This table reflects the first session (50 minutes) on derivative conducted by instructor X in fall 2013.

In this session, the teacher introduced the formal definition of derivative at a point as $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, and solved several exercises. In all the exercises, the functions are expressed through equations (for example, $f(x) = x^{1/3}$; $f(x) = \frac{x}{x-2}$ etc).Then, she related derivative to the slope of the line tangent to the graph of f at a given point. Many things can be noticed from this table. It shows the time (shaded in blocks) that the teacher spent on each representation as well as the teaching methods used. When shading is done under each type of representation and during the same interval of time, then there is connection between the representations. For example, in this session instructor X spent a total of 30 minutes on symbolic representations alone with no connection to other types. This is calculated by adding the time from (2: 00 → 2:07) + (2:12 →2:35) as shown in the table. Moreover, she spent 5 minutes linking the three types of representation together (symbolic, graphical and numerical) as shown from 2:07 to 2:12. Finally, she spent 15 minutes linking the symbolic to graphical representation as shown from 2:35 to 2:50.

Concerning the teaching method used, she spent 7 minutes asking students questions about previously learned materials, 28 minutes conducing lecture as shown in the table (2: 07 to 2:20 and from 2:35 to 2:50), and 15 minutes allowing students to solve exercises individually (2:20 to 2:25). In addition, it is noticed that there was no group work, nor was there any use of technology.

4.2.3.1 Time spent on representations in the control group.

After the analysis of all of the observation logs as in the previous example, the following points were noted in the control group (Figure 20 includes a bar graph showing overview of time spent on representations):

- 51.6 % of the time was spent on symbolic representations (S) alone (using equations)
- 11.2 % of the time was spent on graphical representations (G) alone
- There was no use of numerical representations (N) alone in the given statement of the exercises (no use of tables)
- 16 % of the time was spent on connecting symbolic and numerical representations $(S \rightarrow N)$. In this case, the functions were given symbolically (through equations), and they require finding the rate of change.
- 15.2% of the time was spent on connecting symbolic and graphical representations $(S \rightarrow G)$. In this case, the functions were given symbolically, and they require finding the slope of the tangent line or slope of the curve at a given point.
- There was no connection between graphical and numerical representations
- 6 % of the time was spent on linking the three types of representation together

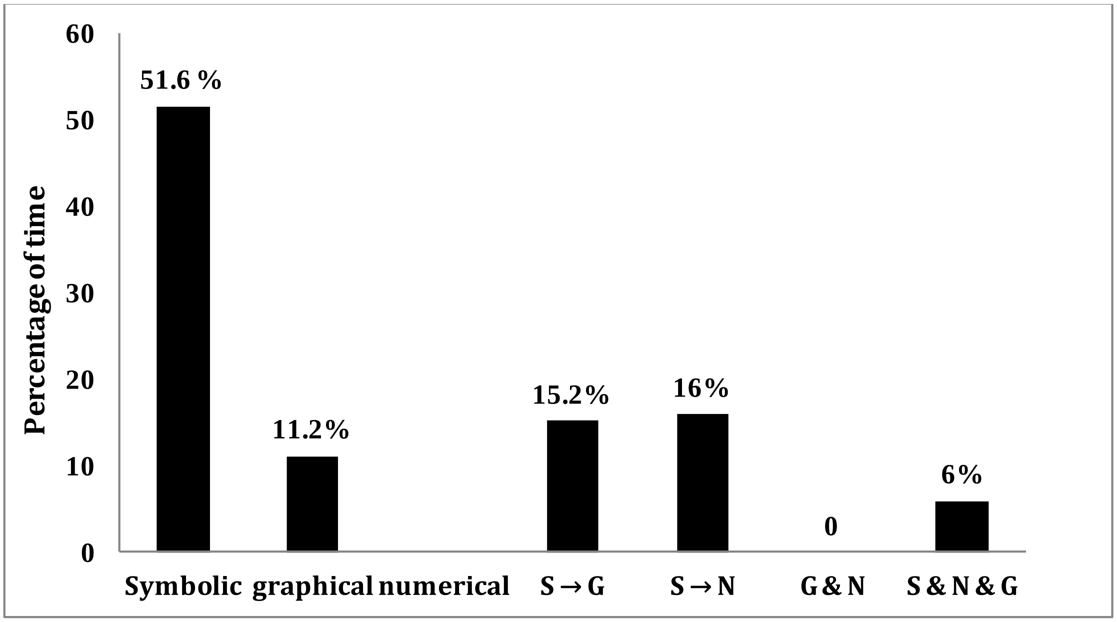


Figure 20. Overview of the time, in percentage, spent on representations (individually or connected) in all sessions of the control group for the derivative concept.

Note., S → G = translation from symbolic to graphical, S→ N = translation from symbolic to numerical, G&N= connection between graphical and numerical (covers both directions), S & N& G = connection between the three types of representation

4.2.3.2 Time spent on the different types of representation in the experimental group.

Table 9 reflects the second session (50 minutes) on derivative conducted by instructor Y in spring 2014. In the first session, the idea of derivative as a rate of change) was explained. In this session, students performed several activities (*Activity 2 → Activity 4*) to deduce the geometric meaning of derivative as the slope of the tangent line. Many things can be noticed from this table. For example, in this session instructor Y spent 25 minutes linking the symbolic to graphical representation as shown from (2:15 to 2:40), and she spent 8 minutes linking the symbolic to numerical representation as as shown from (2:08 to 2:15) and (2:40 to 2:44). Moreover, she spent 11 minutes linking the three types of representations together (symbolic, graphical, and numerical). Finally, she spent 6 minutes on the graphical representation alone.

Table 9

Types of Representations & Teaching Methods used for teaching derivative in the second session of the experimental group

TIME	**Symbolic (S)**					**Graphical (G)**					**Numerical (N)**				
	L	GW	IW	Q	T	L	GW	IW	Q	T	L	GW	IW	Q	T
2:00 - 2:08															
2:08 - 2:15															
2:15 - 2:30															
2:30 - 2:40															
2:40 - 2:44															
2:44 - 2:50															

Notes:

Name of Instructor = Y, Class duration: 50 minutes, Session 2: Derivative as slope of tangent line

L= Lecture; GW = group work; IW = Individual work; Q = asking questions on materials learned in the preceding session, summing up the new lesson and answer questions related to homework; T = use of technology

Class notes*:*

- *First 8 minutes:* Recall previous session + discussion part of homework I

- *Next 8 minutes:* Solve Activity 2 (individually) + came up with the definition: average rate of change= slope of secant line

- *Next 15 minutes:* Solve Activity 3 (pair work)

- Next 10 minutes: Solve Activity 4 (*Autograph*)

- *For 4 minutes*: Class discussion; $\boldsymbol{f'(x)}$ **= slope of tangent line = instant. rate of change**

Average rate of change= slope of secant line

- *remaining time:* Solve an exercise as an application + assign homework II.

Concerning the teaching method used, 15 minutes were spent on group work solving *Activity 3*, 10 minutes on using technology, and 13 minutes were given to students to solve individually *Activity 2* and start with their homework. In addition, instructor Y spent 12 minutes on asking students questions about previously learned material and summing up the new lesson.

After the analysis of all of the observation logs as in the previous example, the following points were noted in the experimental group (Figure 21):

- 11.08 % of the time was spent on symbolic representations (equations) alone
- 16.62 % of the time was spent on graphical representations alone
- 8 % of the time was spent on numerical representations alone
- 26.46 % of the time was spent on connecting symbolic and graphical representations. It is worth to mention that some of the translations occurred from symbolic to graphical representation and others from graphical to symbolic

  (S ↔ G)
- 12.92 % of the time was spent on connecting symbolic and numerical representations. It is worth to mention that some of the translations occurred from symbolic to numerical representation and others from numerical to symbolic

  (S ↔ N)
- 4.62 % of the time was spent on connecting graphical and numerical representations. It is worth to mention that some of the translations

occurred from graphical to numerical representation and others from numerical to graphical

(G ↔ N)

- 20. 3 % of the time was spent on linking the three types of representations

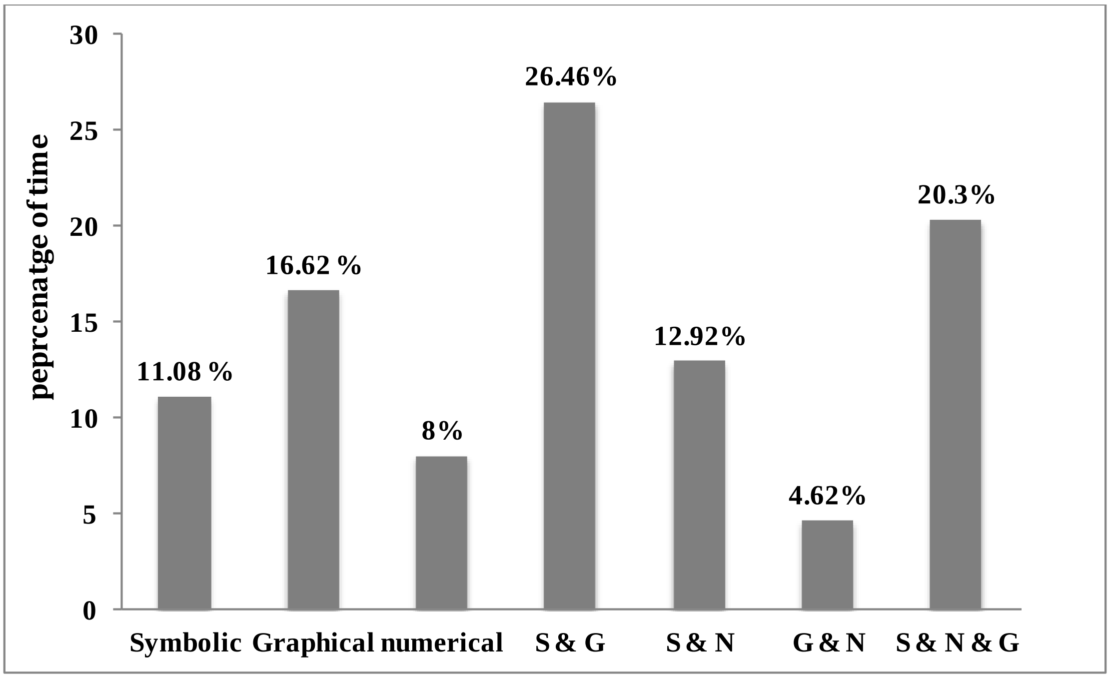


Figure 21. Overview of the time, spent on representations (individually or connected) in all sessions of the experimental group for the derivative concept

Note., S & G = connection between symbolic and graphical (covers both directions), S& N = connection between symbolic and numerical (both directions), G &N = connection between graphical and numerical (both directions), S & N& G = connection between the three types of representation

4.2.4. Teaching and learning methods used in the two groups.

Based on the observation of the two groups (control and experimental) and based on the analysis of the observation logs as in the previous examples (Table 8 and Table 9), the researcher estimated the time spent on each type of the teaching methods (lecture, group work, individual work, questioning technique, and use of technology) used by the two instructors. Figure 22 includes a bar graph that shows the percentages of the different types of teaching methods used in the two groups.

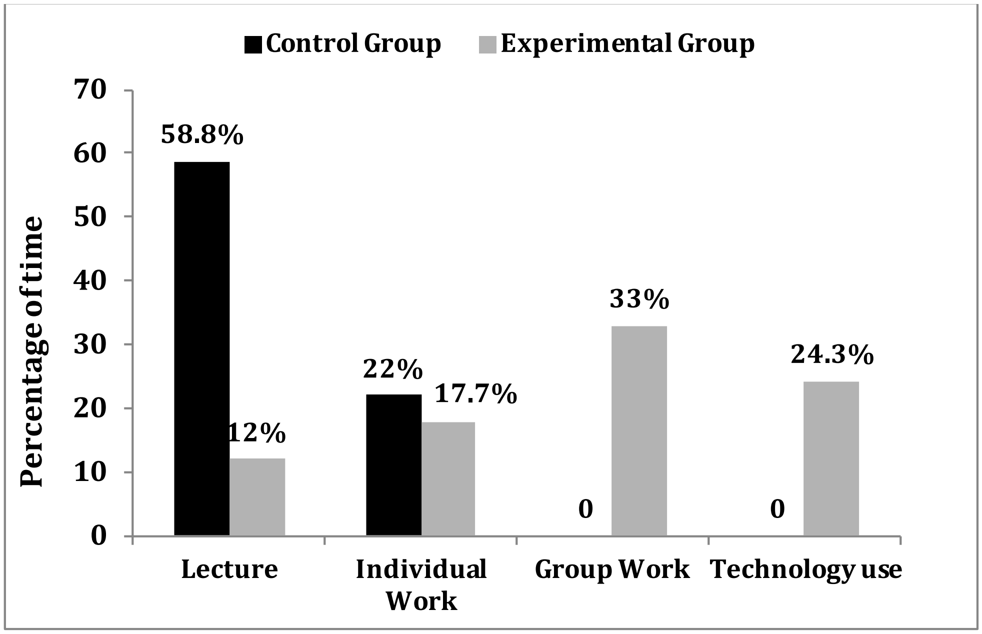


Figure 22. Comparisons of different types of teaching methods used in the two groups

In the control group, the lecture method was frequently used where 58.8% of the time was spent on this method. The knowledge is transmitted directly from instructor X to her students who, most of the time, passively receive information, listen, take notes, and mimic her procedures to solve related problems. Next, is the individual work where 22 % of the time was given to students to solve exercises and problems related to derivatives. It is noticed that technology and group work were never used at all. The remaining time (19. 2 %) was devoted to answer questions asked, at the beginning of every session, by instructor X to her students about the materials they learned in the previous session (revision of the concepts). In addition, some time was spent on answering students' questions related to homework and clarifying the material that they did not understand. In general, the teaching approach used seems teacher- centered; the teacher was the center of attention for 58.8 % of the time compared to 22 % devoted to students. However, sometimes instructor X tried to engage her students by asking them questions during the lecture and inviting them to come to the board and solve exercises at the end of each session.

However, in the experimental group ,instructor Y used several teaching methods such as group work, individual work, lecture, questioning technique and technology. The lecture method was infrequently used (12 %); instead, students completed set activities (paper -pencil based) working in groups (33 %), compared to 17.7 % working individually. Through these activities, students were actively involved in constructing their knowledge and exploring the derivative concept. In addition, 24.3 % of the time was spent on using technology (*Autograph*). Due to time limitation, instructor Y conducted the *Autograph* based activities but students were asked to observe, analyze, make conjectures and interpret the problem until an appropriate conclusion was reached. Finally, the remaining time was spent on the revisions that were conducted at the beginning of each session and on answering students' questions on the materials that they did not understand or questions related to homework.

In general, the teaching approach used in the experimental group seems student-centered; 50. 7 % of the time (group work + individual work) was devoted to students for discovering and constructing their knowledge compared to 12% devoted to lectures (teacher). Moreover, despite the fact that instructor Y conducted the *Autograph* based activities (24.3%), most of the attention was given to students, which supports the student-centered approach as well.

4.3- Analysis of Students' Responses on the Derivative Questionnaire

A questionnaire on *“Students’ perceptions of the notion of derivatives”* (Appendix B) was administered to all students before the instruction on derivative. It aims to record students’ dominant image of the concept of derivative. In addition, it aims to examine students’ conceptions related to derivative before the implementation of the unit on derivatives. The questionnaire was administered again, immediately after the end of the implementation of the unit on derivatives to investigate the development and the progress of students' conception of derivative. The questionnaire consists of three parts (I, II & III).

The first part (I) is an open-ended question that asks students to freely write what they already know about derivative; "*What do you know about derivative (derivative function and derivative at a point)? Explain as much as you can*".

Parts (II) and (III) are multiple – choice items that ask students to choose the correct statements concerning the derivative concept. Part II includes 8 items (a → h) mostly related to the derivative of a function at a pint or on an interval, and part III includes 4 items (a → d) two of which relate to the concept of the derivative function, and two to the role of the derivative in determining the sense of variation of a function. The correct statements for part (II) are items: *a, d, e,* and *g* while the correct statements for part (III) are items: *b* and *c*.

In the following, three major sections are presented: analysis of the results of the control group, analysis of the results of the experimental group and comparison between the two groups. Data are analyzed both qualitatively and quantitatively.

4.3.1. Analysis of the Results in the Control Group

This section presents analysis of the control group's results before and after defining the derivative concept. A paired t - test is used to check whether there is a significant difference in the mean scores before and after the implementation of the unit on derivatives.

4.3.1.1. Qualitative analysis of all parts of the questionnaire (before and after)

Results of Part I of the Questionnaire for the control group (Before and After). Students' answers on part (I) are classified into categories, and then the number of responses that reflect the symbolic representation alone (related to rules and formal definition of derivative), graphical alone (related to slope of tangent/ curve), numerical alone (related to rate of change) or a combination of two or three types of representation are calculated. In addition, students' responses are qualitatively classified in order to investigate whether they focus on "procedural", "formal" conception or a more meaningful understanding.

Table 10 presents control group responses to part I of the questionnaire. Their responses were categorized into seven categories based on representations (individual or connected). Twenty- six students completed the questionnaire before and after the implementation of the unit on derivative. When asked about the meaning of derivative in the first open ended- part, students' responses varied as shown in Table 10.

Table 10

Control group responses to part I of the questionnaire, before and after the implementation of the unit on derivatives

	S	G	N	S & G	S &N	G &N	S &N& G	No answer	Number of students
Before	16	2	1	3	0	0	0	4	26
After	11	2	0	7	0	0	6	0	26

Notes.

S = symbolic representations alone (rules); G = graphical representations alone (slope of tangent or curve); N = numerical representation alone (rate of change); S & G = symbolic and graphical; S &N = symbolic and numerical; G& N = graphical and numerical; S &G &N = symbolic, graphical and numerical representations

The analysis of the questionnaires revealed that before the implementation of the unit on derivative, 16 students defined derivative through giving examples of functions with their derivatives and listed some rules of differentiation as compared to 11 students after the implementation of the unit on derivatives. Examples of students' responses were as follow (before the implementation of the unit on derivative):

> *If* $f(x) = 3$, *then* $f'(x) = 0$, *if* $f(x) = 2x^2 + 3x$, *then* $f\,'(x) = 4x + 3$. *Other students listed some derivative rules (e.g.* $(x^n)' = nx^{n-1}, \frac{d}{dx}(sin\ x) = cos\ x$, $[f(x).g(x)]' = f\,'(x)g(x) + g'(x)f(x)$, *and others*

This result is affected by students' previous experiences of the derivative concept, taught in their high schools. After instruction, 11 students defined derivative using similar examples as above, listed some rules of differentiation, and mentioned the formal definition of derivative using limit, $f'(x) = \lim_{h\to 0} \frac{((f(x+h) - f(x))}{h}$. This is perhaps because of the teaching approach (Book 1 and emphasis on the symbolic representation of derivative) that has used in the control group. Then, the second types of representation that exist in students' mind, after instruction, are both the graphical and symbolic representations connected together. It is noticed that the number of students who connected derivative to the slope of tangent line and to the symbolic representation increased from three to seven after the implementation of the unit on derivatives. Finally, it is noticed that after the implementation of the unit on derivatives, six students mentioned the three modes of representation together while none mentioned all of them together before the implementation of the unit on derivative. Here is an example of students' responses:

$f'(a)$ = *slope of the line tangent to the graph at* $x = a$

$f'(x)$ = *instantaneous rate of change of* ***f*** *with respect to x.*

$$f'(x) = \lim_{h\to 0} \frac{((f(x+h) - f(x))}{h}$$

$$(u.v)' = u'v + v'u$$

$$\left(\frac{u}{v}\right)' = \frac{(u'v - uv')}{v^2}$$

Although there is a little improvement in students' responses after the implementation of the unit on derivatives, the majority of students' conception is limited to the 'procedural' conception of the derivative concept that consists of rules and algorithms learned by memorization. It is noticed that none of the students in the control group spoke about real life applications of derivative. Therefore, the majority of students' concept images, which consists of "all pictures, properties and processes associated with the concept" (Fischbein, 1987; Tall & Vinner, 1981), are limited to their ability to work with the symbolic representation.

Results of Part II of the Questionnaire for the control group (Before and After). Since students can pick up more than one choice, it is worth noting that some students might select both the correct and wrong options showing contradictions between their choices.

As shown in Figure 23, there is an improvement in students' answers. This is reflected by the increase in the number of correct answers on each item ,after the implementation of the unit on derivatives

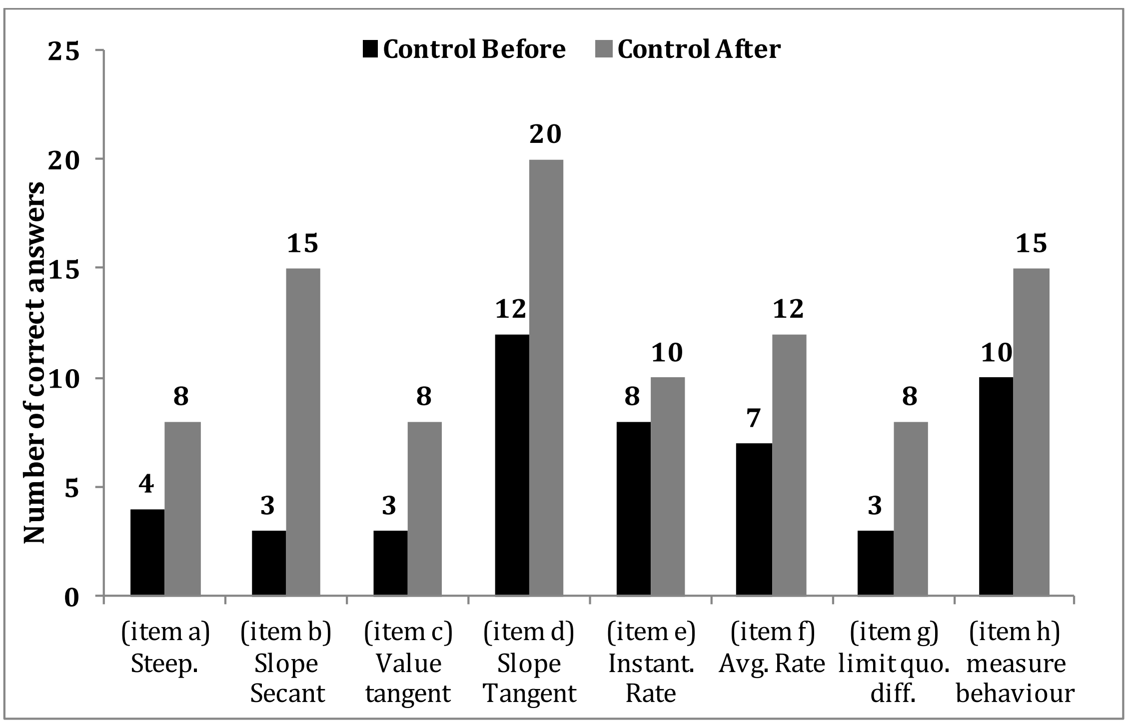


Figure 23. Number of students' correct answers to each item on Part II of the questionnaire before and after the implementation of the unit on derivatives, in the control group.

Many things can be observed from Figure 23: First, items a, c, e and g got the lowest correct responses before and after the implementation of the unit on derivatives. For example, eight students answered item a correctly after the implementation of the unit on derivatives as compared to four students before the implementation of the unit on derivative.

Many students missed the fact the derivative does not measure the steepness of a function at a point (item a). In addition, many students have a misconception about derivative relating it to the value of the tangent equation (item c) at a point. The number of correct responses to this item confirms this result (only eight students solved item *c* correctly, after instruction). Also, although students worked extensively in class on the formal definition of the derivative ($f\,'(a) = \lim_{h \to 0} \frac{((f\,(a+h) - f\,(a))}{h}$), item g obtained a low response (three and eight before and after the implementation on the unit on derivatives respectively). One explanation for this result may be that students did not understand the wording in item g (*derivative is equal to the limit of the quotient difference of the ordinates and the abscissas of two points as the distance between them approaches* 0) and how it relates to the formula. Moreover, few students (10) related derivative to the instantaneous rate of change (item *e*).

On the other hand, 20 students out of 26 related derivative to the slope of tangent line (item d) after the implementation of the unit on derivatives. This gives an indication that many students seem to have the graphical representation of derivative (as a slope of tangent line) in their concept image while the ideas of steepness, instantaneous rate of change are not included in their mind. In fact, these results are not surprising because the approach used in the control group emphasizes the symbolic and the procedural approach to derivative.

Finally, the analysis of students' questionnaires revealed that some contradictions exist in students' choices even after the implementation of the unit on derivatives:

- Nine students think that the derivative is equal to both the slope of the secant line drawn between two points (*item b*) and the slope of the tangent line at a point *(item d)*.
- Seven students think that the derivative is equal to both the average rate between two points (item f) and the instantaneous rate of change at a point *(item e*)

One explanation for the existence of contradictions in students' choices is related to the approach used in the control group where students were not given opportunities to distinguish between the average rate and the instantaneous rate of change or between the secant line and the tangent line.

*Results of Parts III of the Questionnaire for the Control Group (Before and After)***.** Figure 24, represents control group responses to part III of the questionnaire before and after the implementation of the unit on derivative.

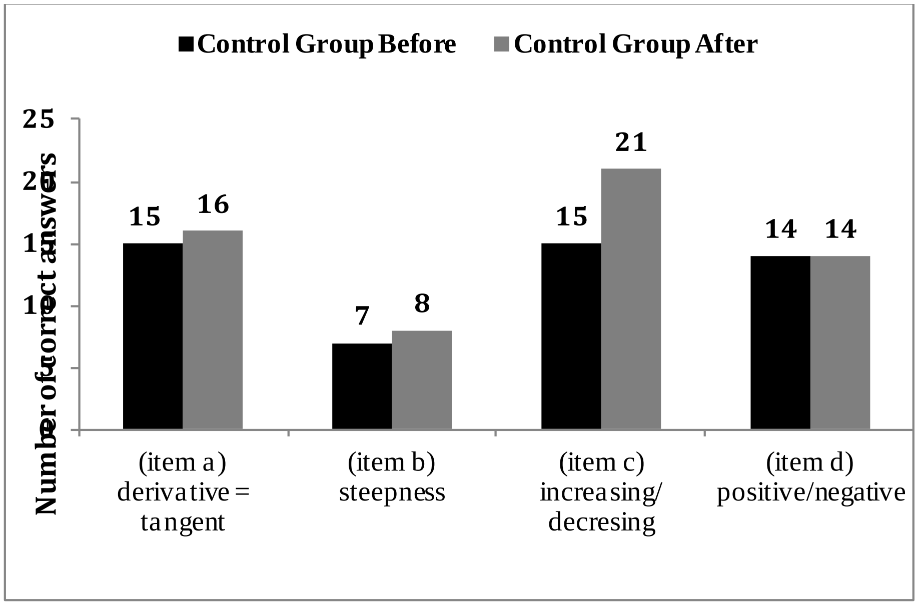


Figure 24. Number of correct answers on each item on Part III of the questionnaire before and after implementation of the unit on derivatives, in the control group.

As shown in Figure 24, there is a little improvement in students' answers as shown by the slight increase in the number of correct answers to each item after the implementation of the unit. For example, an increase of only one correct answer is observed for items *a* and *b*, while no improvement at all was observed for item *d* before and after the implementation of the unit on derivatives.

Concerning students’ conceptions of derivative, it is observed that item *b*, which states that the derivative function measures at every point the steepness of graph of the function at that point, gets the lowest frequency of correct responses (seven and eight before and after the implementation of the unit on derivatives respectively). Concerning item *a*, it is noticed that only 16 students answered this item correctly after the implementation of the unit, which means that 10 students have a misconception assuming that the derivative function is equal to the tangent equation and not equal to the slope of the tangent line. This indicates that the concepts tangent line and its slope are not clear in students' minds. Therefore, more emphasis in teaching should be given to differentiate between the slope of the tangent line and the tangent line itself. It is noticed that there is no improvement in the number of correct responses for item d. This means that those 14 students still think, after the implementation of the unit on derivatives, that the sign of a function (positive/ negative) is the same as its variation (increasing/ decreasing). In addition, the analysis of students' questionnaires revealed that eight students have showed contradictions in their choices by selecting both items c and d. In other words, these students have a misconception concerning the relation between $f'(x)$ and $f(x)$, and hence, fail to distinguish between the sign of a function (positive/ negative) and the variation of the function (increasing/ decreasing). This is despite the fact that item c, which states that the derivative function $f'(x)$ indicates the increase or decrease of $f(x)$, gets the highest response (15 and 21 before and after the implementation of the unit on derivatives respectively). One explanation to these facts is that some students selected item c randomly. Another explanation is that some students have poor reading comprehension skills.

4.3.3.2 Quantitative analysis using Paired t - Test Statistics for the Control Group (Before and after***)***

The following section presents the quantitative analysis of the questionnaire based on the hypothesis that there is a significant difference between the two groups in terms of their performance in favor of the experimental group. The null hypothesis claims that there is no difference in the mean scores between the two groups.

There are eight items in parts II and 4 items in part III. Therefore, the grade of the questionnaire is out of 12, where 1 point is assigned to each correct answer and 0 for each wrong answer. Table 11 shows the results of the paired- t test statistics conducted for the control group.

Table 11

Results of Paired t-Test statistics, on the questionnaire, for the control group (before and after)

	Control Group							
	Before the implementation of the unit on derivative		After the implementation of the unit on derivative					
	Mean	SD	Mean	SD	N	df	T	Sig.(p-value)
Questionnaire	3.96	1.34	6	1.56	26	25	-7.8	0.00

Notes. The questionnaire is out of 12. A t- test was performed at $\alpha = 0.05$ level of significance

Paired t-test measures whether there is a significant difference in the mean scores from the same individuals measured both before and after some intervention or treatment. The assumptions of the t-test were examined.

Dependent variable: Scores on the questionnaires

Independent variable: control group (matched pairs) measured twice; before and after the implementation of the unit on derivatives

Normality: The differences between the pairs are approximately normally distributed using the Shapiro -Wilk and Kolmogorov -Smirnov tests in *SPSS* software

According to Table 11, the comparison of students' scores on the questionnaire before the implementation of the unit on derivative on derivative (M = 3.92, SD = 1.29) and after the implementation of the unit on derivatives (M = 6, SD = 1.79) revealed that there is a significant difference in the mean scores at the 0.05 level ($p = 0.00 < 0.05$). On average the mean score improved by 2.04. It is worth noting that for many students, the derivative concept was a new topic, which explains the low score before the implementation of the unit on derivative on derivative (M= 3. 96). Moreover, although the difference in scores is statistically significant, the mean score after the implementation of the unit on derivatives is considered low (M = 6 out 12). Therefore, the approach adopted in the control group (emphasizing the symbolic representation of derivative) needs improvement.

4.3.2. Analysis of the Results in the Experimental Group

This section presents analysis of the experimental group's results before and after the implementation of the unit on derivatives. A paired t - test is used to check whether there is a significant difference in the mean scores of students before and after the implementation of the unit on derivatives.

4.3.2.1 Qualitative analysis of all parts of the questionnaire for the Experimental Group (before and after)

Results of Part I of the Questionnaire for the Experimental Group (*Before and After*). Twenty- six students completed the questionnaire before and after the implementation of the unit on derivatives. Students were given enough time (10- 15 minutes) to complete it. When asked about the meaning of derivative in part I, students' responses varied significantly after the implementation of the unit on derivatives, as shown in Table 12.

Table 12

Experimental group responses to part I of the questionnaire, before and after the implementation of the unit on derivates

	S	G	N	S & G	S & N	G & N	S &N& G	No answer	Number of students
Before	15	3	0	3	0	0	1	4	26
After	2	1	1	1	2	2	17	0	26

Notes.

S = symbolic representations alone (rules); G = graphical representations alone(slope of tangent or curve); N = numerical representation alone (rate of change); S & G = symbolic and graphical; S &N = symbolic and numerical; G& N = graphical and numerical; S &G &N = symbolic, graphical and numerical representations

It is noticed, from Table 12, that the symbolic representation of derivative (related to rules of differentiation and formal definition of derivative) dominated the thinking of most students (15 students) before the implementation of the unit on derivative. This result was very close to the results of the control group (16 students). For example, one student wrote the following: if $f(x) = 4,$ then $f'(x) = 0,$ which is basically the same example given by a student in the control group, with the only different in the constant value of the function chosen. Another student wrote if $f(x) = 6x,$ then $f'(x) = 6.$ Other students listed some derivative rules (e.g. $(x^n)' = nx^{n-1}$, $[u.v]' = u'v + v'u,$ $f'(x) = \lim_{h\to 0}\frac{((f(x+h) - f(x))}{h}$ and others. However, after the implementation of the unit on derivatives, the three types of representation (slope of curve/tangent line, instantaneous rate of change/ rules of differentiation and formal definition of derivative) dominated the thinking of most students (17 students). This result could be explained by the teaching approach (Book 2, visualization and emphasizing the multiple-representations of derivative) used in the experimental group. Moreover, the analysis of the questionnaires revealed that 10 students mentioned real life applications of derivative (economy, biology, physics, and chemistry). Samples of students' responses are presented below:

- *Velocity = (distance)'*
- *Marginal cost = (cost of production)'*
- *Derivative is used to study change in population with respect to change in births, deaths*

Therefore, as opposed to the control group students' concept definitions and images are not limited to the symbolic representation of derivative (rules for differentiation, formal definition). They exceed to include the graphical representation of derivative as the slope of the tangent line, the numerical representation as the instantaneous rate of change, the formal definition of derivative using limit, and its application in real life.

Results of Part II of the Questionnaire for the Experimental Group (*Before and After*). As shown in Figure 25, there is a significant improvement in students' answers as reflected by the increase in the number of correct answers on each item after the intervention. Before the implementation of the unit on derivative, item *g* got the lowest number of correct responses (three) followed by items *f* (four correct responses), *b* (five correct responses), *a* (eight correct responses), *e* (eight correct responses), *c* (five), *h* (11), and *d* (16) respectively. However, after the intervention, most of the items were solved correctly by most of the students. For example, 26 students think of derivative as the slope of a tangent line (item *d*) and as the limit of the quotient difference of the ordinates and the abscissas of two points as the distance between them approaches 0 (item *g*).

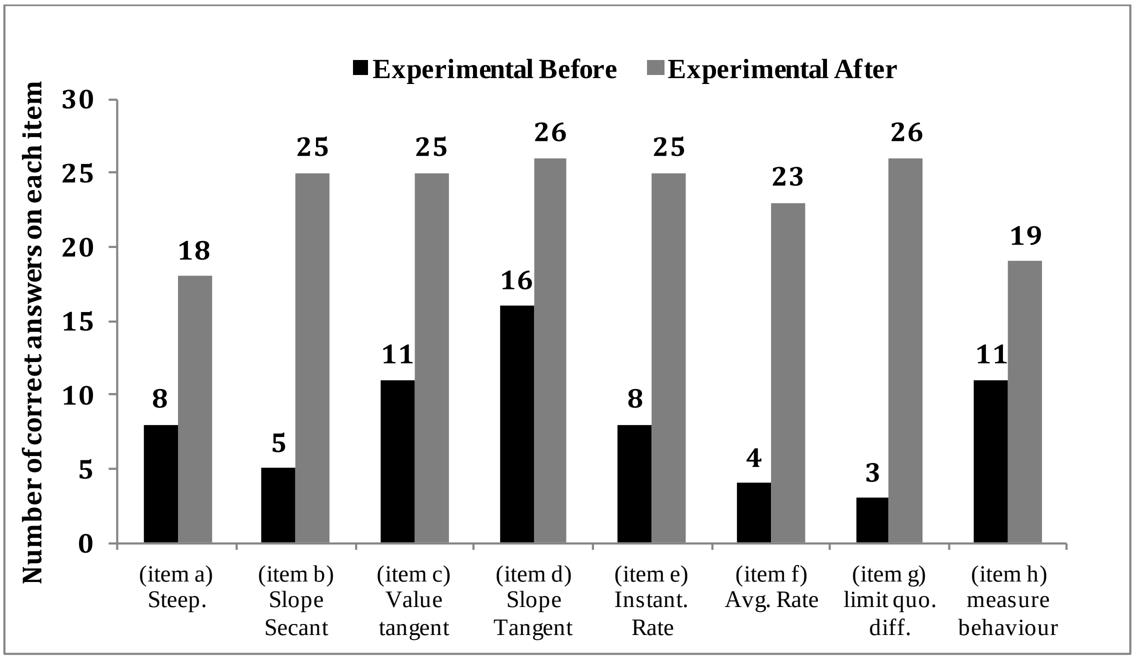


Figure 25. Number of correct answers to each item on Part II of the questionnaire before and after the implementation of the unit on derivatives, in the experimental group.

The items that get the least correct responses are items *a* (18 correct responses), and *h* (19 correct responses) respectively. There are few students where the idea of steepness of the graph of the function is not clear in their mind.

Finally, the analysis of students' questionnaires revealed that the contradictions that existed in some students' choices before the implementation of the unit in derivative disappeared completely after the intervention:

- Before the implementation of the unit on derivative, eight students mentioned that the derivative at a point is equal to both the slope of secant line and slope of tangent line. However, after the intervention, this contradiction disappeared. No student selected both item together.
- Before the implementation of the unit on derivative, eight students thought that the derivative at a point is equal to both the average rate of change between two points and the instantaneous rate of change at a point. However, after the implementation of the unit on derivatives this contradiction disappeared. This means that the approach used in the experimental group (Book 2, multiple representations of derivative) seems effective. Also, most of the students had a clear understanding of the difference between slope of secant line (item *b*) and slope of tangent line (item d), and instantaneous rate of change (item *e*) and average rate of change (item *f*).

Results of Part III of the Questionnaire for the Experimental Group (*Before and After*). As shown in Figure 26, there is a good improvement in students' answers as shown by the increase in the number of correct answers on each item, except for item *b*, after the implementation of the unit on derivatives.

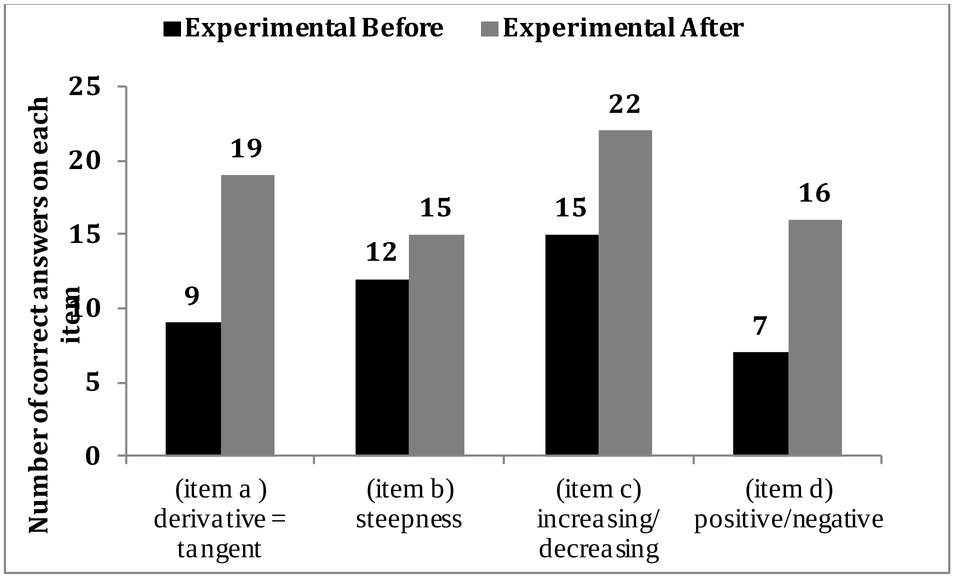


Figure 26. Number of correct answers on each item on Part III of the questionnaire before and after the implementation of the unit on derivatives, in the experimental group.

The number of correct answers to item a increased by 10, followed by item d which increased by nine, and then item c which increased by seven. It is noticed that item b, which relates derivative to the steepness of the graph of the function, did not increase significantly (12 and 15 correct responses before and after the implementation of the unit on derivatives). This means that more emphasis in teaching should be given to the idea of steepness and the derivative concept. Concerning item d, it is noticed that only 16 students out of 26 solved it correctly which means that there are some students who are are still confused about the relation between $f'(x)$ and $f(x)$. The analysis of the questionnaires revealed that there are eight students who showed contradictions in their choices by selecting both items c and d. Thus, again as in the control group, the difference between the sign of a function and its variation should be more emphasized when teaching derivative.

3.4.2.2 Quantitative analysis using paired t- Test for the Experimental Group(before and after)

Results of Paired t- Test for the Experimental Group (Before and after). Table 13 presents the results of the paired t- test statistics applied on students' scores on the questionnaire (score out of 12) before and after the implementation of the unit on derivatives, in the experimental group. This test measures improvement over time. The assumptions of the t-test were examined (Shapiro -Wilk normality test using SPSS).

Table 13

Results of *Paired t-test statistics, on the questionnaire, for the experimental group (before and after)*

	Experimental Group							
	Before the implementation of the unit on derivative		After the implementation of the unit on derivative					
	mean	SD	Mean	SD	N	df	T	Sig.(p-value)
Questionnaire	4.15	1.22	9.89	0.95	26	25	-21.75	0.00

Notes. The questionnaire is out of 12. T- test was performed at α = 0.05 level of significance

According to Table 13, comparison of students' scores on the questionnaire before the implementation of the unit on derivatives (M = 4.15, SD = 1.22) and after the intervention (M = 9.89, SD = 1.12) revealed that there is a significant difference in the mean scores at the 0.05 level (p = 0.00). Since the derivative concept was a new topic for many students, the mean score before the implementation of the unit on derivative was low (M = 4.15). However, after the implementation of the unit on derivatives, the mean score increased to 9.89. On average, the mean score improved by 5.74. Therefore, the approach adopted in the experimental group (emphasizing multiple representation of derivative and use of technology, *Autograph*) seems to be very effective.

4.3.3. Comparison between Control and Experimental Groups

This section presents a comparison between the two groups. The data are analyzed both qualitatively and quantitatively. An independent t- test is used to investigate whether the mean scores of the two groups (control and experimental) are significantly different from each other.

4.3.3.1 Pre- control versus Pre- experimental

First, it is important to compare and note the similarity of the results between the two groups before the implementation of the unit on derivative on derivative. The results of the three parts of the questionnaires (I, II, and III) revealed that the symbolic representation of derivative dominated the thinking of both groups (16 students (62 %) in the control group and 15 students (58 %) in the experimental group). Both groups had a dominant "procedural" conception of derivative that consists of rules of differentiation (e.g. $(x^n)' = nx^{n-1}$, $\frac{d}{dx}(\sin x) = \cos x$, and $[f(x).g(x)]' = f'(x)g(x) + g'(x)f(x)$ and others) learned by memorization. Moreover, the results revealed that both groups had deficiencies in their understanding and misconceptions on derivative; majority of students thought that the derivative of a function at a point is equal to the slope of a secant line between two points, the value of the tangent equation at a point, and the average rate of change between two points. In addition, for most students the relation between the steepness of the graph a function and the derivative concept or the relation between a function $f(x)$ and its derivative $f'(x)$ were ambiguous. Finally, the independent t- test (Table 14) revealed ($p = 0.59 > 0.05$) that there is no significant difference in the mean scores of the questionnaire between the two groups before the implementation of the unit in derivatives.

Table 14

Independent t-test comparing control and experimental groups' mean scores on the questionnaire, before the implementation of the unit on derivative

	Two Groups					
	Control Group		Experimental Group			
	mean	SD	Mean	SD	T	Sig.(p-value)
Questionnaire	3.96	1.34	4.15	1.22	0.54	0.59

4.3.3.2 Post- control versus Post - experimental

The results of students' responses on the questionnaires revealed that there are significant differences between the groups after the implementation of the unit on derivatives.

- The symbolic representation dominates the thinking of 11 students in the control group as compared to only two students in the experimental group.
- None of the students in the control group talked about real life applications of derivative while 10 students in the experimental group mentioned that the derivative is used in many aspects of life (biology,

velocity and acceleration in physics, growth population, marginal cost in economics)

- When students were asked about the meaning of derivative, only six students in the control group mentioned the three types of representation (rules of differentiation, slope of curve/ tangent line and instantaneous rate of change) in their responses as compared to 17 students in the experimental group.
- Concerning parts II and III, it is noticed that there is an increase in the number of correct answers to each item for the two groups with a higher frequency for the experimental group.
- In part III, item d, which states that the derivative of a function indicates whether the function $f(x)$is positive or negative, gets the lowest response by both groups. It is suggested that the difference between the sign of a function (positive/ negative) and its variation (increase/ decrease) should be more emphasized during teaching. To many students, the two terms are the same, while in fact they are not.
- In both groups, the contradictions in students' choices decreased after the implementation of the unit on derivatives with higher frequency for the experimental group.
- Students in the experimental group outperformed students in the control group on the questionnaire. The difference in mean scores between the control group (M = 6) and experimental group (M = 9. 89) is 3.89 which is statistically significant ($p = 0.00 < 0.05$) as shown in Table 15.

Table 15

Independent t-test comparing control and experimental groups' mean scores on the questionnaire after the implementation of the unit on derivatives

	Two Groups						
	Control Group		*Experimental Group*				
	mean	*SD*	*Mean*	*SD*	*df*	*T*	Sig.(p-value)
Questionnaire	6	1.56	9.98	0.95	50	-10.72	0.00

Notes. Twenty -six students on each group completed the questionnaires. *The test was performed at α = 0.05 level of significance. The assumptions of the test were examined.*

In conclusion, based on the above results, it is clear that students in the experimental group showed better understanding that students in the control group. It seems that the concept definitions and images of many students of the control group are dominated by the procedural symbolic representation of derivative (rules of differentiation and formal definition) before and after the implementation of the unit on derivatives. However, some students have the graphical (slope of tangent) and numerical representations (rate of change) of derivative in their minds. On the other hand, students in the experimental group have a richer concept image of the derivative concept. Their responses, which reflect their concept images, include the derivative as the slope of tangent line, the derivative as an instantaneous rate of change, the derivative as rules of differentiation and limit of quotient differences, the derivative as a measure of steepness of the graph of a function at a point. Finally, they includes the relation between $f'(x)$ and $f(x)$. This result is interpreted by the fact that the approach used in the experimental group (Book 2, multiple representation approach visualization, *Autograph)* for teaching derivative seems more effective than the approach used in the control group and enhances students' understanding of the concept.

4.4. Analysis of the 'Derivative Test'

A test (Appendix D) on derivatives was administered to all students in both, the control and experimental groups. The test includes five conceptual – understanding based problems on derivatives. Two associate professors of mathematics and mathematics education reviewed and approved the test. Students did the exam two weeks before the end of the semesters, but after the implementation of the derivative units in the two groups. Students were given 65 minutes to complete the test and were asked to justify their answers and reasoning. The researcher was present to make sure that everything was clear and to answer any technical questions not related to the solutions. It is worth noting that students in the experimental group, who were taught using a multiple- representation approach, Book 2 and technology (*Autograph)*, had not solved questions similar to the test's questions in class. This ensures the fairness between the two groups.

Usually typical (traditional/ procedural) questions on derivative are dominated by the symbolic approach where the functions are expressed through equations and no tables or graphs are used. It is worth to mention that students , in both groups, had solved a quiz (Appendix I) composed of three traditional questions related to derivative. For example, these questions vary to cover some or all of the following:

- Find the derivatives of a set of functions (using rules of differentiation such as *power rule, chain rule, quotient and product rules, etc*).

- Find the derivative of $f(x)$ using the formal definition of derivative (limit of quotient of differences).
- Find the linearization of a function $f(x)$ at a given point, where $f(x) \approx L(x) = f(a) + f'(a)(x - a)$)
- Find the equation of the line tangent to the graph of a function at a given point.
- Sketch the graph of a function using a sequence of steps (finding the domain of definition, limits, asymptotes, table of variation, critical points, inflection point, and x and y- intercepts).

However, in this study, the content and the type of questions included in the test differ from those of a "traditional" test. In this test (appendix D), three questions out of five include graphs of functions without their equations, one question includes a symbolic representation of derivative (equations of functions), and one question presents the function in a table form. These questions aim to identify students' difficulties concerning the different representations and test their conceptual understanding and their abilities to translate among the different representations of derivatives.

The analysis of the test was conducted having the following questions in mind: Are students comfortable with the graphical and numerical representations of derivative? In the absence of an equation for the function, are students able to work with derivatives using only information from the graphs or table? Or do they view graphs and table as just meaningless pictures that do not convey mathematical meaning?

In the following, three major sections are presented:

- Section (4.4.1) presents an a priori analysis of the test based on the genetic decomposition of the derivative concept. In addition, it focuses on the objective(s) behind each question/ sub-question, presents the best solution expected and its *APOS* level, and then other possible solutions with the *APOS* level for each are provided.
- Section (4.4.2) presents a qualitative analysis of students' responses on the test based on the a priori analysis.
- Section (4.4.3) presents a quantitative analysis of students' scores on the test using descriptive and t test statistics.

4.4.1. A priori analysis of the Derivative Test

Question I: The graph of the function f is given (without its algebraic expression) and a tangent line (L) is drawn at the point (5, 3) as shown in Figure 27. This question tests students' graphical understanding of a function f and its derivative at a point. It includes three sub- questions (1, 2, and 3). This question was adobpted from a study conducted by Asiala et al. (1997). The representation involved in this question is the graphical representation.

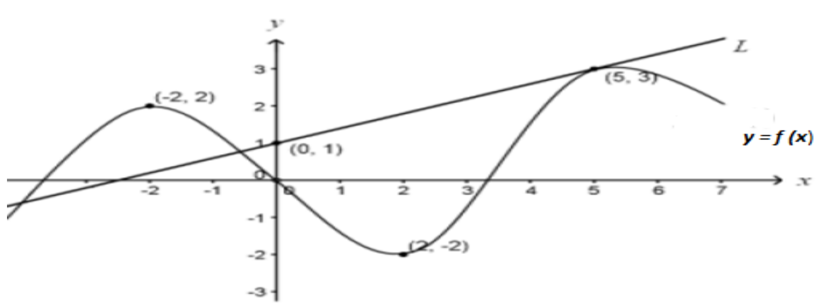


Figure 27. Graph of f and its tangent line (L)

Sub- question 1: (*Find* $f(5)$. *Justify your answer*). This question tests students' ability to use the graph in order to find the value of *f (x)* at x = 5. To solve this question, students are expected to link the y- coordinate of the point (5, 3) on the graph with the function value at x = 5. In other words, $f(5) = 3$. These students tend to have a *process* conception of the concept.

Other Possible Solutions Expected to be solved by Students:

- Some students may not be able to find the answer directly because they do not see the graph as a source of such information, thus the graph seems meaningless to them. Since the point (5, 3) is also on the tangent line (L), some students are expected to find the equation of (L), $y = ax + b$, using the two points (0, 1) and (5, 3) and then substitute x = 5. The equation of (L) is : $y = \frac{2}{5}x + 1$; *where a = slope of L =* $\frac{3-1}{5-0} = \frac{2}{5}$ *and b = 1; that is the y-intercept where the line (L) cuts the y-axis*. It is noted that some students might find the value of *b* through calculations by substituting one of the points (0, 1) or (5, 3) in the equation of (L). Therefore, $f(5) = (2/5 * 5) + 1 = 3$. These students are using step-by-step procedures and thus tend to have an *action* conception of the *function* concept; they cannot work in the absence of the algebraic expression.
- Some students might try to find the equation of the function $f(x)$ or guess an equation by a trial- and- error process. These students lack both, the *process* and *object* conceptions of a function. They cannot interpret

the graph, extract the necessary information or use it in the problem solving process. They think of a function as an expression that evaluates something when numbers are plugged into the equation.

<u>Sub- question 2 :</u>*(Calculate the value of* $f'(5)$. *Justify your answer).*This question tests students' understanding of the relationship between the derivative of a function at a point and the slope of the tangent line at that point. In the absence of the equation for the function, are students able to find $f'(5)$? Two different methods may be used to calculate the slope of the tangent line (L). The first method is the slope formula = $\frac{y_2 - y_1}{x_2 - x_1}$ using the two points (0, 1) and (5, 3) given on the line, and the second method is graphical by counting the rise /run between two points (drawing a right triangle).Therefore, the slope of (L) = $\frac{3-1}{5-0} = \frac{2}{5} = f'(5)$. Students who use such a solution tend to have an *object* conception. In terms of representations, this involves translation from a graphical (slope of tangent) to a symbolic representation ($f'(5)$).

Other Possible Solutions:

- Some students might solve this question wrong by finding the equation of the tangent line (L) and then plugging $x = 5$ to get $f'(5) = \frac{2}{5}(5) + 1 = 3$. These students have a misconception as they think that the derivative function is equal to the equation of the tangent line. These students lack an *object* conception of the derivative concept since they are not able to relate the derivative a point with the slope of the tangent line at that point.

- Some students may be aware of the relation between the slope of a tangent and the derivative at a point, but might use a wrong formula for the slope as $\frac{x2-x1}{y2-y1}$ instead of $\frac{y2-y1}{x2-x1}$. These students lack the *process* conception, since they don't understand the idea of rise/run. They also lack the *action* conception as they apply wrongly the formula.
- Some students might write $f'(5) = 0$ because they might think that the point (5,3) is the maximum point of the curve of $f(x)$. These students lack an *object* conception of the derivative concept because at a maximum point (derivative = slope of tangent = 0) the tangent line must be a horizontal line, which is not the case in this question.

Sub- question 3: (*Find* $f(5.1)$. *Justify your answer and reasoning. Be as accurate as possible).* This question tests whether students are able to apply the linearization concept to approximate $f(5.1)$, using the equation of the tangent line (L) at the point (5, 3). The graph and the tangent line on a small interval around the point (5, 3) are almost confounded. Thus, the tangent line (L) at x = 5 can be used to estimate $f(5.1)$. So to solve this question, students are expected to substitute $x = 5.1$ in the equation of (L) to get $f(5.1) = \frac{2}{5}(5.1) + 1 = 3.04$. Students who are able to solve this question correctly, with an indication of understanding, have an *object* conception..

Other Possible Solutions:

Many students are expected not to use the concept of linearization. Therefore, some students might solve it as follow:

- Some students might use the right hand limit for estimation: $f\ (5.1) = f\ (5) = 3^{+}$. These students are aware that the function is increasing, and thus the value must be close to 3. These tend to have a *process* conception. It is important to note that many students are expected to write $f(5.1) = 3.1$. These students seem to have an *action* conception. An action remains, for the individual, as externally driven.
- Some students might solve this question wrong by substituting x = 5.1 in the equation of $f(x)$ which was guessed or calculated by trial - and - error in sub-question 1 (*note: the equation of the function is hard to be determined*). These students tend to have an *action* conception of the derivative concept.

<u>Question II</u>: In this question, the graph of the function $f(x) = -x^4 + 4$ is given without its equation, as shown in Figure 28. This function is increasing over the interval (-∞, 0[and decreasing on the interval]0, ∞). Students are asked to determine whether the derivative of this function is increasing or decreasing using the rate of decrease and increase of the function. This question tests students' understanding of the relationship between the derivative of a function at a point and the rate of change as well as the slope of tangent line at that point. This question is important because a "typical/ traditional" question would require determining the interval(s) where the derivative of a function, given graphically, is positive or negative and NOT increasing or decreasing.

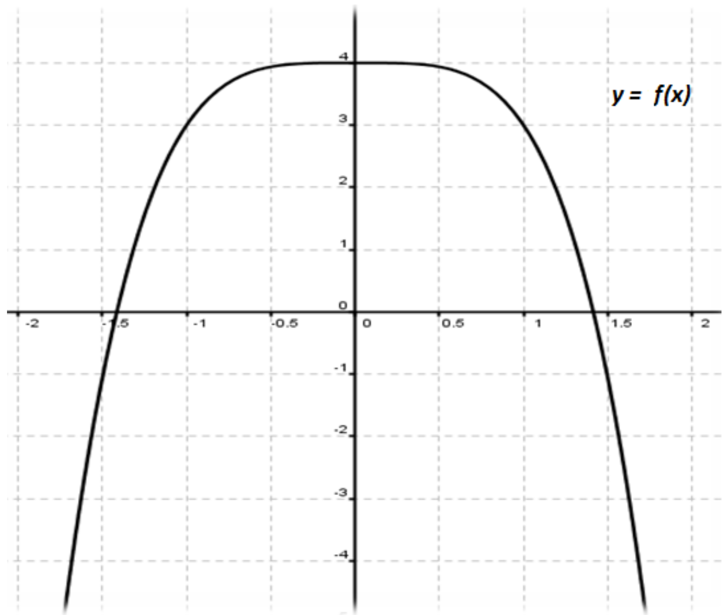


Figure 28. Graph of f(x)

One way to solve this question is to draw several tangents on the graph at different values of *x*, and observe the direction and the steepness of the tangent lines. In fact, the larger the absolute value of the slope, the steeper or vertical the line is. Thus, students are expected to conclude that the derivative is decreasing throughout the interval (-∞, ∞) since:

- Over the interval (- ∞, 0[, the function f is increasing, thus all the tangent lines are increasing lines which means $f' > 0$. Therefore, the slopes of the tangent lines drawn at different values of x, equal to the rate of change, are positive. In addition, the steepness of the tangent lines is decreasing where the slopes of tangent line decrease from almost ∞ (vertical tangent) to zero (horizontal tangent) at x = 0. Thus, the derivative is decreasing on the interval (-∞, 0[.

- On the interval $(0, \infty)$, the function f is decreasing which means $f' < 0$. Therefore, the slopes of the tangent lines, equal to the rate of change, are negative. In addition, the steepness of the tangent lines is increasing with negative slopes of tangent lines .Thus, the derivative is decreasing as well on the interval $(0, \infty)$.

Students who are able to answer this question tend to have a good *schema* of the *derivative* concept. These students understand that the derivative of a function is a function itself and several concepts such as rate of change, slope of tangent lines, and steepness of tangent lines are used in their thinking process. Moreover, they have good visual abilities since they make connections and relations between the properties of the original function f and its derivative f', through the slopes of several tangent lines at different points.

Other Possible Solutions:

- Some students might guess or find the equation of the function $f(x)$, find its derivative $f'(x)$ ($where\ f'(x) = -4x^3$) and then discuss its variation (by taking different values of x, substituting them in the derivative function and observing that the values of $f'(x)$ are decreasing). Others might plot the derivative function and then draw out the conclusion. Their solution may involve translation from graphical to symbolic representation. These students tend to have an *action/ process* conception for what was required. This is because they focus on carrying out step-by-step procedures where each step triggers the next one. Such students are unable to interpret the graph of f or use it as a source of information.

- Some students will fail to answer this question. Some students are expected to write that the derivative is first increasing on the interval (-∞, 0[and then decreasing on the interval]0, ∞). Such incorrect answers occur because such students are unable to differentiate between the graph of a function and the graph of its derivative. To them, they are the same. Thus, these students lack an *object* conception of the derivative concept. It is worth noting that some students will make mistakes in writing the intervals of increase and decrease of the function; that is writing the intervals using y- values instead of x- values.
- Some students are expected to write that, since f is increasing on the interval (-∞, 0 [, its derivative f' is positive and hence it is increasing. Similarly, on the interval] $0, \infty$), the function f is decreasing, thus its derivative is negative and hence f' is decreasing. Such students have a misunderstanding making them unable to distinguish between the variation of the derivative function (increasing or decreasing) and its sign (positive or negative).

Question III: This question consists of three sub-questions (1, 2, and 3). The graph of the function $G'(x) = x^2 - 4$ is given without its equation, as shown in Figure 29. This question aims to test students' understanding of the derivative function G' given graphically, its relation to the graph of G and what each tells about the other. This type of question requires students to read, analyze and interpret the graph of G', gain information about the sign of G' (positive or negative) and the roots of G' in order to relate them to the original function G.

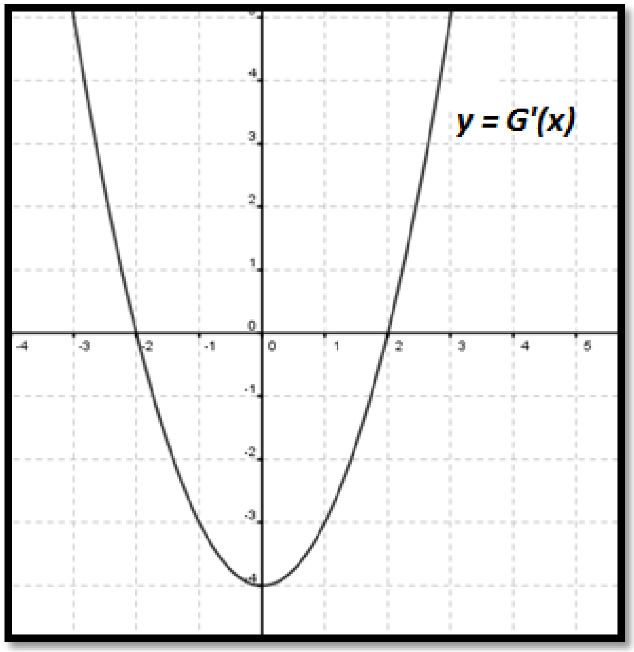


Figure 29. Graph of G '(x)

<u>*Sub-question 1*</u>: (*On what interval is $G(x)$ increasing? decreasing?*). The question tests students' understanding of the relationship between the sign of the derivative function $G'(x)$ (positive or negative) and the variation of the original function G (increasing or decreasing). The function G is increasing when $G'(x)$ is positive; the graph of G' is above the x - axis. The function G is decreasing when $G'(x)$ is negative; the graph of G' is below the x - axis. Therefore, according to the graph of G', the function G is decreasing over the interval]-2, 2[because the derivative is negative in this interval. G is increasing when x ∈ (-∞, -2[∪]2, +∞) because in these intervals the derivative is positive.

Students who are able to use the properties of the graph of G' and relate them to the graph of G have an adequate *schema* of the *derivative* concept.

Other Possible Solutions:

- Some students might find the equation of the function $G\,'(x)$, and then set up the table of variation of G using step-by-step procedures. These students tend to have an *action/process* conception of the *derivative* concept. In addition, their solution involves translation from graphical to symbolic representation.
- Some students are expected to fail in answering this question because they are unable to differentiate between the graph of a function and its derivative; to them they are the same. They would say that G' is decreasing over the interval (-∞, 0[and increasing over the interval]0, ∞). These students lack an *object* conception of the derivative concept. Such students, do not understand that the derivative function is an object or an entity that has its own properties. It is worth noting that some students might write the intervals using y- values instead of x- values. For example, some might write that $G'(x)$ is decreasing over t (-∞, - 4 [and increasing over the interval]- 4, ∞).

<u>Sub- question 2</u>:(Determine the critical points of G(x). Justify your answer). This question tests students' understanding of the relationship between the roots of $G'(x)$ and the critical points of G. In general, critical points include maximum and minimum points of a function (where its derivative function = 0) and points where the derivative function is undefined. In this question, $G'(x)$= 0 when the graph of G' cuts the x- axis. Therefore, the critical points of G occur at $x =$

-2 *and* $x = 2$. Students who are able to answer this question correctly, with an indication of an understanding, have a good *schema* of the derivative concept.

Other Possible Solutions:

- Some students might get the critical points mechanically by finding the equation of $G'(x)$ and solving for x. That is $x^2 - 4 = (x-2)(x+2) = 0$ at x = -2 and x = 2. Their solution involves translation from graphical to symbolic representations. These students tend to have an *action* conception of the derivative concept; they cannot reach a conclusion unless performing step-by-step procedures. Also, they cannot work in the absence of the algebraic expression.
- Many students are expected to say that the minimum point of G', which is (0, - 4) ,is the critical point of $G(x)$. These students lack an *object* conception of the derivative concept. To them, the graphs of a function and its derivative are the same. They seem not to understand that the derivative function is an object or an entity on its own that has its own properties.

<u>*Sub- question 3*</u>: (*Which critical point is a local maximum/ local minimum?)* This question is related to sub-question 2 where students need to discuss the nature of the critical points; that is determining which point is a maximum point and which is a minimum point. A maximum point occurs when G' changes sign from positive to negative or when G changes variation from increasing to decreasing.

A minimum point occurs when G' changes sign from negative to positive or when G changes variation from decreasing to increasing. Therefore, $x = -2$ is a local maximum and $x = 2$ is a local minimum. Students who are able to answer this question correctly have a good *schema* of the derivative concept.

Other Possible Solutions:

- Many students, in particular those who answered sub-question 2 incorrectly, are expected to say that the point (0, - 4) is the minimum point of $G(x)$. Again, these students lack an *object* conception of the derivative concept.
- Some students might say that x= -2 is the minimum point, while x = 2 is the maximum point. They seem to have misconceptions concerning the maximum and minimum points of a function and the maximum and minimum values. It seems that the highest number is taken as a maximum and the other one as a minimum.

<u>Question IV:</u> Given $F(x)$; Let $G\,(x) = F(x) + C$, where C is any constant. Clearly, $G' = F'$ since $\frac{d}{dx}$(C) = 0 (derivative of a constant = 0). Now, explain geometrically why the two derivatives are equal. First, this question tests whether students can make a relation between the symbolic and graphical representations of the derivative. In other words, it tests students' understanding that the derivative is just the slope of a curve at a point or the slope of the tangent line to the curve at that point. Then, they should extend this relation and definition from a single point to the derivative function.

In this question, it is proved symbolically that the derivatives of the two functions are equal because the derivative of a constant is zero. It is required to graphically to prove this fact. To solve this question, students need to realize that the two functions have equal derivatives if their graphs have equal slopes for all values of x. Moreover, students need to recognize that the function $G(x)$ is obtained by shifting the graph of F vertically C units (either upwards or downwards depending on the sign of C). In addition, the slopes of the tangent at any point a on the graph of F and G are the same since the graphs are identical and just translated along the vertical direction, thus the tangent lines are parallel. Students who answer this question as stated above have developed an *object* conception of the *derivative* concept.

Other Possible Solutions:

- Some students are expected to think of polynomials as an example of functions. For example, $f(x) = x^2$ and $g(x) = x^2 + 2$. This example is a prototype example of functions as Bakar and Tall discussed in *Students' Mental Prototypes for Functions and Graphs* (1991). In the absence of an algebraic expression of a function, students are able to work only by thinking of functions such as y = x^2 or polynomials in general. Then, students are expected to find their derivatives using rules of differentiation, plot the functions, and notice that they coincide. These students' understanding is limited to an *action* conception because they need to work with equations first and then do step-by-step procedures to come up with conclusions. This means that these students exhibit a complete reliance on formulas (*action level)*, and that they cannot make connections between the graphical and algebraic representations of derivative.

- Many students are expected to write examples of functions using polynomials such as $y = x^2$ and $y = x^2 + c$, where c is any number, and then state that the derivatives of the two functions are equal because the derivative of a constant is zero. This means that these students have a' procedural' perception of derivative; that is they can work using the symbolic representation through algebraic expressions, equations and rules of differentiation. Their "procedural" knowledge might be good, but their "conceptual" understanding is weak. In other words, these students do not make connections between symbolic and graphical representations of the derivative. These students lack an *object* conception of derivative as slope of the curve or slope of the tangent line.

Question V: In this question, $C(t)$ represents the concentration of a drug in the bloodstream at time t (min), as shown in Table 16. Some values of the function $C(t)$ are given in a table form (numerical). Students are asked to complete another table for $C'(t)$, derivative of $C(t)$.

Table 16.

Table of values of C(t)

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
C(t)	0.84	0.89	0.94	0.98	1	1	0.99	0.9	0.79	0.63

This question tests whether students can read the table of values and get information about the function to estimate the derivative at a point numerically. In other words, it tests students' understanding that the derivative of a function at a point is equal to the instantaneous rate of change, which is approximated by calculating the average rate of change using two nearby points. For example, $C\,'(0) = \frac{C\,(0.1) - C(0)}{0.1 - 0} = \frac{0.89 - 0.84}{0.1} = 0.5$ and $C\,'(0.2) = \frac{C\,(0.3) - C(0.2)}{0.3 - 0.2} = \frac{0.98 - 0.94}{0.1} = 0.4.$ Students, who are able to do so, tend to have an *object* conception of the *derivative* concept.

Other Possible Solutions:

- Some students are expected to plot the graph of $C(t)$ and then estimate the derivative at point *t* by calculating the slope of the secant line passing through the points $(t, c(t))$ and $(t + h, c\,(t + h))$ where h = 0.1, the increment in time (t). Their solutions involve translation from numerical to graphical representation of derivative. Students, who are able to do so, tend to have also an *object* conception for what was required.
- Some students are expected to fill the table $C\,'(t)$ with all zeros justifying their reasoning by saying that the derivative of a constant is zero. These students lack both the *process* and *object* conceptions of the function and derivative concepts. They are unaware that the table represents some values of a continuous function for different values of x. that has its own properties. The function $C(t)$ increases when $0 < t < 0.4$, remains constant over the interval] 0.4, 0.5[, and then decreases when $t > 0.5$. Thus, $C'(t)$ has to be positive, zero and negative respectively.

- Some students are expected not to solve it at all because they think they need the equation of $C(t)$ in order to find the derivative function $C'(t)$ using rules of differentiation, and then substituting the different values of *t*. This means, if the equation of the function is given, such students tend to have an *action* conception of the derivative concept.
- Some students might fill the table of $C'(t)$ with the same values as those of $C(t)$. Such students lack an *object* conception of the derivative function. They think that the function $C(t)$ and its derivative $C'(t)$ are the same. Such students seem not to be aware that the derivative of a function is an object and a function itself that has its own properties.

4.4.2. Qualitative analysis of students' results on the Derivative Test

This section presents qualitative analysis of students' responses on each question/sub- question of the test as described in the a priori analysis (see section 4.4.1).

Question I. This question involves graphical representation of a function $f(x)$. It includes three sub-questions.

Sub- question 1 (Find $f(5)$. *Justify your answer*). The analysis of students' responses showed that 23 students in the control group, compared to 26 students in the experimental group, used the graph of f to answer this question. These students linked the y- coordinate of the point (5, 3) on the graph with the function value at x = 5. These students have *a process* conception of the function concept.

On the other hand, two students in the control group have an *action* conception for what was required, since they used step-by-step procedures (finding the equation of the tangent line (L) by using the two points (0, 1) and (5, 3), and then substituting x =5 to get $f(5) = (2/5 * 5) + 1 = 3$). Finally, one student in the control group failed to solve this question. She assumed that the given graph represents the function $f(x) = sinx$, and thus writing f(5)=sin 5. This gives us an indication that some students cannot work in the absence of equations.

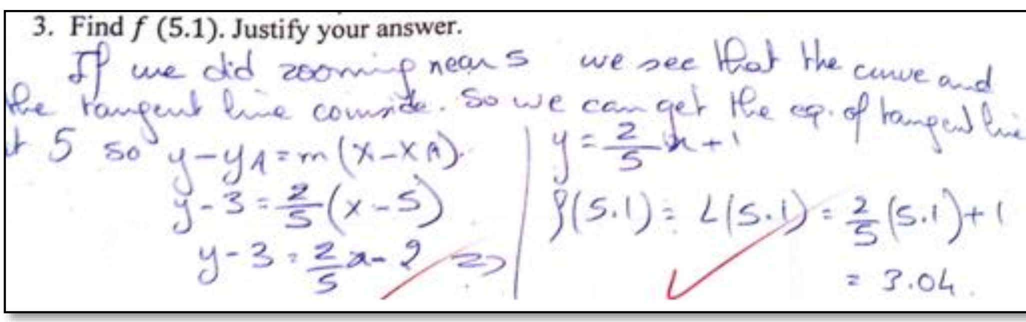
Sub- question 2. (Find $f'(5)$. *Justify your answer*). It is noticed that 10 students in the control group, compared to 26 students in the experimental group, answered this question successfully. These students tend to have an *object* conception of the derivative concept since they related $f'(5)$ with the slope of the line (L) tangent to the graph of $f(x)$ at the point (5, 3). It is noticed that all students in the control group, whose answers were correct, calculated the equation of the tangent at x = 5 (which is not necessary); however, none of the students in the experimental group computed the equation of the tangent line. This gives an indication that students in the control group feel more comfortable with working on equations rather than on graphs.

On the other hand, 10 students in the control group were not able to provide a correct answer. Students' attempts to find $f\,'(5)$ resulted in a variety of errors because of inappropriate visualization and inappropriate association between the slope of the tangent line at a point and the derivative of the function at that point. For example, four students assumed that the point (5,3) is the maximum point of the graph, thus writing $f\,'(5) = 0$. These students lack an *object* conception of the derivative concept because at a maximum point (derivative = slope of tangent = 0) the tangent line must be a horizontal line, which is not the case in this question.

In addition, four students assumed that the value of the function at a point is equal to the derivative at that point, thus writing $f'(5) = 3$. These students lack an *object* conception because they think that a function and its derivative are the same. Finally, two students didn't provide an answer, commenting that the equation of the function is missing.

<u>*Sub- question 3.*</u> (Find $f(5.1)$. *Be as accurate as possible*).

This question was solved correctly by 4 students in the control group, compared to 16 students in the experimental group. They used the equation of the tangent line (L) to the curve at x = 5 to estimate *f (5.1)*.It is noticed that many students in the experimental group provided a good comprehensive explanation as they used the concept of ' linearization' and the word 'zooming' in their answers as shown in Figure 27. Thus, these students seem to have an *object* conception since they used the derivative at point x= 5 as an object to estimate the function value at x = 5.1.


3. Find *f* (5.1). Justify your answer.

If we did zooming near 5 we see that the curve and the tangent line coincide. So we can get the eq. of tangent line at 5 so $y - y_A = m(x - x_A)$

$y - 3 = \frac{2}{5}(x - 5)$

$y - 3 = \frac{2}{5}x - 2 \Rightarrow$

$y = \frac{2}{5}x + 1$

$f(5.1) = L(5.1) = \frac{2}{5}(5.1) + 1$

$= 3.04$

Figure 30. An answer, from a student in the experimental group, to sub-question 3 of Question I revealing a good understanding of the linearization concept

It is noticed, however, that 14 students in the control group, compared to 5 students in the experimental group, wrote that f *(5.1)* $\approx$ *3.1* justifying their answers by using the right- hand limit. However, students who mentioned that the value must be close to three, have a *process* conception thinking of a function as a machine that maps an input to an output.

Finally, some students in the control group attempted to find the equation of the function and then substitute x = 5.1 in $f(x)$. This means that these students can't work using graphs and that they need equations to calculate the function value at a point, showing that their understanding is at the *action* level.

Question II. In this question, the graph of the function $f(x) = -x^4 + 4$ is given without its equation. The question requires discussing whether the derivative of the function is increasing, decreasing or both.

The two groups answered this question poorly. Only one student in the control group and seven students in the experimental group recognized that the derivative is decreasing throughout the whole interval $(-\infty, \infty)$. Those students managed to provide an acceptable explanation by discussing how the variation of the function, the sign of its derivative and the steepness of the tangent lines to the curve at different values of x are changing (see Figure 28).

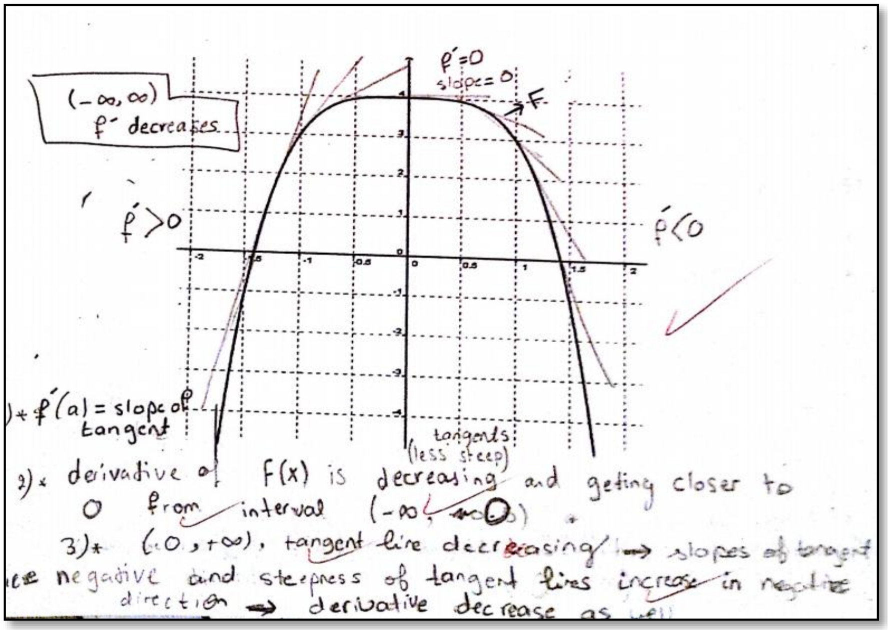


Figure 31. An answer, from a student in the experimental group, to question II revealing an appropriate schema of the derivative concept

These students seem to have good visual and analytical skills and a rich *schema* of the *derivative* concept since they combined several ideas together: derivative at a point, slope of a tangent line, steepness of tangent lines, and increase and decrease of a function with the sign of the derivative function.

On the other hand, incorrect answers occurred for several reasons:

- Sixteen students in the control group and 10 students in the experimental group assumed that the graph of $f'$is the same as that of f. In other words, they used the intervals of increase and decrease of $f(x)$ to determine the increase and decrease of $f'(x)$; that is writing $f'(x)$ is increasing on the interval (-∞, 0[and then decreasing on the interval] 0, ∞). These students lack an *objec*t conception for what was required. It is worth noting that several students committed mistakes in writing endpoints of the intervals; using y- values instead of x values. For example, several students wrote that f' is increasing over the interval (-∞, 4[and decreasing over the interval,]4, ∞).

- Nine students in the control group and nine students in the experimental group were aware that the given graph is for f and not for f', but they assumed that the sign of a function (positive or negative) is the same as the variation of the function (increasing or decreasing). Here is an example of students' responses:

  -On the interval (-∞, 0), $f(x)$ is increasing, thus $f'(x) > 0$ and hence $f'(x)$ is increasing

  - On the interval (0, ∞), $f(x)$ is decreasing, thus $f'(x) < 0$ and hence $f'(x)$ is decreasing

- Three students in the control group mentioned that the given information is missing because they do not have the equation of the function $f(x)$. Two of them explained that if the equation of $f(x)$is given, they would derive $f'(x)$ using rules of differentiation, then plot the function or discuss its variation by substituting several numbers. This gives us an indication that these students have an *action* conception of the derivative concept since they used step-by- step procedures and could not work without equations.

Question III. In this question, the graph of the derivative function_$G'(x)$ is given without its equation ($G'(x) = x^2 - 4$). It includes three sub- questions.

<u>Sub- question 1.</u> (On what interval is $G(x)$ increasing? decreasing?). This question was answered successfully by 8students in the control group, compared to 18 students in the experimental group. These students managed to relate the intervals where G' is positive (above x- axis) with the intervals where $G(x)$ is increasing, and then relate the interval where G' is negative (below x- axis) with the interval of decrease of $G(x)$. Therefore, according to the graph of G' , the function $G(x)$ is decreasing over the interval]-2, 2[because the derivative is negative in this interval. $G(x)$ is increasing when x $\in$ (-∞, -2 [∪]2, +∞) because in these intervals the derivative is positive. These students seem to have good visualization abilities and an adequate *schema* of the *derivative* concept since they are able to use the properties of the graph of G' and relate them to the graph of G.

On the other hand, the remaining students failed to answer the question (17 and 8 students in the control and experimental groups respectively). Incorrect answers occurred because these students were not aware that the given graph is for G' and not G. To them the two graphs are the same. Therefore, they used the intervals of increase and decrease of $G'(x)$, and thus assumed that $G(x)$ is decreasing over the interval (-∞, 0[and increasing over the interval]0, ∞). Figure 29 provides a sample of a student's incorrect answer on this question.

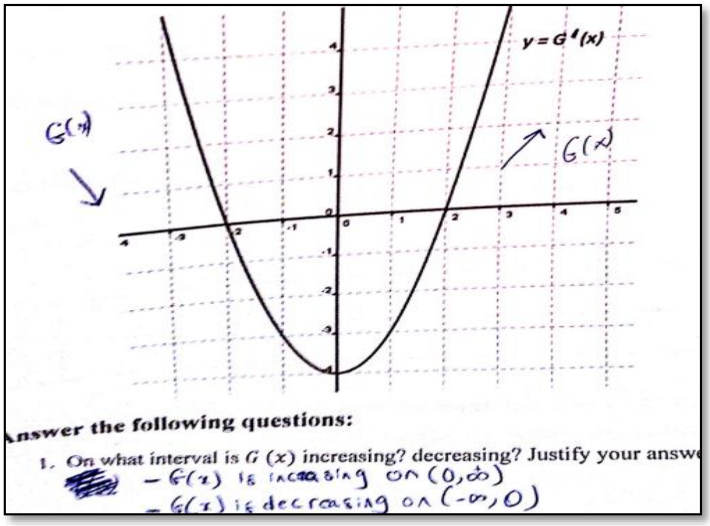


Figure 32. A sample of a control student's answer to sub-question 1 of question III, showing lack of an object conception

Thus, these students lack an *object* conception of the derivative concept. Finally, the analysis of students' papers revealed that some students have misconceptions in writing the interval notation where they used the y- values instead of the x- values. For example, some wrote that $G'(x)$ is decreasing on the interval (-∞, - 4[, and it is increasing on the interval]- 4, ∞).

<u>Sub- question 2.</u> *(At what point(s) does* $G(x)$ *have critical points?)* This question requires determining the critical points of $G(x)$; that is when $G'(x) = 0$ or undefined. Students who managed to answer this question correctly (10 students in the control group and 17 students in the experimental group) noticed that $G' = 0$ or the graph G' cuts x- axis at x = -2 and x = 2. It is noticed that two students in the control group solved this question mechanically, that is by finding the roots of $G'(x) = x^2 - 4 = (x-2)(x+2)$. Their answers involved translation from graphical to symbolic representation. These students seem to have an *action* conception of the derivative

concept since they cannot work without equations and their problem solving requires using step-by-step procedures.

On the other hand, the majority of students who failed to provide a correct answer assumed that the point (0, - 4), which represents the minimum point of $G'(x)$, is the critical point of G(x). These students lack an *object* conception of the derivative concept. They are not aware that the derivative function is an object or an entity that has its own properties.

<u>*Sub- question 3.*</u> *(Which critical point is a local maximum/ local minimum?)* This question is related to sub-question 2, which requires discussing the nature of the critical points as maximum and minimum points.

Students who managed to answer this part correctly (8 students in the control group 15 students in the experimental group) used the graph of G' and recognized that x= -2 is a maximum point since G' changes sign from positive to negative while x= 2 is a minimum point since G' changes sign from negative to positive. Unlike to other students who solved this question wrong, these students seem to have an *object* conception of the derivative concept since they operated on the graph of G' as representing function. They realized that the point (0, - 4) is the minimum point of G' and not G. Thus, they seem to have good visualization and an appropriate schema of the derivative concept since they related the graph of G' to that of the original function G.

The incorrect answers occurred mostly because the majority of students (18 students in the control group and 4 students in the experimental group) assumed that the point (0, - 4), which is the minimum point of G', is the minimum point of G as well. These students lack an *object* conception of the derivative function. Finally, it is noticed that four students in the experimental group reversed the maximum and minimum points. They justified their reasoning by considering the minimum value (-2) as corresponding to the minimum point, and the bigger value (2) as corresponding to the maximum point. These students lack an *object* conception and have an inappropriate schema concerning the maximum and minimum points of a function.

Question IV. This question tests students' ability to prove geometrically (graphically) why the two functions $f(x)$ and $g(x)$ have equal derivatives, where $g(x) = f(x) + c$.

Only three students in the control group managed to provide acceptable correct answers compared to 15 students in the experimental group. Their answers involved translation from symbolic to graphical representation and a connection between them. Out of these 18 students, 12 thought of polynomials (in particular y = x^2 and y = $x^2 + c$) as shown in Figure 30, sketched the graphs of both functions, drew tangent lines at different values of x and noticed that the slopes are equal since the graphs are identical and just translated along the vertical direction, thus the tangent lines are parallel.

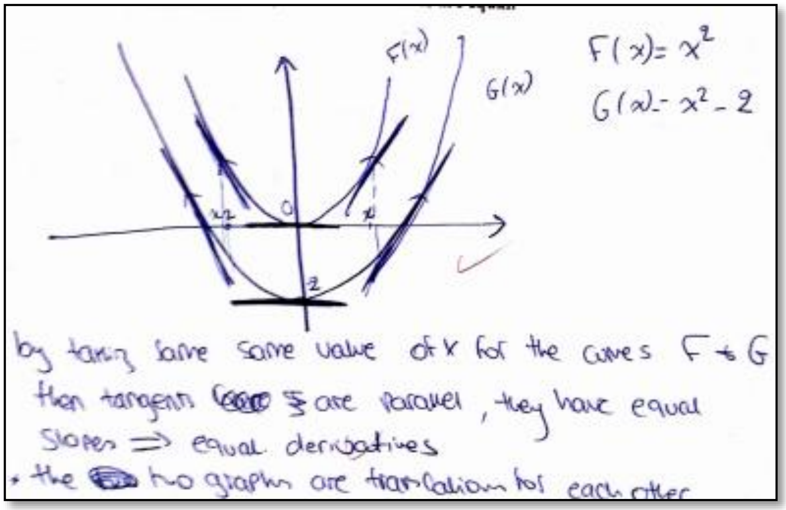


Figure 33. A sample of an experimental student's answer to question Iv taking polynomials as an example of functions

Notes. The answer shows an *object* conception of the derivative concept and involves a translation from symbolic to graphical representation.

However, six students in the experimental group sketched graphs of two arbitrary functions, vertically translated along the vertical axis, used similar procedures as their colleagues, and then came up with the conclusion. Students who solved this question correctly seem to have an *object* conception of the derivative concept. First, they interiorized the fact that the action of producing the derivative at a point; that is seeing the derivative as the correspondence: $a \rightarrow$ slope of tangent line at the point $(a, f(a))$. Then, they encapsulated that process to extend the concept of derivative from a single point to the derivative function.

The remaining students (15 students in the control group and 8 students in the experimental group) gave examples of polynomials, in particular y = x^2 and y =$x^2 + c$, differentiated the functions using rules of differentiation and then concluded that the derivatives are equal since the derivative of a constant is equal to zero as shown in Figure 31.

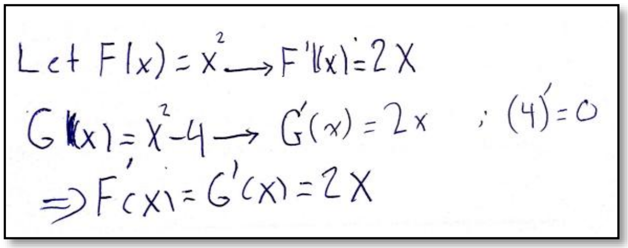


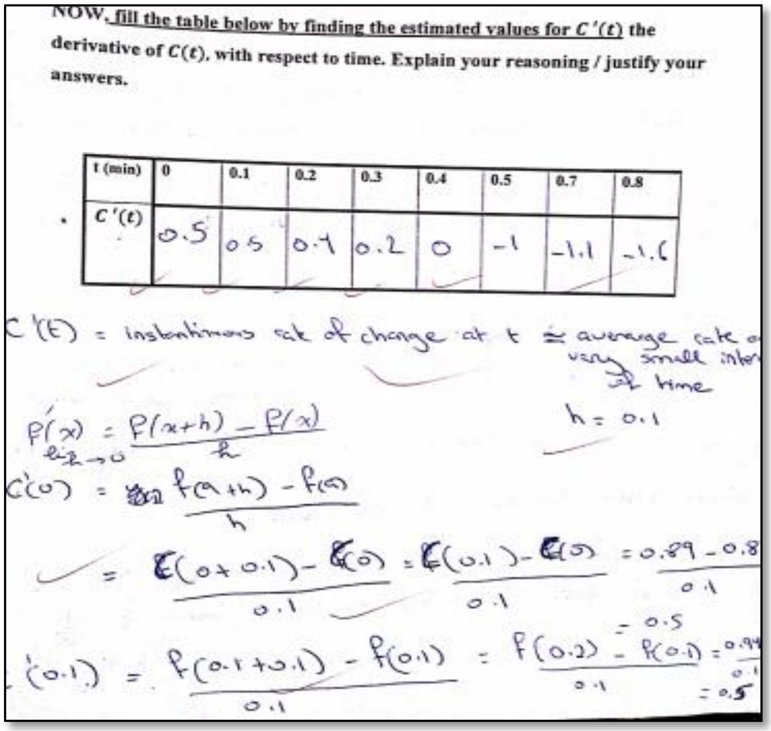
Figure 34. Sample of a student's answer to question IV, showing procedural understanding

These students solved this question symbolically and not geometrically as required. This means that they have a 'procedural' perception of the derivative concept, that is their conceptions are limited to their ability to work with the symbolic representation (using rules and formulas for derivation).

It is noticed that seven and three students in the control and experimental group respectively did not provide an answer. This gives an indication that these students, either did not understand the question, or most probably, do not see any connection between the graphical/ geometrical and the symbolic representations of the derivative.

Question V. This question involves a numerical representation (table of values) of a function $C(t)$ describing the concentration of a drug in the blood at time t (min.). Students were asked to fill in a table for *C '(t)*, derivative of *C (t)* at different values of time (t).

This question was solved successfully by nine students in the control group, compared to twenty students in the experimental group. This question tests students' understanding that the derivative of a function at a point is equal to the instantaneous rate of change at that point. Students managed to solve this question by realizing that the change in time iss very small (h = 0.1) and thus, over short intervals of time, the instantaneous rate of change can be approximated by calculating the average rate of change using two nearby points. Figure 32 presents a sample of a student's correct answer to this question.



NOW, fill the table below by finding the estimated values for $C'(t)$ the derivative of $C(t)$, with respect to time. Explain your reasoning / justify your answers.

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.7	0.8
$C'(t)$	0.5	0.5	0.4	0.2	0	-1	-1.1	-1.(

Figure 35. Sample of a student's response to part V of the test

These students have an *object* conception of the derivative concept since they were able to interpret the table of values as representing a continuous function, and then view the derivative as the correspondence $a \rightarrow$ instantaneous rate of change at the point $(a, f(a))$. It is worth to mention that out of the 20 students in the experimental group, 6 students used graphical and numerical representations in their answers. Those students sketched the graph of *C(t)*, drew tangent lines at different values of t , and then related derivative to the slope of tangent lines and instantaneous rate of change. These students seem to have a rich schema and a good understanding of the derivative concept.

On the other hand, students who were not able to solve this question committed different mistakes. The analysis of students' papers revealed the following:

- Eight students in the control group and two students in the experimental group filled the table representing $C\,'(t)$ with all zeros, justifying their answers by the fact that the derivative of any constant is zero. For example, since C (0) = 0.84, then $C\,'(0) = 0$, and since C (0.1) = 0.89, then C '(0.1) = 0. These students lack an *object* conception of the derivative concept since they are unaware that the given table represents a continuous function, and hence its derivative is a function as well. Also, they have poor visual abilities since they didn't recognize that the function C(t) is increasing between t = 0 and t = 0.4, remains constant in the interval]0.4, 0.5[, and then decreases for t > 0.5. Thus, its derivative function $C\,'(t)$ is positive, zero and negative respectively.
- Three students in the control group and four students in the experimental group filled the table of $C\,'(t)$ with the same values as $C(t)$. For example, C (0.2) = 0.94, thus its derivative $C\,'(0.2) = 0.94$, similarly, $C(0.4) = C\,'(0.4) = 1$ etc. Again, these students lack an *object* conception of the derivative concept since they are unaware that the derivative of a function is a function itself that has its own properties.

- The remaining students commented that the equation of the function $C(t)$ is missing, and thus they cannot find its derivative function $C'(t)$ nor the derivative at a point. This gives an indication that some students cannot work in the absence of algebraic equations of the functions.

Conclusion. In general, students in the experimental group seem to have better conceptual understanding of the derivative concept as compared to students in the control group. This is evidenced by the number of correct responses on each question/ sub- question in the two groups as shown in Figure 33.

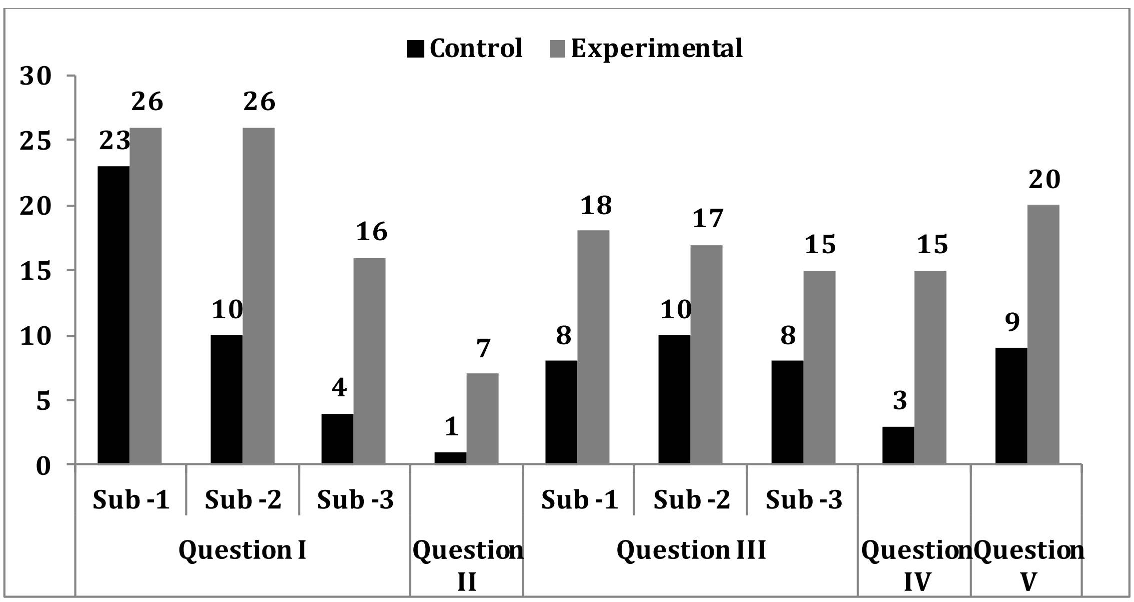


Figure 36. Comparison of the Number of correct responses on each question/ sub-question of the test, in the two groups.

It is noticed that many students in the control group have deficiencies in their graphical understanding of the derivative as the slope of the curve or the slope of the tangent line. The low number of students solving correctly sub-question 2 of question I and question IV supports this conclusion. Moreover, they have weaknesses in their understanding of the derivative as a rate of change; they are unable to estimate the derivative numerically (Question V). Moreover, many students, particularly in the control group, lack an *object* conception of the derivative function as they are unaware that the derivative of a function is a function itself that has its own properties. This is evidenced by the low number of correct responses on each sub-question of question III. This question requires interpreting the graph of the derivative function $G'(x)$ and relating it to the original function $G(x)$. In addition, the analysis of students' papers revealed that the graphs and the tables of values given in the test without their algebraic expressions are meaningless for many students in the control groups, compared to few students in the experimental group. The use of comments by students (interviewees and non- interviewees) in the control group such as" the equation of the function is missing", or "how can we solve it without being able to find the equation of $f(x)$ or $f'(x)$", indicates their understanding is rather procedural than conceptual.

Finally, it is noticed that question II was solved poorly by both groups (one student in the control group and seven students in the experimental group). This question requires higher-order thinking, good visualization and maturity in thinking. One reason for failing this question by most students is that they are used to deal with questions that require discussing the sign of a derivative (positive/ negative) rather than its variation (increasing/ decreasing). Moreover, short time was spent on teaching derivatives given graphically.

4.4.3. Quantitative analysis of students' results on the Derivative Test

Following is a quantitative analysis of both groups' scores on the test using descriptive (mean, standard deviation) and t- test statistics. The t- test is appropriate because it assesses whether there is a significant difference between the means of the two groups. The t- test was performed at $\alpha = 0.05$ level of significance. The null hypothesis claims that there is no difference in the mean scores of the test between the two groups.

Descriptive Statistics. The test was graded over 22. The mean score for the control group was 7.4 with a standard deviation 4.14, indicating that students varied widely in their achievement scores. On the other hand, the mean score for the experimental group was 14.26 with a standard deviation 3.86, indicating that students varied as well in their achievement scores. A mean difference of -6.9 was noticed between the two groups indicating that the control group scored much lower than the experimental group. In order to examine if this mean difference is significant, an independent t-test was conducted on the data.

Independent t-test. The assumptions of the t-test were examined using *SPSS* program (Table 16).

Dependent variable. The dependent variable (scores) were measured on a continuous scale

Independent variable. The scores come from two different groups: control and experimental

Normal distribution. The scores in the two groups are normally distributed as shown by Shapiro -Wilk test for normality

Control: $p = 0.506 > 0.05$

Experimental: $p = 0.349 > 0.05$

Homogeneity of variance. This assumption was met using Levene's test ($F(50) = 0.028, p = 0.87$)

Table 17

Independent t-test comparing control and experimental groups' mean scores on the test

		Levene's Test for Equality of variances		t- test for Equality of Means			Mean Difference
		F	Sig.	t	Df	Sig. (2 tailed)	
Scores on the Test	Equal variances assumed	0.028	0.87	-6.17	50	0.00	- 6.9

Note. The t-test was conducted at $\alpha = 0.05$ level of significance.

As shown in Table 17, the independent t- test revealed that there is a significant difference in the mean scores between the two groups (t (50) = -6.17, p = 0.00 < 0.05). The t- test showed a difference in the average scores between the two groups of -6.9, which means that the experimental group scored significantly higher than students in the control group on average 6.9. Therefore, the approach adopted in the experimental group (multiple- representations of derivative, activities and the use of technology, *Autograph*) seems to be more effective than the approach used in the control group (Book 1, and emphasis on symbolic approach).

4.4.4 Summary

In general, the qualitative analysis of students' responses on the tests revealed that many students in the experimental group seem to have an *object* conception and a good schema of the *derivative* concept, while their counterparts lack an *object* conception of the concept. Most of the students in the experimental showed a complete graphical understanding of the derivative as the slope of the tangent line, the derivative as the instantaneous rate of change, and an appropriate association between a function and its derivative. This suggests that the intervention is effective and that the multiple-representation approach, Book 2, and the activities used in the experimental group enhanced and deepened students' conceptual understanding. The quantitative analysis using the independent t-test supports this conclusion.

4.5. Analysis of the Interviews

Interviews are very useful in math education since they allow the researcher to understand students' thinking and problem solving processes. In the context of this paper, it allows, in addition, to obtain a clear and explicit picture of students' conceptual understanding of derivative.

Each interview conducted consists of four parts. In the first part, students were asked questions about their educational background and their attitudes toward math in general and derivative in particular. Part two, investigates students' attitudes toward the teaching approach (lecture, group work, technology etc) used in their classes. The third part is an open- ended question that asks students about the meaning of derivatives. The last past is designed around a series of five problems of the *Derivative Test (Appendix D).* In part four, students' responses to each question on the test are analyzed according to APOS (*Action- Process- Object- Schema*) model and based on the a priori analysis of the test (section 4.4.1).

Table 17 presents the characteristics of students who were interviewed in the two groups.

Table 18

Characteristics of the interviewees in the two groups (control and experimental)

Control Group	Experimental Group
CA1: first high- achieving student of the control group (grade A)	EA1: first high- achieving student of the experimental group (grade A)
CA2: second high- achieving student of the control group (grade A)	EA2: second high- achieving student of the experimental group (grade A)
CB1: first good- achieving student of the control group (grade B)	EB1: first good- achieving student of the experimental group (grade B)
CB2: second good- achieving student of the control group (grade B	EB2: second good- achieving student of the control group (grade B)
CC1: first low-achieving student of the control group (grade C)	EC1: first low- achieving student of the experimental group (grade C)
CC2: second low- achieving student of the control group (grade C)	EC2: second low- achieving student of the experimental group (grade C)

Notes: Students ' levels of achievements were determined based on their average grades in calculus I course (prior their final exam)

Twelve students, six students in each group, were interviewed individually before the final exam periods. Whereas some students volunteered to take part in this study, the selection of the other students was based on the following criteria:

- Combination of girls and boys (5 boys, 7 girls)
- Having different levels of achievement (grades A, B and C)
- Change significantly in attitudes and interests toward math after the implementation of the unit on derivatives (such as EC1). This student hated and resisted the approach used at the three sessions of the intervention. However, after the implementation of the unit on derivative, she showed liked the approach particularly the visual part and showed positive attitude

It is noticed that one of the students is repeating the course (EB2). This student who took D in Calculus I in the previous semester (Fall 2013), is repeating the course to improve his grade. In spring 2014, he took the course with instructor Y. Therefore, he was exposed to the two different approaches derivative (symbolic versus multiple representations approach). Thus, it is interesting to investigate his attitudes and preferences between the two approaches.

After the implementation of the unit on derivatives, a common written test, consisting of five conceptual- understanding based problems on derivatives (Appendix E) was administered to all students to solve during class time. The test duration is 65 minutes. Concerning the interviews, they revolved around the five problems as well. The researcher and the interviewees agreed on the place and the time of the interviews. Two of the interviews were conducted one day before the test, three of the interviews were conducted at the same day of the derivative test, and three interviews were conducted one day after the test. The remaining four interviews (two students in each group) were conducted with students who did the test with their classmates during class time. These students were presented later with their test papers and were asked to re- examine their solutions and describe their solutions and thinking process.

The researcher interacted with the interviewees by providing probes and questions to obtain a clear picture of each student's conceptual understanding of derivatives. There was no induction of answers or any indications to the correctness of the solutions. It is worth to mention that the interviewees were also asked questions about their educational background and their attitudes toward math. In addition, they were asked to explain or describe what they know about derivatives.

Even though the researcher explained to the interviewees the purpose of the interview, and assured them both confidentiality and anonymity, she could not audiotape nor videotape the interviews upon their request. Detailed notes were taken. To increase the validity and the reliability of the interviews, the researcher asked a friend, holder of BS degree in computer science, to take notes of both the interviewer's questions and students' responses.

The analysis of the interviews is divided into five parts.

4.5.1. Educational Background and Attitudes toward mathematics

Except for EA1 and EB2, all of the interviewees had their school education in Lebanon. EA1 received her school education in Saudi Arabia, while EB2 received her school education in Al Bahrain. CB1, CC1, CC2, EC1 and EA2 are holders of the official Lebanese Baccalaureate, Economics and Sociology (ES) track. CA1, CA2, and EB1 are holders of the International Baccalaureate (IB), while CB2 and EC2 are holders of a high school diploma.

When asked about their opinions toward mathematics, four of the six students in the control group said that they hate mathematics and think it is boring and difficult, while two students said that they like math. In the experimental group, two students said that they like math and believe it is useful in life, three students said that they started liking math since they started taking derivative, and one gave a neutral attitude (neither like nor dislike).

4.5.2 Attitudes of students toward the teaching approach used

In the control group, five students out of six agreed that the approach used in their class is neither motivating nor interactive, that is lacks technology and includes many rules and formulas that need to be memorized. One student said that he likes instructor X's teaching style and finds it clear and structured. However, in the experimental group, five students out of six agreed that the approach used for teaching derivative is interactive, fun, and emphasizes the multiple representations of the derivative concept. Moreover, they like and favor the visual element of the technological software, *Autograph*. Students confirmed that *Autograph* helped them make connections between the graphical representation, the table of values and the algebraic expressions of functions. They also added that one can experiment, make and test conjectures. Moreover, the '*animation*' feature of *Autograph* allowed them to clarify the idea of average rate of change becoming an instantaneous rate of change; in addition, the 'slope function' and the 'slow plot' buttons allow students to plot gradually the derivative of the selected function showing the moving tangent on the original function. Also, they mentioned that they can visualize, on the spot, the effect of a change in the parametesr of the functions. One student, however, stated that the approach especially the activities used were difficult and demanding, and required daily practice. He mentioned that memorizing rules and following set of algorithms and procedures are easier. Here is a sample of students' responses:

- CA2: *Well, one can pass the course easily by just studying the rules and at the last minutes.*

- CB2*: I cannot wait to finish this course. I hate math. I think it is complicated and it is all about formulas and rules to memorize. I find this course useless. I do not think I am going to use it in life or in my major.*

- EC1: *In general, I used to hate math a lot and I used to fail in all my exams. Now, I am happy. I am passing my exams. I feel that I am more confident and I can understand because of the visual element of the course. We can use graphs and tables and not only equations.*

- EB1: "*I am repeating this course for the second time. I don't remember anything from the first time because it was all rules and equations.., but now I feel it is a new way. I can remember more because it includes more graphs, visual ...*"

4.5.3 Students' responses on the meaning of derivative

Question: What do you know about derivative? Explain as much as you can.

When interviewees were asked about the meaning of derivative, three out of the six students in the control group mentioned only the symbolic representation of derivative including rules of differentiations and the formal definition of derivative. Two students (CA2 and CB2) mentioned both the symbolic and the graphical representation as slope of the tangent line. Only one student (CA1) mentioned the three types of representation together, in addition to the relationship between the sense of variation of a function with the sign of its derivative.

- CA2: *Derivative is about rules and formulas like product rule, quotient rule and others that we apply them to find the derivative of functions. It is equal to the slope of the tangent line.*

- CC*1: Derivative includes many rules that we need to memorize in order to find the derivative of a function.*

- CA1: *Derivative is a concept that has many interpretations. It is equal to the slope of a curve or slope of the line tangent to the curve at a point. It is the instantaneous rate of change of a function with respect to x. We can use the formal definition of derivative using the limit or the rules (power, product, quotient rules and others) to find the derivative mechanically.*

However, all students in the experimental group mentioned the three representations in their definition. In addition, four of them mentioned that the derivative is used to find the critical points of a function, the maximum and the minimum, and to determine the intervals where a function is increasing or decreasing. Moreover, five out of six students mentioned that derivative is used in real life, where as none of the students in the control group spoke about real life applications of derivatives. For example:

- EC1: *Well, derivative is used in biology to study the rate of change of some drug in the blood with respect to time.*

- EB2: *It can be used in physics to study the velocity and the acceleration of a moving car.*

- EA2*: Derivative is used in many real life situations. For example, derivative can be used to explain the rate at which a certain population is changing with respect to the change in the number of births, deaths etc. It is used in economic as well to calculate the marginal cost...*

In general, based on students' responses on the meaning of derivative, one can notice that students in the experimental group have a better conceptual understanding of the derivative than those of the control group who showed a "procedural" perception of derivative.

4.5.4. Analysis of students' responses on the test

Discussions with the interviewees then revolved around the five conceptual-understanding based problems on derivative. Students' responses and their ways of thinking were categorized based on *APOS* model and on the a priori analysis of the test (section 4.4.1). That is one student may have an *action* conception of derivative and another have an *object* conception. For example, an individual is restricted to an *action* conception of the derivative concept if he / she is unable to interpret a situation as a function unless a formula o ran algebraic expression of the function is given; for such students, having a table or a graph that represent a function without an explicit equation is meaningless. However, students who have an *object* conception of the derivative concept understand that the derivative of a function is itself a function; for example, they can act of the graph of the derivative function and relate it to the graph of the original function. Moreover, they can extract information from a graph to relate derivative to the slope of the line tangent to the curve, and interpret a table of values to relate derivative to the instantaneous rate of change. Section 4.4.1 provides detailed analysis of students' thinking according to all levels *APOS*.

4.5.4.1 Summary of students' thinking processes and conceptions in the control group

The interviews' results show that all students in the control group were not comfortable working with functions without their algebraic expressions. Some students expressed frustrations. They mentioned that the problems would be much easier if the equations of the functions were given. Moreover, four students out six revealed difficulties in reading and interpreting graphs and tables. Also, they have deficiencies in their graphical understanding of derivative as slope of a curve and slope of a tangent line (Question I (2) and Question IV). In addition, they have weak visualization skills. They were not able to use the graph of the derivative function and relate it to that of the original function (Question II and Question III). Finally, all students except for AC1 showed lack of an *object* conception of derivative. Their understanding remained at the action/ process levels.

Question I (This question corresponds to question Ion the derivative test).

Concerning question I, the graph of of f and its tangent line at x= 5 are given; it requires determining $f'(5)$ and $f(5.1)$. Three students in the control group demonstrated lack of understanding of the relationship between the derivative of a function at a point and the slope of the tangent at that point. Here is a sample of one student's response:

> Researcher: *What is* $f'(5)$? *How can you handle this problem?*
>
> CB1: (thinking...): *Well, we don't have the equation of* $f(x)$, *and thus I can't find* $f'(x)$.
>
> Researcher: *What about the graph and its tangent? Do they include any useful information?*
>
> CB1: *(thinking...).No, I cannot see any relation between the graph and the derivative. This question is hard.*

The remaining three students (CA1, CA2, CB1) were successful in their answers. They mentioned that the derivative of $f(x)$ at x = 5 is equal to the slope of the line tangent to the curve at x= 5. They used the two points (0, 1) and (5, 3) on the curve and calculated the slope ($f'(5) = 2/5$), thus showing an *object* conception. Then, when they were asked to find $f(5.1)$, only CA1 and CA2 out of the six students managed to provide an accurate answer. They used the equation of the tangent line at x= 5 to find $f(5.1)$. When asked to justify their answers, they used the concept of linearization, which approximates a function at a point using a tangent line near that point. The remaining students mentioned that $f(5.1)$ should be close to 3. When asked to provide a more accurate answer, they replied they do not know since they do not have the algebraic expression of the function, thus showing an *action* conception of derivative.

Question II (This question corresponds to the second question on the derivative test).

In question II, the graph of $f(x) = -x^4 + 4$ is given without its equation, and the requires determining whether its derivative is increasing or decreasing. It is noticed that none of the students in the control group solved this question, thus showing a lack of an *object* conception. Four of them failed to recognize that the derivative of a function is a function itself which has its own properties. Also, two students demonstrated poor understanding of the relationship between the sign (positive, negative) of a function and its variation (increasing, decreasing). Here is a sample of students' responses:

Example 1

Researcher: *Is the derivative of this function is increasing/decreasing or both?*

CB2: *Well, from (-∞, 0) the derivative is increasing (going up), and from (0, ∞) it is decreasing (going down).*

Researcher: *Are you aware that this graph is for* $f(x)$ *and not* $f'(x)$?

CB2: *(thinking...).So what, I think that its derivative acts the same way.*

Example 2

CC2*: from (-∞, 0) the function is increasing, thus its derivative is positive and hence increasing. On the interval (0, ∞), the function is decreasing, thus the derivative is negative and hence decreasing.*

Researcher: *So, are you saying that a positive function is the same as an increasing function?*

CB3: *Yes, and a negative function means a decreasing function*

Question III (This question corresponds to question III on the derivative test). In this question, the graph of $G'(x) = x^2 - 4$ is given without its equation and three questions on $G(x)$ are posed (intervals where $G(x)$ is increasing/ decreasing, the critical points of $G(x)$ and the nature of the critical points). Four students out of the six failed to answer this question, thus showing lack of an *object* conception of the derivative concept. Even though the researcher made it clear that the given graph is for G', the interviewees did not realize that G' is itself a function, an object that has its own properties. They assumed that the intervals where $G'(x)$ is increasing / decreasing and the critical point of $G'(x)$ are the same for $G(x)$. However, two students (CA1 and CA2) solved this question successfully. It is noticed that CA2 solved this question mechanically, thus showing an *action* conception. First, he found the equation of $G'(x)$, found its critical points (G'(x) = 0) and then set up the table of variation of G(x) to discuss the nature of the critical points.

Question IV (This question corresponds to the fourth question on the derivative test). In this question, students were asked to explain geometrically why the derivatives of the two functions $F(x)$ and $G(x)$, where $G(x) = F(x) + C$, are equal. Four students in the control group (CB1, CB2, CB3, CC1) demonstrated an *action* conception for what was required. These students wrote examples of functions using polynomials, and then stated that the two functions have equal derivatives since the derivative of any constant is zero. Here is a sample of CB1's response:

> Researcher: *Why do the two functions have equal derivatives*?
>
> CB1: *Well, suppose that* $F(x) = x^2$ *and* $G(x) = x^2 + 3$.
>
> Researcher: *Do the functions have to be polynomials*?
>
> CB1: (*thinking).This come to my mind when dealing with functions. I think yes.*
>
> Researcher*: Proceed.*
>
> CB1: *The derivative of* $F(x)$ *is* $F'(x) = 2x$ *and* $G'(x) = 2x$ *since the derivative of 3 is zero. Therefore, they are equal.*

Researcher*: The question asked you to explain geometrically and not symbolically. Can you give me a graphical representation of the derivative?*

CB1: *I don't know*

However, two students (CA1, CA2) recognized that the curve of *G* is the vertical translation of *F* along the y- axis, and that the slopes of the two curves must be equal in order for the derivatives to be equal. However, they were not able to reach out a conclusion showing *process* conception for what was required.

Question V (This question corresponds to question Ion the derivative test). In this question, a table of values for the function $C(t)$ is given, students were asked to complete another table for $C\ '(t)$. Four students failed to answer this question showing lack of *object* conceptions of both the function and the derivative concepts. In other words, these students failed to use the table of values for $C(t)$ as an object, and hence estimate the derivative at a point by calculating the average rate of change over small interval. Here is a sample of students' responses:

CB1: *Well, using the information from the table, C (0) = 0.84, C (0.1) = 0.89 and so forth. Therefore,* $C\ '(0) = 0$ *and* $C\ '(0.1)$ *=0. Same for other values.*

Researcher*: Why zero?*

CB1*: because the derivative of any constant is zero.*

Researcher*: It is given that C(t) is a continuous function*

CB1: Yes, it is given I know.

Researcher: What does a continuous function mean?

CB1: Its curve is smooth curve and does not include any jumps

Researcher: Okay. So, is this your final answer?

CB1: (Thinking...) This is my final answer.

Moreover, it is noticed that CA2 was not able to provide a complete correct answer, but he demonstrated an *object* conception of the function concept. He noticed that $C(t)$ increases in some interval, remains constant and then decreases. Also, he mentioned that the derivative of the function must be positive, equal to zero, and negative and negative respectively. However, he failed to calculate the rate of change using two nearby points, and hence failed to find the derivative at a point numerically. Finally, two students (CA1 and CB2) solved this question correctly showing an *object* conception. They realized that they can take two consecutive points and calculate the slope which approximates the derivative at a point. When they were asked if there is a way to check the correctness of their answers, they did not suggest any. One way for example is to check the intervals where the function $C(t)$ is increasing, decreasing or constant with those where its derivative is positive, negative and zero respectively.

4.5.4.2 Summary of students' thinking processes and conceptions in the experimental group

In general, students in the experimental group were more comfortable working with functions represented graphically and numerically than students in the control group. Most of the interviewees outperformed students in the control group, demonstrated a good understanding of the derivative concept and showed evidences of having an *object* conception of derivative. They were able to explain the relationship between the derivative of a function at a point with the slope of the tangent line and the instantaneous rate of change at a point, in addition to the relationship between a function and its derivative. The interviews' results showed the following:

Question I tests students' graphical understanding of a function and its derivative. When asked to find $f'(5)$, the six students managed to relate the derivative of the function $f(x)$ at x =5 with the slope of the line tangent to the curve at that point, thus showing an *object* conception. Then, when asked to find $f(5.1)$, five students out of the six used the equation of the tangent to the curve and substituted x = 5.1 in the equation. They used the idea of 'zooming in' near the point (5, 3) and explained that the tangent line and the curve near their region are the same. However, one student mentioned that $f(5.1)$ must be close to 3. When she was asked to provide an accurate answer, she was not able to reach a conclusion, thus showing a *process* conception.

Question II was solved poorly by the two groups. Given the graph of $f(x)$, students were asked to determine whether its derivative is increasing or decreasing. Two students in the experimental group (EA1 and EA2) solved this question successfully (compared to none in the control group) showing good visualization skills and an appropriate schema for what was required. These students managed to provide an acceptable explanation by discussing how the variation of the function $f(x)$, the sign of its derivative and the steepness of the tangent lines drawn to the curve at different values of *x* are changing. They explained that on the interval (-∞, 0) the slopes of the tangent lines are positives and decreasing (less seep), thus the derivative is decreasing. Similarly on the interval (0, ∞) the slopes of the tangent lines are negative and decreasing (more steep in negative direction), thus the derivative is decreasing as well. The remaining four students were not able to reach a conclusion. They mentioned that the question is tricky.

However, they were able to read the graph of f, determine the intervals where the f is increasing or decreasing, and then relate it to the sign of $f'(x)$. The demonstrated *process/object* conception, unlike most students in the control group who assumed that a function and its derivative behave in the same way and have the same properties.

Question III requires students to use the graph G' as an object and answer questions related to the graph of G. Five students out of the six (compared to two students in the control group) solved this question successfully, and managed to use the properties of the graph G' to determine the intervals where $G(x)$ is decreasing and increasing, find its critical points and discuss their natures. Thus, they have good visualization skills and an appropriate schema of the derivative concept. When asked about the difference between a positive and an increasing function, they provided a good explanation and made it clear that a positive function does not imply an increasing function and vice versa, unlike most students in the control group. However, one student (EC1) demonstrated lack of an *object* conception of the derivative concept. He assumed that the given graph is for G, and thus answered accordingly. When the researcher pointed out that given graph is for G' and not G, he replied that they are the same. Therefore, like most students in the control group he is not aware that the derivative of a function is a function itself that has its own properties (graph, critical points, etc)

Concerning question IV, four students out of the six showed an *object* conception for what was required compared to none in the control group. These students were able to prove geometrically why the two functions $F(x)$ and $G(x)$ have equal derivatives. Three of them thought of polynomials as an example of functions before they proceeded in their work while one thought of arbitrary functions. Then, they sketched the graphs, recognized they are vertically translated, and drew several tangent lines. They mentioned the slopes of tangent lines at any point a on F and G are equal since the tangent lines are parallel. However, two students (EB2 and EC2) were not able to use the relationship between the slope of a tangent line and the derivative at a point. They mentioned that the two functions are equal since the derivative of a constant is zero.

Concerning question V, five students out of six demonstrated an *object* conception of both the function and the derivative concepts compared to two students in the control group. They managed to solve this question by realizing that the change in time in very small (h = 0.1) and thus, over short intervals of time, the instantaneous rate of change can be approximated by calculating the average rate of change using two nearby points. Moreover, three of them defended their answers by relating the sign of the derivative on some interval with the variation of the function. Here is a sample of a student's response.

> EA1*: We have a table here that represents some function $C(t)$. The function seems to increase first from t = 0 to t = 0.4, then it remains constant from t = 0.4 to t = 0.5, and finally decreases from t = 0.5 to t = 0.9. Therefore, I expect the derivative to be positive first, then zero, and then negative.*
>
> Researcher: *Can you tell me why?*
>
> EA1: *derivative represents the rate of change of the function at some point, which is equal to the slope of the curve or the slope of the tangent line at that point. So, when the slope of the tangent line is positive, then the line is increasing, and hence the function is increasing as well.*
>
> Researcher*: Continue.*

EA1: *Well, since the points are very close to each other,* $\Delta t = 0.l$*. Then, we can estimate the derivative of the function C (t) at time t by taking two points and calculate the average rate of change.*

Researcher: *Can you give an example*

EA1: *For example, we have the points (0, 0.84) and (0.1, 0.89).* $C'(0) \approx \frac{0.89-0.84}{0.1}$

However, only student (EB1) was not able to solve this question. He tried to find the equation of the function using trial and error, and he realized that the derivative must be positive first, then zero then negative. He did not realize that derivative can be approximated by calculating the average rate of change over small intervals, thus showing lack of an *object* conception.

In general, the results of the interviews show that the approach used in the experimental group (Book 2, *Autograph*, and visual- multiple representations) seem to have positive effects on students' conceptual understanding. The interviewees in the experimental group were more flexible and comfortable working with the different representations of derivative than in the control group. Students in the control group showed weaknesses not only in the derivative concept but in the function concept as well. For most of them, they seem they cannot visualize that the derivative of a function is another function that has its own properties. Moreover, most of those of were successful in answering the questions based their answers on memorized facts and not understanding.

4.6 Summary of Chapter Four and Evaluation of the Genetic Decomposition

The findings obtained from the observations, the questionnaires, the tests, and the interviews revealed that students in the experimental group showed better understanding of the derivative concept than students in the control group. The intervention that was implemented in the experimental group, which placed emphasis on the visual multiple-representation approach, has enriched and deepened students' conceptual understanding of derivatives. The intervention has successfully allowed the researcher to instill in students' minds many aspects of derivatives such the slope of the tangent line, the derivative as the instantaneous rate of change, the derivative function, the formal definition of derivative, and the relation between a function and its derivative and others. Also, the use of technology, *Autograph*, in the experimental group has improved students' visual, critical and analytical skills. The activities that were solved mostly as group works and the use of *Autograph,* have improved students' abilities in reading and interpreting graphs and table of values of functions , interpreting graphs of derivatives, and making connections and associations between the properties of a function and its derivative. On the other hand, many students in the control group were not comfortable working with functions without their algebraic expressions as shown in their test papers. Even some students expressed frustrations as showed during the interviews. Students in the control group revealed deficiencies in the understanding of derivatives and were dominated by the procedural symbolic representation of derivative, before and after the implementation of the unit on derivatives. In general, most of the students in the experimental group significantly outperformed their counterparts on both the questionnaire and the test.

Moreover, the results obtained from the aforementioned instruments serve as useful tools for the evaluation of the genetic decompositions that were developed for both the *function* and the *derivative* concepts (sections 3.4.2 and 3.4.3). The results show that the activities designed for the experimental group seemed to be successful because they helped the formation of the mental structures as described in the genetic decompositions of the *function* and *derivative* concepts. This can be also interpreted by the interviews' results and the significant difference in students' performance on the questionnaire and the derivative test between the two groups. However, few modifications should be made since there are students in both groups, particularly the control group, who assumed that a function and its derivative are the same objects that have the same properties. Moreover, the relationship between the properties of function and its derivative should be more addressed. Therefore, the researcher suggests adding more activities that include:

1. Matching graphs of functions with the graphs of their derivatives
2. Sketching the graph of the derivative from the graph of the function.
3. Sketching the graph of a function from the graph of its derivative
4. Asking questions on the original function given the graph of its derivative
5. Comparing the derivatives of two functions given graphically (in terms of which curve is more steeper, increasing faster and decreasing faster etc)

6. Finding the equation of the line tangent to the curve of a function at a given point, given only the graph of the derivative of the function
7. Finding the equation of a function (simple functions such as linear functions) given the graph of its derivative

Therefore, such activities will enhance students' visualization, deepen their understanding of the derivative function, and emphasize the relation between a function and its derivative.

CHAPTER FIVE

Conclusions and Recommendations

This chapter consists of three parts. The first part summarizes all the answers to the research questions obtained from the content analysis of the books (Book 1 and Book 2), the observations, the derivative questionnaires, the derivative tests and the interviews. Limitations of the study are discussed in part two. Part three presents recommendations for further research.

5.1. Discussion of the Results of the Research Questions

This study aimed to examine how the concept of derivative is presented and developed in two different calculus textbooks (Book 1 and Book 2) , which adopt two different approaches. The control group was taught using Book 1 that emphasizes the symbolic approach of the derivative concept, while the experimental group was taught using Book 2 that emphasizes the multiple- representation approach. The second purpose was to investigate the types of difficulties that students face when learning derivative. In addition, it aimed to compare the impact of two different instructional methods, one using a multiple- representation visual approach and the other using a formal symbolic approach, on students' conceptual understanding of derivatives. Finally, it aimed to examine whether the use of technology and a multiple-representation approach improves students' motivation and attitudes toward mathematics.

5.1.1. Research question 1

How is the concept of derivative presented and developed in two different calculus textbooks (Book 1 and Book 2)?

The selected sections on derivative from each book share the same objectives; however, they differ in their content, presentation of the material, structure, and modes of representation and translations used in the *Text* and the *Exercises* sections of the books. As shown in section 4.1, the approaches adopted by the two books for presenting and developing the derivative concept are different. Book 1 starts with the formal definition of derivative, $f'(x) = \lim_{h\to 0} f(x+h) - f(x))/h$, proceeds to link it to the slope of the tangent line, and then propose rate problems as an application of the formal definition, and provide model solutions. Book 1 follows the 'traditional' approach of teaching where the abstract definitions of the concepts are directly stated at the beginning of the 'Text' without giving students the opportunity to be active learners or to construct their knowledge. However, Book 2 introduces the idea of derivative using real-life problems. In Book2, the formal definition of derivative develops from the investigation and exploration of two practical problems (the *rate-of- change problem* and the *tangent- line problem*. First, the derivative is defined as the rate function as opposed to the amount function using real-life examples (rate of change of distance with respect to time, rate of change of balloon’s height with respect to time, etc). Then, the geometric meaning as the slope of a curve and the slope of a line tangent to the graph of a function at a given point are discussed using the "zooming in" strategy numerically and graphically. Tall (2008) argues that the 'local' straight approach is important since it allows students to discuss the differentiability of a function, and discuss the cases where a function fails to have a derivative at a point (existence of corner, cusp, discontinuity etc). After that, Book 2 describes how the graphs of f and its derivative $f\,'$ are related, and what each tells about the other. Finally, it arrives to the formal definition of derivative. Book 2 follows the constructivist approach of teaching where students are independent learners who are active in constructing their knowledge. It is important to note that the approaches adopted by the two books for developing and presenting the derivative concept have an effect on students' motivation and interest to learn mathematics as discussed later in research question 3.

A closer analysis showed that, in Book 1, the symbolic representation is the most dominant mode used in the *Text* and the *Exercises* sections (65 % and 81.46 % respectively) as most of the functions are expressed through algebraic expressions. All the given algebraic expressions of functions are used as tools to: find the derivative of a set of functions using the formal definition of derivative or the rules of differentiation, and to find the slope of a curve or the slope of the line tangent to a curve at a given point. Moreover, part of the questions in Book1, require solving rate problems, and finding the linearization of a function at a point using the equation $L(x) \approx f(x) = f(a) + f'(a)(x-a)$. The graphical and the numerical representations are rarely used. For example, none of the functions in the "given" part of the exercises is presented in a table form. Moreover, ordered pairs (e.g., (2, 1), (-2, 7)) or phrases like" rate of change"," instantaneous rate of change", or "average rate of change" are not used in the given statements of the exercises. Moreover, Book 1 includes few exercises (17.07%) that require using graphs to find the slope of a tangent line, sketch the graph of the derivative, or find the instantaneous rate of change of a function at a point. Instead, graphical and numerical representations are requested and developed as resulting from the algebraic expressions of the functions. It is noticed, from Table 6, that 34.49 % of the translations in the control group take place from symbolic to graphical representations, and 17.01 % from symbolic to numerical representation. Research (Knuth, 2000; Leinhardt et al 1990) has shown that emphasizing translation from symbolic to graphical representation might cause learning difficulties and restrict students' thinking since it does not allow them to see graphs or tables as a source of information to be used in problem solving. This fact agrees with the results of this study as many students in the control group, compared to few students in the experimental group, have faced difficulties in reading, interpreting and extracting information from graphs and tables of values, which are later discussed in research question 2 and research question 4. If students are not prepared for, or trained in reading graphs or tables of values, their visual abilities will be limited and they will develop serious misconceptions (Malaty, 2006).

On the other side, the use of the three modes of representation (symbolic, graphical and numerical), in the *Text* and the *Exercises* sections of Book 2, is rather balanced with more emphasis on graphical representations. Translations take place among and within each mode of representation, thus deepening and improving students' understanding of the concepts (NCTM, 1989, Arcavi 2003, Gagatsis & Shiakalli, 2004). This improves, as well, students' visualization skills and increases their flexibility to move among representations. This fact is observed in students' responses to the tests, which are discussed in research question 4. Unlike to the control group, many students in the experimental group successfully answered questions I, III, IV and V which involve different representations of derivatives. Finally, it is noticed that Book 2 includes a set of exercises that begin with the words "Explain", "Interpret", "verify", "Why" all of which require critical thinking and the synthesis of several ideas, while Book 1 uses the verbs "find", "calculate", "sketch", all of which require procedural knowledge. This fact has an impact on students' conceptual understanding of the derivative concept as shown in research question 4.

5.1.2. Research question 2

What types of difficulties do first- year (Freshmen) students face when learning and applying the various notions of derivatives?

The observation of the sessions on derivative, as well as students' participation and work in both control and experimental groups provided insight about the difficulties that students faced while learning the derivative concept.

In this study, for many students the derivative concept was their first exposure to the topic. For others, the derivative concept was discussed in their high school classes graphically and symbolically, with more emphasis on the rules of differentiation (*power rule, product and quotient rules, chain rule* etc). However, in schools the formal definition ($f'(x) = \lim_{h\to 0}\frac{((f(x+h)-f(x))}{h}$) is not included in the calculus curriculum. Due to this fact, many students in both groups were resistant to the approach; they did not like the approach used and wanted to use the rules of differentiation directly. Ironically, many students in the control group who spent too much time working on the formal definition of derivative ($f'(x) = \lim_{h\to 0}\frac{((f(x+h)-f(x))}{h}$) had difficulties in applying the formula. Common mistakes include writing the expression wrongly as

$[f(x) + h - f(x)]$ or $[f(x + h) + f(x)]$ in the numerator instead of $[f(x + h) - f(x)]$. This fact was observed also in the experimental group, but with less frequency. This gives an indication that such students have poor conceptual understanding of the formal definition of derivative. It is noticed that during the interviews, the six students in the control group stated correctly the expression of the formal definition of derivative, but only two out of the six students knew how it works and how it is related to other representations. They were unaware of the fact that the expression $\frac{((f(a+h) - f(a))}{h}$ is equivalent to the average rate of change and to the slope of the secant line passing through the two points $(a, f(a))$ and $(a + h, f(a + h))$, and that the limit of the difference quotient is equivalent to the instantaneous rate of change and to the slope of the line tangent to the curve at point a. However, in the experimental group five students out of the six interviewed students were able to provide clear explanations about the aforementioned relationships. Thus, in this study, the approaches that are used in the two books for developing the derivative concept (as discussed earlier in research question 1) have affected students' conceptual understanding of the formal definition of derivative , as the formal definition in Book 2 was developed from two problems (the 'tangent-line' problem and the 'rate-of-change' problem). This result is similar to the findings of a study conducted by Zandieh (1998) on nine students, who were clinically interviewed, to investigate their conceptual understanding of the formal definition of derivative. In her study, Zandieh asked her students a set of questions aiming to investigate whether they are able to relate the formal definition of derivative with other aspects (e.g. slope of tangent line, instantaneous rate of change). She noticed that five students out of nine had memorized the formal without being able to relate it to other aspects of derivative.

Moreover, the 'animation' feature of *Autograph* has visually clarified to students in the experimental group the limiting process of how the secant line becomes a tangent line, and how the average rate of change becomes the instantaneous rate of change. Finally, it is worth mentioning that some students in both groups committed arithmetic and algebraic errors while working with the formal definition (e.g., not distributing the sign over a factor, mistakes in rationalizing, and mistakes in taking common denominators etc). This fact was noticed in a quiz, solved by the two groups, about finding the derivatives of a set of functions using the formal definition. Also, students' results in the diagnostic test revealed that many students have weaknesses in their prior knowledge.

In the experimental group, and during the first two sessions, students were showing resistance to the approach used (many activities, use of multiple representations: graphs, tables of values, and equations) and they did not want to change their previous procedures about derivative. The majority of students' conception of derivative was procedural consisting of rules and algorithms that were learned by memorization. This is due to the fact that the intervention in the experimental group was only done in the derivative chapter, while the approach was 'traditional' in the earlier chapters, which caused some resistance in the beginning. Gradually, students in the experimental group got acquainted with the new approach, showed more motivation and were convinced that learning derivative using graphs and tables of values is as important as using rules of differentiation. However, some students had difficulties in reading and interpreting functions and derivative functions presented graphically and numerically, and some were not able to understand that the derivative of a function is itself a function. One explanation for such difficulties is that the time spent on the study is short, and this sudden transition to a teaching approach that requires visual thinking is not easy for many students. A study conducted by Habre (2006), aiming to explore students' understanding of functions and derivatives in an "experimental calculus course", showed similar results. Habre noted that even with the instructor emphasis on the graphical/visual approach of functions and derivatives, many students preferred the symbolic procedural approach and faced difficulties in interpreting the graphs of functions and their derivatives that were given in their exam.

For many students, particularly in the control group, the graph of a function and its derivative are the same. Such students are unaware that the derivative of a function is itself a function that has its own properties. This fact was noticed in the test (Question II and Question III). It is noticed that many students were not able to handle these questions since they assumed that the derivative function behaves the same way as the original function; for them both both functions have same graphs, same signs, same critical points, and both increase and decrease during over the same intervals, and common properties have others. This result is not surprising in the control group because the idea of derivative as a function was not emphasized graphically. Most of the time, students were asked to differentiate functions using the formal definition. Moreover, as shown in research question 1, the graphical representations were rarely used as tools, thus limiting students' understanding of derivatives as objects. Moreover, students in the control group did not have the opportunities to work with *a graphing software* , and hence visualize that the derivative of a function is another function that has its own properties (a graph, sign, sense of variation, maximum and minimum points etc). The use of *Autograph* in the experimental group allowed the generation of the table of values, the graph and the equation of the functions. Moreover, the 'slope function' and the 'slow plot' buttons of *Autograph* allowed plotting gradually the derivative of the selected function showing the moving tangent on the original function graph. These features affected positively the conceptual understanding of students in the experimental group. Such results concur with those by Tall (2008) argued that the visual element of a dynamic math software enables students to visualize the changing slope of the graph as a function.

5.1.3. Research question 3

Does the use of technology and multiple- representation approach improve students' motivation and attitudes toward math?

Group work, technology (*Autograph)*, and student- centered activities were the main characteristics of the instructional approach used in the experimental group. The activities were designed to promote students' construction of the mental structures as described in the genetic decomposition of the derivative concept. However, group work and technology were not used at all in the control group. Instead, the lecture method was frequently used in the classroom.

Based on the interviews and the observations of students' participation and work, the researcher noticed that the teaching methods used in the two groups have affected students' attitudes and motivation toward calculus. A simple definition of attitude is: a positive or negative emotional opinion towards math; It can be seen as a combination of several factors such as motivation, confidence, values and beliefs in mathematics (Tapia, 2000). During the first two sessions in the experimental group, many students were not motivated, resisted the approach used and found it difficult. Some of the students' comments were as follows:

- "Why do we need to study through graphs and tables?, the derivative rules and equations are much easier."
- "We are not used to visualize things."
- "When are we going to start with the rules?"
- "We hate the approach and we don't care about the use of derivative in our life".

However, gradually later, most students got acquainted with the new approach. They started interacting and became more motivated about the topic. They were working in groups with excitement, discussing and sharing ideas with their peers, asking questions, and exploring the problems at hand. At the end of the instructional treatment, the majority of students' comments and attitudes toward the approach used were positive. Some of students' comments (interviewees and non-interviewees) were as follows:

- "I am repeating this course for the second time. I do not remember anything from the first time because it was all rules and equations, but now I feel it is a new way. I like the approach and I can remember more because it includes more graphs, visual."

- "Derivative is a nice and interesting topic that describes the rate of change of any quantity with respect to another. It has many life applications (biology, chemistry, economic etc). The approach used in class is interactive, enjoyable and interesting. It depends on analyzing. It helps us to understand the big picture and improves our visual thinking".

- "It is a rich approach. It is hard though. Each exercise and activity is different and exciting. I think I will get good results in the test".

In general, all students in the experimental group liked the visual part of the approach and agreed that technology helped them make connections between the graphical representation, the table of values and the algebraic expressions of functions. It is noticed that many weak students liked the approach and performed well in the test. Such results of the study are consistent with previous studies conducted on the positive relationship between the use of multiple- presentation approach with the use of dynamic math software, and students' attitudes toward mathematics (Tseng, Chang, Lou, & Chen, 2011; González and Rodríguez, 2011). However, few students in the experimental group did not like the approach and commented that the approach was demanding, required lots of work, and that the activities solved in class were difficult and needed more time. They added," working with graphs is not an easy job, we have to interpret the graphs, extract information and then relate them to the function and its derivative. However, rules are easy; we just need to memorize them". One interpretation of such attitude may be that the time spent using this approach was short, thus such students needed more time to get used to it, to think visually and change their attitudes. One obstacle to visual thinking is the traditional instructional methods that place a strong emphasis on formulas, equations, memorizing rules and manipulating symbols (Dreyfus and Eisenberg, 1991; Habre, 2001). In many cases, this traditional instructional method carries over from the school years; thus this sudden shift to a teaching method that requires visualization, critical thinking and analysis is not easy for students. Moreover, some students are just not visualizers (Presmeg, 1986).

On the other hand, most students in the control group were not motivated and were passive learners; they spent their time taking notes, and very few of them were participating and engaging in the class discussions. This passivity is due to the fact that the approach and the teaching methods used were 'traditional'. Some of students' comments were as follows:

- "I am counting the days to finish this course. I am passing each exam though. I think that math is complicated, boring, and not necessary in my major and it is all about formulas and rules to memorize".
- "I hate math, it requires memorization more than the history course."
- "I never liked math courses and I still hate math. It is useless and includes many formulas and rules that are meaningless. I know how to apply the rules and formulas, but I do not have any clue about the meaning of the concepts.

Most students in the control group showed dissatisfaction with the approach used in their class, and agreed that math is about rules and formulas that need to be memorized. This is due to the fact that the book used in the control group is dominated by the symbolic representation of the mathematical concepts. Moreover, most of the exercises that were solved by student are abstract and purely mathematical and not related to real-life examples. Moreover, Hughes-Hallett (1991) argued that for many students also for some instructors, learning calculus is equivalent to manipulating symbols and rules. Therefore, the use of a book that emphasizes one representation of a concept, at the expense of others, limits students' conceptual understanding, as shown later in research question 4.

5.1.4. Research question 4

What are the differential effects of the two approaches (formal symbolic approach and multiple-representation visual approach) on students' conceptual understanding of derivatives?

To answer this question, results from the derivative questionnaires, the derivative tests and the interviews are considered. Procedural knowledge refers to memorized facts, rules, procedures, methods and formulas, while conceptual understanding refers to learning with understanding and emphasizing the relationships and connections between ideas and meanings. It cannot be based only on memorization, but it needs reflective and thoughtful thinking.

As shown in section 4.3, the qualitative analysis of the questionnaires showed that the experimental group has better conceptual understanding of the derivative concept than the control group. Making connections between related ideas and emphasizing the multiple representations of the concept lead to conceptual understanding (Zorn & Ostebee 1999, NCTM, 2000). When students were asked in part I of the questionnaires about the meaning of derivatives, the symbolic representation dominated the thinking of most students in the control group (before and after the implementation of the unit on derivatives). Although some students in the control group gave a geometric description of the derivative as the slope of the tangent line at a point and related it to the instantaneous rate of change, the majority of students' concept image was limited and included some notions of the derivative concept. However, in the experimental group, students' dominant image of the derivative concept has changed after the implementation of the unit on derivatives. Before the implementation of the unit on derivative, 62 % of students mentioned in the questionnaires, that the derivative is about rules and formulas and none of the students mentioned the three representations (numerical, graphical and symbolic) in their definitions. After the intervention, only 8 % of students spoke of the rules while 65% mentioned the three representations (derivative as slope of a curve, slope of a tangent line, instantaneous rate of change, and derivative as rules of differentiations and formal definition), in addition, they mentioned the relationship between a function and its derivative. Moreover, many students in the experimental group provided real-life examples of derivative, compared to none in the control group. The quantitative analysis of the questionnaires was consistent with the qualitative analysis where students in the experimental group outperformed students in the control group. The difference in the mean scores between the control group ($M = 6$) and experimental group ($M = 9.89$) is 3.89, which is statistically significant ($p = 0.00 < 0.05$), as shown in Table 14. Thus, the approach used in the experimental group (Book 2, multiple representation approach, visualization, and *Autograph)* for teaching derivative seems to be effective and to enhance students' understanding of the concept.

Moreover, as shown in section 4.4.2, students in the experimental group outperformed their counterparts in the derivative test. The test aimed to examine students' graphical understanding of derivative as the slope of a tangent line, the relationship between the derivative and the instantaneous rate of change, and finally the relationship between a function and its derivative. The main characteristic of the test is that the algebraic expressions of the functions are not provided; instead, graphs and tables of values of functions are given. The qualitative analysis of students' responses revealed that most of the students in the experimental group seem to have an *object* conception of the *derivative* concept, while their counterparts lack an *object* conception of the concept. The majority of students' conceptions in the control group reached the *action/ process* levels. This is evidenced by the number of correct responses to each question/ sub- question in the two groups, as shown in Figure 32. More than half of students in the control group (even the B students) failed to answer the questions of the test, and some students left some questions unanswered. On the other hand, more than half the students in the experimental group provided correct and complete answers. Moreover, the quantitative analysis of students' tests supports the qualitative results where the difference in the mean scores between the control group (M = 7.4) and the experimental group (M = 14.26) is 6.9, which is statistically significant. It is noticed that many of the control group who failed the test did well in another test (instructor' X exam). Thus, the results of this study suggest that the pedagogy used in the experimental group (cooperative learning, *Autograph*, and activities that emphasize the multiple representations of the concept) is effective in helping students develop a better understanding of the derivative concept. The results of this study are similar to the findings of many studies where students learning in a constructivist- technology- based environment tend to acquire conceptual understanding of the concept learned than students learning using a traditional- based instruction (such as Heid, 1984; Asiala et al, 1997; Naidoo, 2007; Gonzalez & Rodriguez, 2010). In their study, Zulnaid and Zakaria (2012) examined the effects of using multiple- representations approach on the conceptual knowledge of functions and derivatives for 124 students. The students were divided into two groups: control and experimental. The results showed significant differences at $p < 0.05$ between the two groups in favor of the experimental group.

The analysis of students' papers revealed that many students in the control group had deficiencies in their graphical understanding of derivative as the slope of a curve or slope of a tangent line (Question I and Question IV of the test). In question I, the function $f(x)$ and its tangent line at the point (5, 3) are presented graphically and students were asked to find $f'(5)$ and $f(5.1)$. This question was answered by 35 % of students in the control group compared to 100 % in the experimental group who seem to have an *object* conception of the derivative concept. Moreover, when asked to find $f(5.1)$ many students in the experimental group provided a good comprehensive explanation as they used the concept of ' linearization' and the word 'zooming' in their answers. Question I was adopted from a study by Asiala et al (1997). The results to question I were similar to the findings of Asiala et al (1997) where students who took the reformed calculus course (learning using computers and cooperative learning) had developed better graphical understanding of a function and its derivative than students taught in a traditional course. Another example is question IV that tests students' ability to translate from the symbolic to the graphical representation and requires proving geometrically why the derivatives of any two functions $F(x)$ and $G(x)$, where G(x) is a vertical translation of F(x), are equal. It is noticed that only three students of the control group (12 %) were able to provide correct answers while 16 students (62 %) of the experimental group managed to provide complete correct answers. To solve this question, students need to realize that the slopes of the tangent at any point a on the graph of F and G are the same since the graphs are identical and just translated along the vertical direction, thus the tangent lines are parallel. This suggests that the software, *Autograph*, used in the experimental group has positive effects on students' conceptual understanding. The 'slope function' and the 'slow plot' buttons of *Autograph* allowed students to plot gradually the derivative of the selected graph or equation showing the moving tangent on the original function, thus, emphasizing the idea that for every point a on the graph of f, the derivative is equal to the slope of the line tangent to f at that point.

Moreover, in question III, the graph of the derivative function $G'(x)$ is given without its equation and students were asked to interpret the graph of G' to answer questions related to the original function $G(x)$. This question was answered correctly by almost 30 % of students in the control group compared to 69 % in the experimental group. This result was not surprising because the findings of several studies showed that students learning in a traditional setting faced difficulties in interpreting the graph of a derivative function (Maharaj 2013; Ubuz, 2007; Berry and Nyman 2003). The use of technology and multiple representations in the experimental group has improved their visual thinking, as shown by many studies (Gonzalez and Rodríguez, 2011; Pool, 1992). It is noticed that many students in the control group lack an *object* conception of the derivative concept as they were unaware of the fact that the derivative of a function is a function itself that has its own properties. These students assumed that the increase and decrease of $G'(x)$ and its critical points are the same for $G(x)$. In addition, the last question in the test (question V) tests students' ability to read and interpret the table of values of $C(t)$ to find the derivative at a point numerically. The results showed that 35 % of students in the control group, compared to 77% in the experimental group, managed to solve this question by realizing that the change in time in very small (h = 0.1) and thus, approximating the instantaneous rate of change by calculating the average rate of change using two nearby points. It is noticed that many students in the control group failed to answer this question because they did not see that the table of values can be used as an *object* to get information about the function and its derivative. This result in the control group is not surprising because as discussed earlier, the numerical representation of derivative was used as an end and not as a teaching tool. This result was similar to the findings of a study conducted by Naidoo (2007) who developed an interactive and dynamic module for teaching derivative. A group of 33 engineering students was taught using an interactive program, while 30 students were taught using the traditional lecture method. Students were tested on the ideas of average rate of change, instantaneous rate of change, limit of sequence and some rules of differentiation. The findings of his study showed that students in the experimental group scored significantly higher than those in the control group. Moreover, the clinical interviews conducted with some students indicated that students in the experimental group exhibited deep understanding of the concepts while their counterparts had superficial understanding.

Finally, the results of the interviews showed that the approach used in the experimental group (Book 2, *Autograph*, and visual- multiple representations) had positive effects on students' conceptual understanding. The interviewees in the experimental group were more flexible and comfortable working with the different representations of derivative than in the control group. For many control group students, the formulas and equations come first then the graphs. Moreover, the interviewees in the experimental group were able to explain the relationship between the graphical, the numerical and the symbolic representations of derivative, showing an indication of understanding and not just knowledge based on memorized facts as in the control-group students. For example, consider CA1 (an A level control-group student) and EA1(an A level experimental-group student) responses when asked to give a geometric description of the formal definition of derivative:

> Researcher: *Can you explain to me why derivative is equal to the slope of the tangent line at a point?*
>
> CA1: *This is a fact we took it in class. I don't know how to explain the relationship.*
>
> EA1: *Well, both derivative and slope represent rate of change. The tangent line is the limit of the secant line. Suppose we have a secant line that passes through two points A and B, then the slope of the secant line represents the average rate of change of the function between the two points. When A gets close to B, the secant line becomes a tangent line and, thus the average rate becomes an instantaneous rate o change. This explains the formal definition of derivative using limit as h approaches zero.*

Finally, the comments expressed by students (interviewees and non- interviewees) in the control group while solving the test, such as "how can we solve this question without being able to find the equation of $f(x)$ or $f'(x)$", and "how can we use the graph or the table to find the derivative?" indicate that their understanding is more procedural than conceptual.

5.2. Limitations of the Study

This study has few limitations. First, the participants are conveniently selected from one university that adopts an American program, thus they are not considered a representative sample of all calculus I students in all universities in Lebanon. Moreover, the sample size (52) is not big enough to generalize the results.

Moreover, despite the fact that *Autograph* was available on all computers in the university's computer lab, only the instructor in the experimental group used it for presentation and demonstration purposes, due to time limitations. Moreover, even though students in the experimental group performed well in the derivative test, the time devoted to teaching derivative was not enough. The researcher noticed that the visual approach used in the experimental group proved to be satisfying for some students and disliked by others. As Habre (2001) showed, an obstacle to visual thinking is the traditional/ procedural teaching that students are still encountering in their schools.

Moreover, in this study, the use of multiple representations is limited to the course calculus I and to the derivative concept in particular. Therefore, the results of the study cannot be generalized to other calculus topics such as integrals, limits and other calculus courses (Calculus II, Calculus III). Therefore, further studies could be conducted to investigate the effects of using multiple- representation approach and technology on students' conceptual understanding of other calculus topics.

5.3. Concluding Remarks and Recommendations for teaching and future studies

The results of this study suggest that the instructional treatment used in the experimental group is more effective than that of the control group. The use of a book (such as Book 2) that emphasizes the multiple representation of the derivative concept leads to better conceptual understanding of the concept. On the other side, the use of one type of representation at the expense of other types, limits students' understanding and does not address diverse learning styles and needs. The findings of this study revealed that the activities designed in the experimental group helped in the formation of the mental structures as described in the genetic decomposition of the *function* and *derivative* concepts. However, the researcher concluded that, to improve students' *object* conception about derivative, more time should be devoted to sketching the original function based on its derivative graph, matching graphs of functions with the graphs of their derivatives, or asking questions on the original function based on the graph of a derivative.

Cooperative learning is very important because students are provided with opportunities to be actively involved in their learning, to explore, make conjectures, discuss and share ideas with their peers, and ask questions. Moreover, all students in this study confirmed that *Autograph* helped them make connections between different representations such as graphical representation, the table of values and the algebraic expressions of functions, and thus enhance their understanding. Therefore, it would be more interesting to expand this study and investigate the effects of using a multiple visual approach on students' conceptual understanding of other aspects of derivative such as the second derivative, the relation between the first and the second derivative of a function, and the concavity and inflection point of a function.

We are living in a world where young children spend hours on using their Ipads, cell phones, digital games and other technological gadgets. Prensky (2001) referred to the new generation as the "digital natives". Thus, the use of technology, in particular dynamic math software, is a must. All math teachers are recommended to use visual demonstrations in their teaching. Bringing technology (computers, calculators, simulation games etc) into classrooms is not enough, teachers must effectively integrate technology into lesson plans, and activities, and know *when* and *how* to use it (Erhan 2011). Moreover, when students' thinking is dominated by one type of representation, this leads to serious learning difficulties. Therefore, changes in the curriculum of calculus courses and in the book adopted should be considered to encourage the use of multiple representations.

Moreover, further studies are encouraged in order to investigate students' conceptual understanding of other advanced courses that require the use of the derivative concept such as Calculus II (integrals), Mechanics, Numerical Methods, Numerical Analysis and Differential Equations etc.

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APPENDICES

LEBANESE AMERICAN UNIVERSITY

Appendix A: Diagnostic Test

MTH 101-Calculus I

Purpose of the test

This test aims to assess your previous mathematical background knowledge. It will not affect your average grade negatively.

INSTRUCTIONS

- You have 65 minutes to finish it
- There are 9 problems in this test (check that you have all the pages). If more space is needed, use the back of the pages.
- Show all your work.
- Calculators are <u>NOT</u> allowed.

1. **Solve the following equation in the set of real numbers.**

$$(\boldsymbol{x}+\mathbf{3})^{\mathbf{2}}=\mathbf{9}$$

2. **Find the domain of definition for each of the following functions:**

a. $f(x)=\frac{1}{x^2+x-6}$

b. $f(x)=\sqrt{-2x+8}$

3. **Find the average rate of change of the function *f(x)* below the given interval.**

$\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{x}^{\mathbf{2}}$ **;** over interval [-1,1]

4. **You are given different values of angles in both degrees and radians. Complete the following table:**

Degrees	**0**	**30**	**45**	**60**	**90**	**180**	**270**	**360**
Radians	**0**	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	**2** π
Sin Θ								
Cos Θ								

5. **Given the points A (2, 5) and B (0, 3), find the equation of the line passing through :**

a. A and having a slope of 2

b. A and B

c. A and parallel to the line: $y = -5x+1$

d. B and perpendicular to the line: $3x-2y = 8$

6. **Given the parabola $y = f(x) = x^2 - 5x + 6$**

a. Find the x-intercept and y-intercept of the graph of the $f(x)$.

b. Give the ordinates (y- value) of the two points of abscissas: x = 1 & x = 4 respectively.

c. Find the equation of the tangent to the parabola at the vertex A of coordinates (**5/2 , -1/4)**

d. Sketch the graph of the parabola

7. **Given $sinx$= 3/5 ,where x∈[π /2, π]. Find $cos\,x$ and $tanx$.**

8. **Compute the following limits:**

a. $\lim_{x\to 1} \dfrac{x^2 + x - 3}{x} =$

b. $\lim_{x\to 3} \quad \frac{x^2-9}{x^2-2x-3} =$

9. **If you are familiar with derivatives, find the derivatives of the following functions:**

a. $f(x) = 5$

b. $f(x) = x^3 - 2x^2 + 1$

c. $g(x) = (x^4 + 3x - 1)(x^2 + 1)$

Appendix B: Derivative questionnaire

Students Perceptions of the notion of derivative

This questionnaire is made up of three parts (I, II, and III). These three parts aim to check your understanding of the notion of "derivative". Your responses will remain confidential. It takes few minutes of your time, and your participation is highly appreciated.

Thank you

Hana Shatila

Email: hana.shatila@lau.edu

Name: ____

Part- I:

What do you know about derivative (derivative function and derivative at a point)**? Explain as much as you can.**

Part-II:

1. **Circle the statement(s) that describe what you understand by derivative (there may be more than one correct answer).**

   a) The derivative of a function at a given point measures the steepness of a function at that point. *(correct)*

   b) The derivative is equal to the slope of the secant line to the graph of a function drawn between two points.

   c) The derivative at a point is the value of the tangent equation at that point.

   d) The derivative of a function at a given point is equal to the slope of the tangent line to the graph of the function at that point. *(correct)*

   e) The derivative of a function at a given point is equal to the instantaneous rate of change of the function at that point. *(correct)*

   f) The derivative of a function is equal to the average rate of change between two points

g) Derivative is equal to the limit of the quotient difference of the ordinates and the abscissas of two points as the distance between them approaches 0. *(correct)*

h) The derivative describes the behavior of a function near some point.

Part-III:

1. Circle the statement(s) that describe what you understand by the derivative function (there may be more than one answer).

a) The derivative of a function is a function that is equal to the tangent equation.

b) The derivative function measures at every point the steepness of the function at that point. *(correct)*

c) The derivative of a function at a point determines whether the function $f(x)$ is increasing or decreasing in the neighborhood of that point. *(correct)*

d) The sign of the derivative of a function at a point determines whether the function $f(x)$ is negative or positive at that point.

Appendix C: Observation Log

LESSON OBSERVATION SHEET

Date:

Lesson:

Instructor:

Purpose:

The purpose of this observation log is to observe the instructional methods and strategies used in the classroom as well as students' participation and work. In addition, a major attention will be directed toward the different representations of the concept derivative that are emphasized or de-emphasized in the classes.

In other words, this sheet allows the observer to answer the following questions:

- What types of representations (Symbolic, numerical or Graphical) are being used or emphasized?
- Are the representations of the derivative integrated in any way?
- Are students just copying and taking notes? Are they asking questions? Is there any class discussion being conducted?
- Does the teacher provide students (individually or group work) with activities to discuss, think, reflect, analyze or explore certain problems?
- Is technology incorporated in some activities? Is it used by students or by teacher or both?
- Is technology providing students with opportunities to access and deal with the multiple representations of the concept derivative?

Codes used in the observation log

I. Teaching styles/methods used during instruction on derivative

L = lecture	**Q**= Questioning
GW = Group work	**T**= Technology use
IW= Individual work	

L **:** Lecture method characterized by teacher centeredness; teacher talks most of the time, students are passive listeners who take notes.

IW **:** Each student works individually on solving exercises and problems.

GW **:** Two to four students work in groups to solve or discuss certain problem.

T **:** The teacher and/or the students use technology (*Autograph*) for exploring, describing, and visualizing certain ideas, and making connections among different representations.

Q : Questioning technique. It is used for revision purposes, asking questions for clarifying some ideas, and answering questions related to homework

Representations & Teaching Methods/ Styles															
TIME	**Symbolic (S)**					**Graphical (G)**					**Numerical (N)**				
	L	GW	IW	Q	T	L	GW	IW	Q	T	L	GW	IW	Q	T
2:00 ↓ 2:01															
2:01 ↓ 2: 02															
2:02 ↓ 2:03															
......ETC......															
↓ 2:50															

Notes + Comments

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Appendix D: Derivative Test

MTH 101-Calculus I

DERIVATIVE

The purpose of this assessment is to check your understanding of the derivative concept in its three different representations: symbolic, graphical, and numerical. In addition, it aims to assess your ability to translate among the three representations.

INSTRUCTIONS:

- NAME:
- There are five problems in this assessment (check that you have all the pages). If more space is needed, use the back of the pages.
- Show all your work

.

Question I *(5 points)*

(L) is a straight line tangent to the graph of the function $y = f(x)$ at the point (5, 3), as shown in Figure 1.

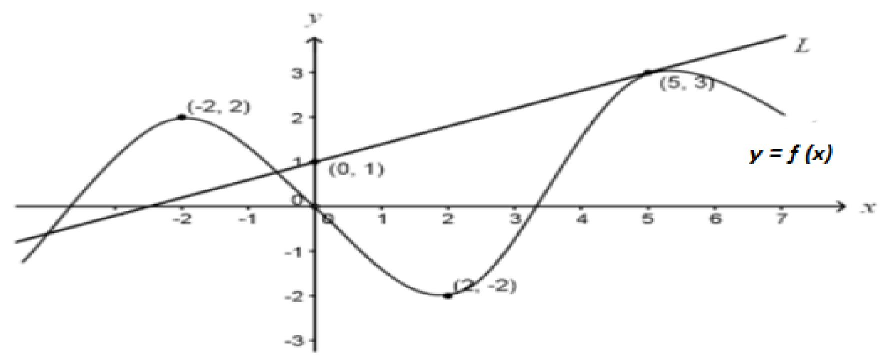


Figure 1: A straight line and a function.

Answer the following questions:

1. **Find $f(5)$. Justify your answer.**

   Solution: $f(5) = 3$. The point (5, 3) is on the curve of f, so its coordinates satisfy the function. In general, for every point (x, y) on the curve we have $f(x) = y$

2. **Calculate the value of $f'(5)$. Justify your answer.**

   Solution: $f'(5) =$ slope of the line tangent to the curve at x= 5, passing through the two points (0, 1) and (5, 3).
   Therefore, $f'(5) = \frac{3-1}{5-0} = \frac{2}{5}$.

3. **Find $f(5.1)$. Justify your answer (**be accurate as possible**).**

   Solution: The graph and the tangent line on a small interval around the point (5, 3) are almost confounded. Thus, the tangent line (L) at x = 5 can be used to estimate $f(5.1)$.

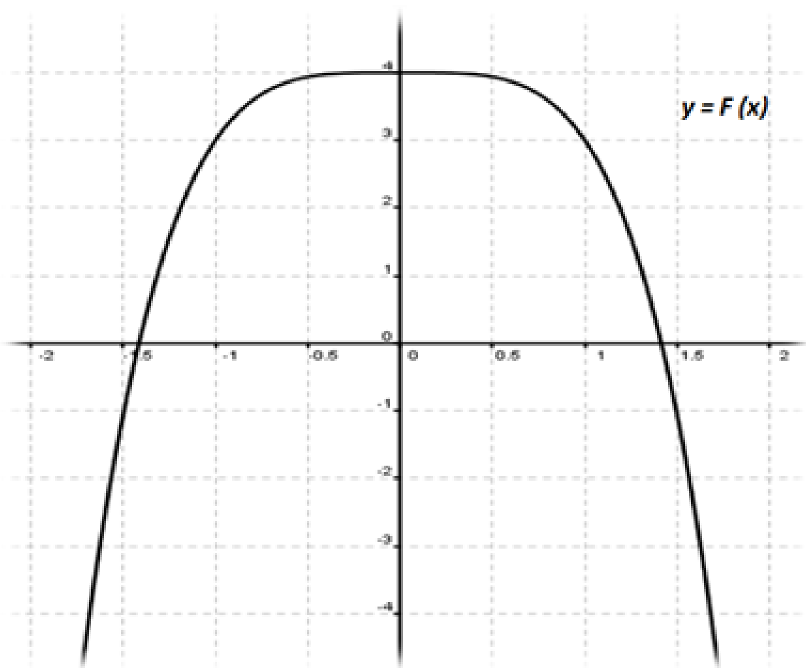
This concept is called linearization. So to solve this question, substitute $x = 5.1$ in the equation of (L) to get $f(5.1) = \frac{2}{5}(5.1) + 1 = 3.04$

Question II

(4 points)

Given a function $F(x)$ that is defined on the interval: (- ∞, ∞).The graph of the function $F(x)$ is shown below.

Now, is the <u>derivative</u> of this function increasing, decreasing or both? Discuss <u>using rate of increase and decrease</u> of the function.



Solution:

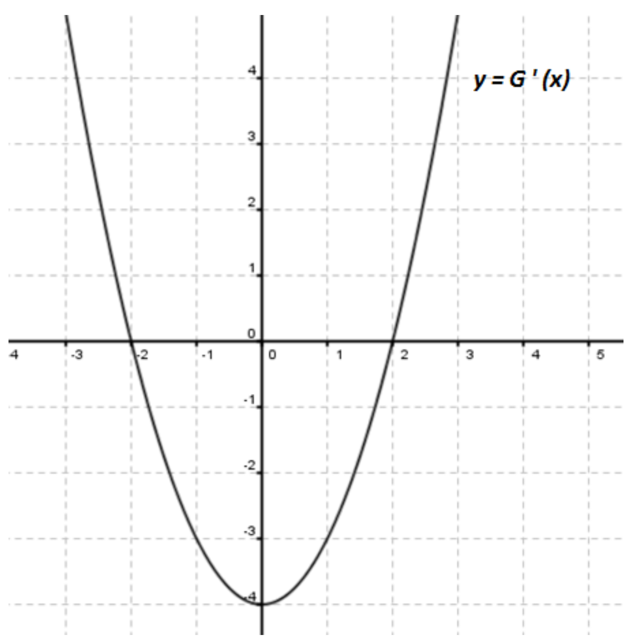
The derivative of this function is *decreasing* because:

- On the interval (-∞, 0), $F(x)$ is increasing, thus the tangent lines drawn at different values of x are increasing. The slopes of tangent lines at each point on the curve are positive and getting less steeper from (-∞, 0), thus the slopes of tangent lines are decreasing. Therefore, the derivative, which is equal to the slopes of the lines tangent to the curve, is decreasing as well.

- On the interval (0, ∞), $F(x)$ is decreasing, thus the tangent lines drawn at different values of x are decreasing. The slopes of tangent lines at each point on the curve are negative and getting steeper in the negative direction. Thus, the slopes of the tangent lines are decreasing. Therefore, the derivative, which is equal to the slopes of the lines tangent to the curve, is decreasing as well.

Question III *(6 points)*

The graph of the function $\boldsymbol{G'(x)}$ (derivative) is shown below. It is defined on the interval (- ∞, ∞).



Answer the following questions

1. **On what interval is $G(x)$ increasing? decreasing? Justify your answer.**

   Solution:

   - $G(x)$ is increasing on the intervals:]- ∞, -2[∪] 2, +∞ [. This is because $G'(x)$ is positive on these intervals; the graph G' is above the x- axis.

   - $G(x)$ is decreasing on the interval:]- 2, +2[. This is because $G'(x)$ is negative on this interval; the graph G' is below the x- axis.

2. **At what point(s) does $G(x)$ have critical points? Justify your answer**

   Solution:

   $G(x)$ has critical points when $G'(x) = 0$. Therefore, at x = - 2 and x = 2

3. **Which critical point is a local maximum / local minimum? Justify your answer.**

   Solution:

   - x = -2 is a maximum point because $G'(x)$ changes sign from positive to negative. In other words, $G(x)$ increases first, then decreases.

   - x = 2 is a minimum point because $G'(x)$ changes sign from negative to positive. In other words, $G(x)$ decreases first, then increases.

Question IV *(4 points)*

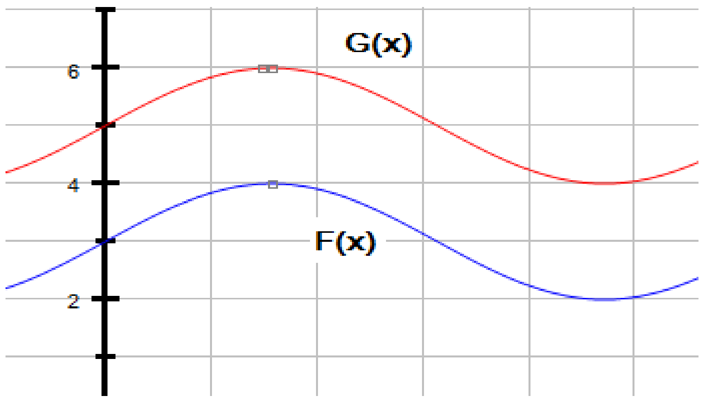
Let $F(x)$ be any function. $G(x)$ is another function defined as:

$\boldsymbol{G(x) = F(x) + C,}$ *where C is a constant.*

Clearly G ′ = F ′ since $\frac{d}{dx}[C] = 0$ (derivative of a constant = 0).

NOW, explain <u>geometrically</u> why the two derivatives are equal.

Solution:



F(x) and G(x) are arbitrary functions. The function $G(x)$ is the vertical translation of the function $F(x)$, on the vertical axis C units upwards or downwards (depending on the sign of C). If we draw tangent lines at different values of x, it is noticed that the slopes are equal since the graphs are identical and just translated along the vertical direction, thus the tangent lines are parallel.

Since the derivative of a function at a point is equal to the slope of the line tangent to the curve at that point, then the two functions F(x) and G(x) have equal derivatives.

Question V *(3 points)*

Suppose the table below gives the concentration (mg/cc) of a drug in the bloodstream at time t (min).

t (min)	**0**	**0.1**	**0. 2**	**0. 3**	**0. 4**	**0. 5**	**0. 6**	**0. 7**	**0. 8**
$\boldsymbol{C(t)}$	0.84	0.89	0.94	0.98	1	1	0.9	0.79	0.63

NOW, **<u>Fill the table below by finding the estimated values for $C'(t)$,</u> the derivative of $C(t)$ with respect to time. Explain and justify your answers.**

t (min)	**0**	**0.1**	**0. 2**	**0. 3**	**0. 4**	**0. 5**	**0. 6**	**0.7**
C ' (t)								

Solution:

$C'(t)$ = instantaneous rate of change of the the function $C(t)$ with respect to time t.

≈ Average rate of change of the function over very small interval of time.

Given the points: $(t, C(t))$ and $(t+h, C(t+h))$, then

$$C'(t) = \frac{C(t+h) - C(t)}{h}, \text{ where h = 0.1}$$

For example, $C'(0) = \frac{C(0.1) - C(0)}{0.1} = \frac{0.89 - 0.84}{0.1} = 0.5$

$$C'(0.1) = \frac{C(0.2) - C(0.1)}{0.1} = \frac{0.94 - 0.89}{0.1} = 0.5$$

$$C'(0.2) = \frac{C(0.3) - C(0.2)}{0.1} = \frac{0.98 - 0.94}{0.1} = 0.4$$

$$C'(0.3) = \frac{C(0.4) - C(0.3)}{0.1} = \frac{1 - 0.98}{0.1} = 0.2$$

………etc…………….

$$C'(0.6) = \frac{C(0.7) - C(0.6)}{0.1} = \frac{0.79 - 0.9}{0.1} = -1.1$$

Observation:

- $0 < t < 0.4$, $C(t)$ is increasing and $C'(t)$ is positive
- $0.4 < t < 0.5$, $C(t)$ is constant and $C'(t)$ is zero
- $0.5 < t < 0.8$, $C(t)$ is decreasing and $C'(t)$ is negative.

Appendix E: Sample exercises on derivative solved in the control group

(Fall 2013)

I. Derivative at a point (formal definition + slope of tangent line)

1. Consider the function $f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Does the graph of $f(x)$ have a tangent at the origin? Justify your answer

2. Given $f(x) = x^{1/3}$**.** Show that $f(x)$ has a vertical tangent at the origin.

3. Given $(x) = \frac{x}{x-2}$. Find the slope of the function's curve at (3,3).

4. Show that the function $f(x) = |x|$ is not differentiable at zero.

II. Derivative function + discussing some of the cases where a function fails to be differentiable

1. Differentiate $f(x)$ using the definition:

   a) $f(x) = x$

   b) $f(x) = \frac{x}{x-1}$

   c) $f(x) = \sqrt{x}$

   d) $f(x) = cosx$

2. Explain cases where the function fails to have derivative at a point.

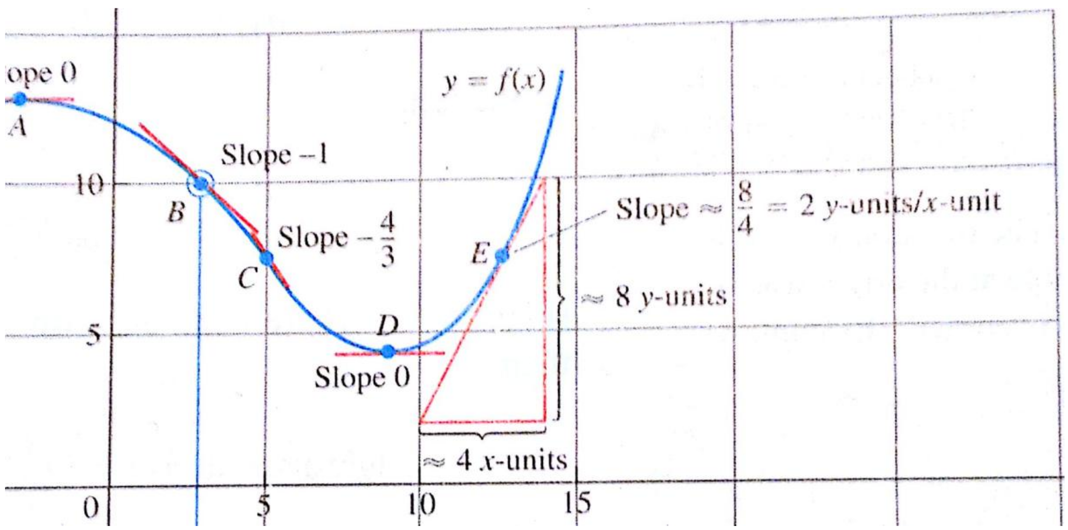
Note: Three graphs were discussed from section 3.2 in Book 1, page 11

3. *Discuss:* The function $f(x) = |x|$ is continuous at a at x = 0. Discuss its differentiability at that point.

III. Derivative function (formal definition + graphically)

1. Differentiate the functions, and find the slope of the tangent line at the given points
   - $f(x) = (x-1)^2 + 1;\ x = -1$
   - $g(x) = 1 - \frac{1}{x}; x = \sqrt{3}$
2. Graph the derivative of the function y = $f(x)$ given its graph below. (no equation)

   (From Book1 page 109)



IV. Instantaneous rate of change

1. The volume of a cylinder of a fixed height $h = 10$ is given in terms of its radius (r) as: **V(r) =** $\mathbf{10.\pi.r^2}$; where *V* is in (m^3) and h in (*m*).

   How fast does the volume change with respect to its radius when r = 5 m?

2. A dynamite blast blows a heavy rock straight up with a launch velocity of 50 m/sec. It reaches a height of $\boldsymbol{s = 49t - 4.9t^2}$ m after *t* sec.

a) How high does the rock go?

b) What are the velocity and the speed of the rock at time 2 seconds?

c) What is the velocity of the rock when it is 78.4 m above the ground?

d) What is the acceleration of the rock at any time?

3. The velocity of a body moving along the horizontal axis is given by: $v = t^2 - 4t + 3$. Find its acceleration.

V. Linearization

1. Consider the function $f(x) = x^2$. Find the linearization of $f(x)$ at x = 1.
2. Consider the function $\boldsymbol{f(x) = \sqrt{1+x}}$.

a) Find the linearization of $f\ (x) at\ x = 0$.
b) Use the linearization in part a) to estimate $f(0.1)$.
c) Use the linearization in part a) to estimate $\sqrt{1.05}$.
d) Use calculator to check your estimations in parts b and c.

3. Consider the function $f(x) = \cos x$. Find the linearization of $f(x)$ at x = $\frac{\pi}{2}$.

Appendix F: All activities solved in the experimental group

Activity -1 (group work)

Suppose an astronaut standing on the moon threw a candy bar with an initial velocity of 53 (m/s) meters per second.

The height of the candy bar (*in meters)* is given by the following equation:

$$h(t) = 58t - 8.3\, t^2$$

where (t) is the number of seconds after the astronaut threw the candy bar.

t(sec)	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
h(t)	31.8	36.5	41	45.6	49.7	53.8	57.6	61.4	64.9

a) Complete the table below by calculating the rate of change of height of the candy bar with respect to time using the given time intervals:

(Note: In each row calculate the increment in time ,Δt.)

Approaching 1 from the left	Δt	**Average Rate of change**		**Approaching 1 from the right**	Δt	**Average Rate of change**
[0. 6, 1]				[1, 1.4]		
[0.7, 1]				[1, 1.3]		
[0.8, 1]				[1, 1.2]		
[0.9, 1]				[1, 1.1]		

b) Reflect:

What happens to the values of the rate of change before and after $t = 1$ as the size of the time interval shrinks or gets smaller and smaller?

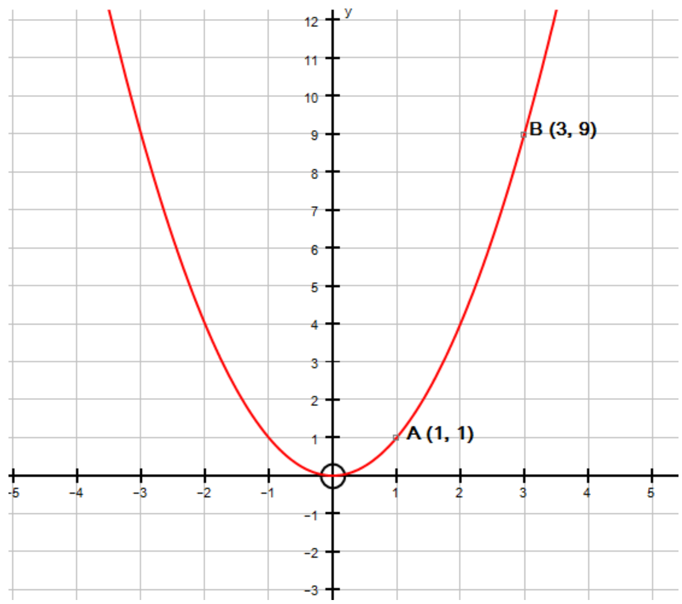
c) Estimate the velocity at $t = 1$; that is find the rate of change at $t = 1$ sec.

d) Communicate your understanding:

- What is the difference between average rate of change and instantaneous rate of change?
- How can the instantaneous rate change be obtained from the average rate of change?

Activity-2 (Individually**)**

The graph of the function $f(x) = x^2$ is given below.



Answer the following questions:

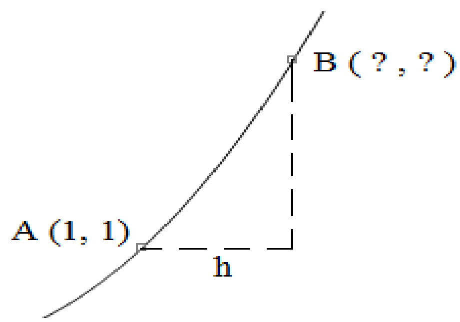
a) Find the average rate of change of $f(x)$ between x =1 and x = 3.

b) Draw the line that passes through the points A (1,1) and B (3, 9). What does this line represent with respect to the curve? (*secant or tangent*)

c) Find the slope of the line (AB) .Compare your result to part (a).

Activity -3 (Group work)

The diagram shows the graph of $y = x^2$ near the point A (1, 1). The point B is a horizontal distance h along from A.



a) Find the coordinates of B in terms of h.

b) Find the slope of the secant line passing through A and B, in terms of h. Simplify your answer.

c) What happens to the secant line as B gets closer and closer to A? What value does the slope obtained in **b)** approach?

Activity - 4

(Conducted by instructor Y using technology, *Autograph*)

Tangent as the limiting position of the Secant

Instructions in *Autograph*

Consider any function $y = f(x)$ and two points A and B on it.

- Click on  to plot any curve y = f(x) , for example $y = x^3 - 2x + 2$. The click Ok

- Click on the point mode  and place any point on the curve. With the point selected, right click on the mouse and select **text box** to name the point as A. Or you can click on the text box button at the top of the tool bar to name it A.

- Select point A then right click the mouse and select **tangent** from the menu. The equation of the tangent will be given in the status bar at the bottom of the screen.

- Add another point on the curve and name it B using text box button(follows steps as before)

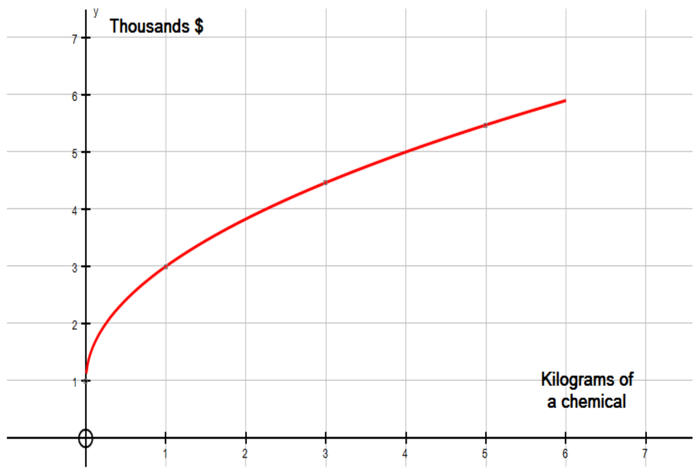
- Select both points A and B, right click on the mouse and select **straight line** from the menu. You can use the **text box** to name the line 'secant line'.

- Select point B, move it manually towards A, and observe what is happening.

  More interesting: select point B and click on the animation button. After choosing the speed and adjusting the parameters, click ok then click play. Observe what is happening.

Activity-5 (group work)

The graph below shows the cost in dollars, $y = C\,(x)$, of manufacturing x kilograms of a substance.



a) Is the average rate of change of the cost greater between x= 0 and x = 3, or between x = 3 and x = 5? Justify your answer graphically.

b) Is the instantaneous rate of change of cost producing x kilograms greater at x = 1 or x = 5? Justify your answer graphically.

Activity 6

(Conducted by instructor Y using technology, *Autograph*)

Slope of a Graph at a point: Tangent Lines

A line has only one slope and any two points are enough to compute it. What about a curved graph? How can we measure the slope of a curve at a point?

Autograph can be used to show how the function at a certain point is locally straight by using the 'zoom in' button.

Instructions in *Autograph*:

- Click on  to plot any curve y = f (x), for example y $=x^2$. Then, click Ok.
- Click on the point mode  and place a point on the curve, say A (1,1)
- Click on the point then right click the mouse and select **tangent** from the menu. The equation of the tangent will be given in the status bar at the bottom of the screen.
- Click on the  rectangle zoom-in button around the point (1, 1) repeatedly and observe what is happening to the graph.
- Choose other points and do the same.

Activity 7

(Conducted by instructor Y)

Let us *discover* through *Autograph* several cases where the function fail to have derivative at a point

Instructions in *Autograph*:

Part I: $y = x\,|sinx|$

- Click on  to plot any curve y = f (x), for example $y = x\,|sinx|$. Then click Ok.
- Click on the point mode  and place a point on the curve, say A (0,0)
- Click on the  rectangle zoom-in button around the point (0, 0) repeatedly and observe what is happening to the graph.

  Note: the scales will automatically adjust to each zoom.
- Choose other point than (0 , 0) and do the same. What do you conclude?

Part II: $y = \sqrt{x}$

- Click on  to plot any curve y = f (x) , for example $y = \sqrt{x}$. Then click Ok.
- Click on the point mode  and place a point on the curve, say A (4,2)

❖ Click on the  rectangle zoom-in button around the point (4, 2) repeatedly and observe what is happening to the graph.

Note: the scales will automatically adjust to each zoom.

❖ Choose other point than (0, 0) and do the same. What do you conclude? What happens to curve?

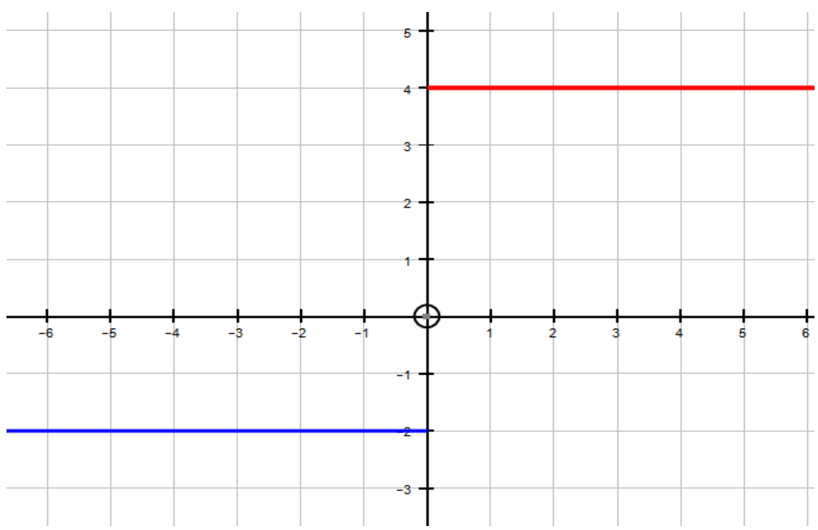
Activity 8 (Class work)

a. Given $f(x) = 2x + 2$. Explain geometrically and in terms of rate why the derivative is the constant function $f'(x) = 2$.

b. For any linear function $f(x) = ax + b$, what would be its derivative? Explain

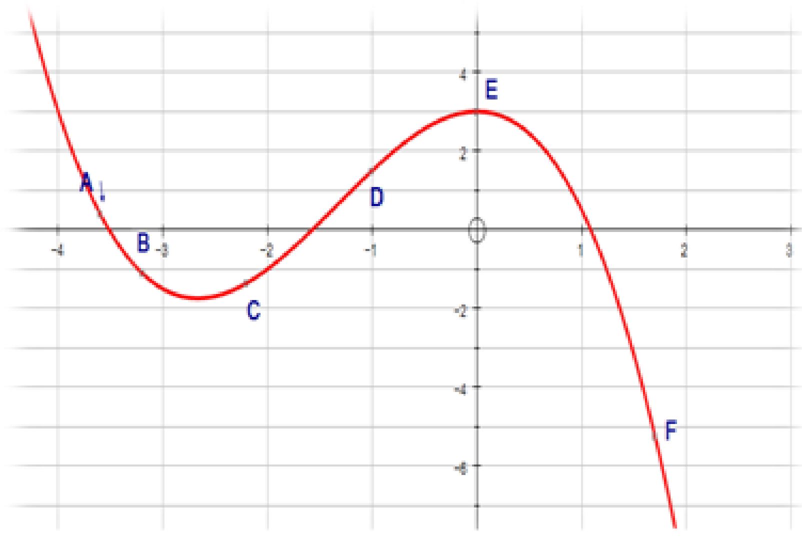
Activity 9 (Class work)

The graph below represents the derivative of $g'(x)$. Can you guess a formula for $g(x)$?



Activity 10 (Group work, then corrected using Autograph)

Below is the graph of $y = f(x)$.



a) Match the points labeled on the curve with the given slopes of the curve in the table below.

Slopes	Points
- 8	
- 5	
-2.5	
0	
1.5	
2.5	

b) Sketch the graph of the derivative of $y = f(x)$ in the same system.

Activity 11 (individual work)

Some values of the derivative of the function $f(x)$ are given in the table below.

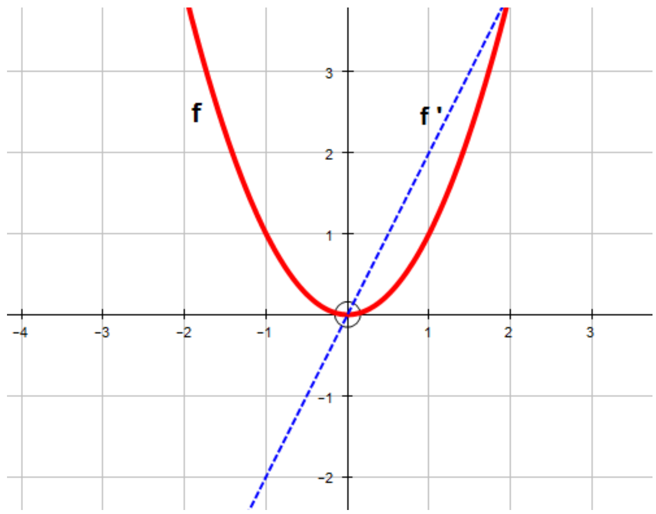
x	**-3**	**-2.5**	**-2**	**-1.5**	**-1**	**0**	**1**	**1.5**	**2**	**2.5**
$f'(x)$	**-12**	**-10**	**-8**	**-6**	**- 4**	**0**	**4**	**6**	**8**	**10**

a) Plot $f'(x)$.

b) Guess a formula for $f'(x)$. Justify your answer.

<u>**Activity 12** (Autograph)</u>

Given the graphs of f and f' below. The red graph is for f and the dotted blue graph is for f'.



a) **Complete the following statements**:

- $f'(x)$ is positive on the interval......................................
- $f'(x)$ is negative on the interval...

b) **Complete the following statements**:

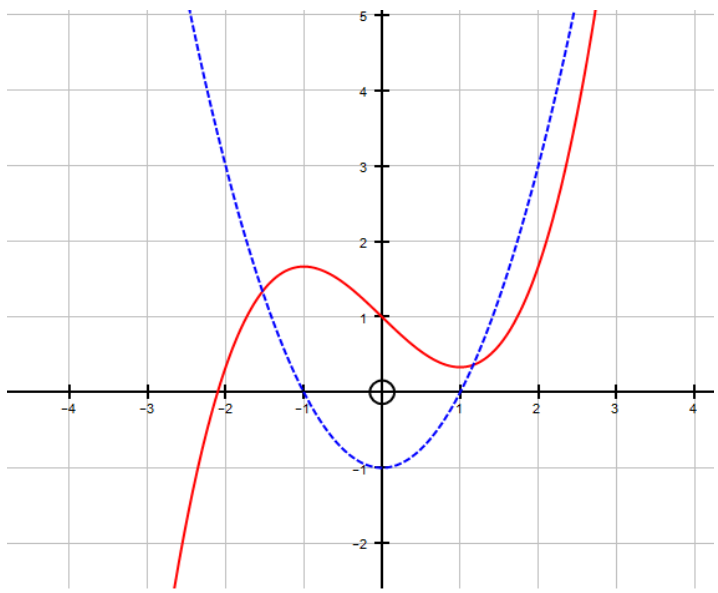
- $f(x)$ is increasing on the interval......................................
- $f(x)$ is decreasing on the interval...

c) **Using the results of parts (a) and (b), to explain the relation between f and f'.**

<u>**Activity 13**</u>

Consider the two graphs below.

The solid curve (red) represents the graph of a function. Explain why the dotted curve represents the graph of is derivative.



Activity 14

For all x in the closed interval [2, 5], the function $f(x)$ has positive derivative. Which could be a table values for $y = f(x)$?

x	Y
2	7
3	9
4	12
5	16

Table 1

x	y
2	9
3	12
4	7
5	16

Table 2

x	y
2	16
3	12
4	9
5	7

Table 3

Activity 15

A function $f(x)$ is defined over [-2, 2].

Its derivative $f\,'(x)$ is defined as follows:

$$f\,'(x) = \begin{cases} 3 & ; \quad -2 \le x < -1 \\ 1 - 2x & ; \quad -1 \le x < 0 \\ -5x + 1 & ; \quad 0 \le x < 1 \\ -4 & ; \quad 1 \le x \le 2 \end{cases}$$

a. Determine the interval(s) where $f(x)$ is increasing.

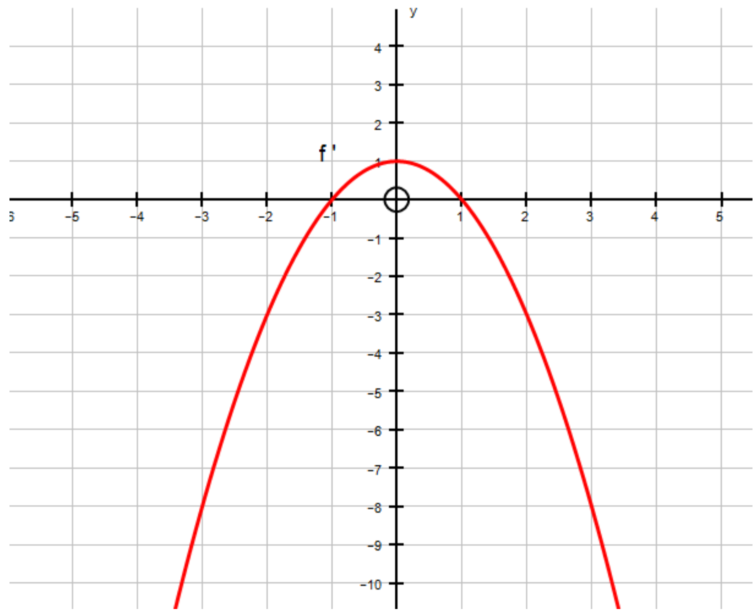
b. Determine the interval(s) where $f(x)$ is decreasing.

Justify your answers

Activity 16

The graph of the derivative function $f'(x)$ is given below.

It is defined from (- ∞, ∞).



a) **On what interval $f(x)$ is increasing?**

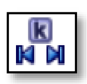
b) **On what interval $f(x)$ is decreasing?**

Activity 17

Formal Definition of Limit

Instructions in *Autograph*

- Click on the 

toolbar button.
- Enter a function f(x): $x^2 - 4x - 3$ and g(x) = [f(x+ h) -f(x)]/ h
- Click on  to plot the curve y = f(x).
- Select y = f(x) then click on  to draw the derivative of f(x).
- Click on  to plot the curve y = g(x)
- Click on 

constant controller and change the value of h and observe what is happening as h approaches 0.
- Do the same steps as before for the function $f(x) = x^3 - 2x^2 + 4$

<u>Activity 18</u>

Part I:

Using the <u>formal definition of derivative</u>, differentiate the following function:

$$\boldsymbol{f(x) = x^2 + 4x - 1}$$

Part II:

The volume of a cylinder of a fixed height $h = 10$ is given in terms of its radius (r) as:

V(r) = $10.\pi.r^2$; where V is in (m^3) and h in (m). How fast does the volume change with respect to its radius when r = 5 m?

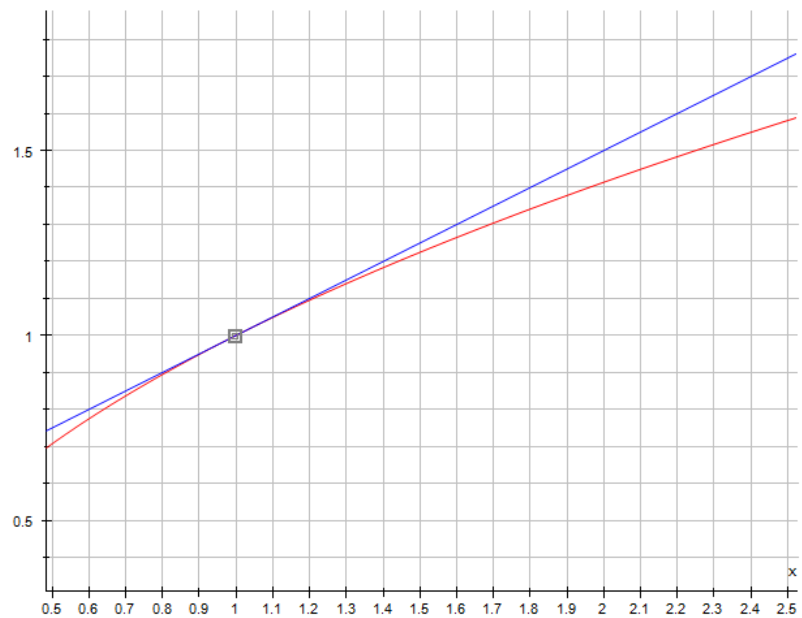
Part III:

Find the slope of the tangent to the curve $\boldsymbol{k(x) = \frac{1}{x+1}}$ at $\boldsymbol{x = 1}$

Activity 19

1. Given $f(x) = \sqrt{x}$. Find the linearization of $f(x)$ at x = 1. Find $\sqrt{1.1}$ without using calculator.

   (*Autograph* was used for explanation)



2. Given $f(x) = \frac{x}{x+1}$. Approximate $f(1.3)$ using a linearization to the function at a suitably chosen integer near x = 1.3

Appendix G: Unit plan in the control group (fall 2013)

The world around us is always changing. Understanding the nature of change and how fast changing occurs is very important. Thus, derivative is a very important and interesting topic that is used to model the behavior of changing quantities such as: rising in prices, growing population, decaying radioactive materials, finding velocity and acceleration of moving objects and others.

Title of the unit: Derivative

General Objectives:

- Find the derivative of a function $f(x)$ at a point $x = a$ using the formal definition ($f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$)
- Compute the derivative of functions using the formal definition.
- Recognize the different notations used for derivative ($f'(x), \frac{dy}{dx}, y', \frac{df}{dx}$)
- Recognize that the derivative of a function at a point is equal to the slope of the line tangent to the curve at that point
- Find the derivative of a function at a point from the graph
- Recognize that the derivative of a function at a point is equal to the instantaneous rate of change of the function that point
- Determine whether a function is differentiable at a given point
- Use the sign of the derivative to determine the variation of a function
- Use the derivative to find the maximum and minimum points of a functions
- Sketch the graph of a function using the properties of its derivative
- Sketch the graph of a derivative from the graph of its original function
- Find the linearization of a function at a point $(f(x) \approx L(x) = f(a) + f'(a)(x-a))$
- Solve real life problems related to derivative

Sessions	Procedure
Session 1	-At first**,** the teacher defined the derivative of a function at a point using the formal definition : $f\ '\ (a)\ = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided the limit exists. - Then, she told students that the derivative of a function $f(x)$ at a point a, denoted by $f'(a)$, is used to deal with problems that require finding: slope of a curve at a point, slope of tangent line and instantaneous rate of change. - The teachers explained step by step how to apply the formula - Finally, students applied the definition and solved four exercises. In all the exercises, the equations of the functions are given where three questions are related to tangent lines/ slope of curve and one requires finding the derivative at a point (*no exercise was related to rate of change)*. It is noticed that few students participated while the majority were just copying from the board. - Assign homework I
Session 2	- Correction part of homework I - Defining the derivative function $f'(x)$ using first the formal definition. $f\ '\ (x)\ = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists. - Apply the definition and solve four exercises (all given their equations). - Introducing the cases where the function is NOT differentiable (existence of corner, vertical tangent and discontinuity). - Discuss the theorem: Differentiable functions are continuous, but a function need not have a derivative at a point where it is continuous. - Assign homework II
Session 3	- Recall the meaning of derivative. It is noticed that many students did not know the answer. - Correct part of the previous homework (two questions)

<table>
<tr>
<td></td>
<td>- Individual work:
Sketch the derivative of a function given its graph. In this exercise, the teacher reminds students that the derivative is equal to the slope of the curve and slope of tangent line. Then, she discussed with her students that whenever a function is decreasing → $f'(x)$ < 0; increasing → $f'(x)$ < 0 and that $f'(x)$ = 0 at the maximum/ minimum point.
- In the remaining time, the teacher introduced some rules for derivative (derivative of a constant, power rule and the sum rule).
- Assign homework III</td>
</tr>
<tr>
<td colspan="2" align="center">In sessions 4 and 5, the teacher continued explaining the rules for derivative with examples:
-Product rule
-Quotient rule
-Derivative for trigonometry
Chain rule
(Note: Rules are NOT part of my study)</td>
</tr>
<tr>
<td>Session 6</td>
<td>- Revision of the rules and formulas
- Then, the teacher gave students real life examples on derivative that involve rate of change (position of a moving car with respect to time, skiing with respect to t time, area of circle with respect to radius etc...)
- Then, the teacher gave students the definitions of three terms : velocity, speed and acceleration
- As an application, the teacher and the students solved two exercises that involve rate of change using two methods: formal definition of derivative and the rules.
• In the first exercise, the equation of the volume of a cylinder is given , where: $\mathbf{V = 10\,\boldsymbol{\pi}\,\boldsymbol{r}^2}$. How fast does the volume of a cylinder change with respect to its radius when r= 5 m?
• Second exercise: A dynamite blast blows a heavy rock and reaches a</td>
</tr>
</table>

<table>
<tr><td></td><td>height of $s(t) = 49t - 4.9t^2$
a) How high does the rock go?
b) What are the velocity and the speed of the rock at ti me 2 seconds?
c) What is the velocity of the rock when it is 78.4 m above the ground?
d) What is the acceleration of the rock at any time?

-Assign homework IV</td></tr>
<tr><td colspan="2">In session 7 and 8, the teacher explained implicit differentiations
(NOT PART OF MY STUDY)</td></tr>
<tr><td>Session 9</td><td>- introduce the idea of linearization (approximating complicated functions with simpler ones that are based on tangent lines)
- Give the definition: If f is differentiable at $x = a$, then the approximating function $L(x) = f(a) + f'(a)(x - a)$ is the linearization of f at a.
$$f(x) \approx L(x) \; near \; a$$
- Explain to students idea that the more we magnify the graph of a function near a point, the flatter the graph becomes which resembles its tangent at that point
- Solve three exercises on linearization given the equations of the functions.
-Assign homework</td></tr>
</table>

Appendix H: Unit plan in the experimental group (spring 2014)

Notes:

In the experimental group, almost the same objectives as in the control group are covered. However, a visual multiple- representations approach was implemented. The derivative of a function was discussed using graphs, table of values and algebraic expression. sTechnology (*Autograph*) was integrated in some activities assisting students in the exploration of problems. In addition, group work was frequently used.

Derivative - Unit Plan

Session - 1

Objectives: (50 minutes)

At the end of this session, students are expected to:

- Give real life examples that involve average rate of change and instantaneous rate of change
- Differentiate between average rate of change and instantaneous rate of change
- Define the derivative of a function at a point as the instantaneous rate of change at that point.
- Use the difference quotient $f(a+h)\ -f\ (a)/h$ to approximate the instantaneous rate of change or the derivative of a function $f(x)$ at a point.

Prerequisites:

Students should know how to calculate the average rate of change of a function over an interval, whether the data is given in a table form or through equation.

Procedure:

- In the first few minutes, gain students' attention by telling them that we are living in a world that is constantly changing, and then discuss with them real life examples that involve rate of change. Such examples include velocity (change in distance over change in time), change in human population with respect to change in the number of births, deaths, migration etc.
- Solve Activity 1. Students will work on this activity in groups of three. This activity allows students to make connections between the symbolic and numerical representations of a function since the data is given through an equation and through table.
- Through Activity 1 and class discussion, students will come up with the first definition of derivative: $f'(a) =$ instantaneous rate of change of f at a, which is approximated by calculating average rate of change over very small interval.
- Do Exercise A as an application of the formulated definition.
- Finally, students will be asked to interpret the following problem:
  At time t, g (t) represents the value of a bank deposit, where the time is in months since initial deposit, and the balance in $.
    - What does the rate function g'(t) represent?
    - What does the statement g' (3.5) = 4.2 mean in context?
- Assign homework I

Session -2

Objectives: (50 minutes)

At the end of this session, students are expected to:

- Make connection between average rate of change and slope of secant line
- Define derivative of a function at a point as the slope of tangent line to the curve at that point.
- Recognize that the instantaneous rate of change is equivalent to the slope of tangent line.

Prerequisites:

Students should know the concepts of average rate of change of a function over an interval, slope of a line, and the concept of derivative as instantaneous rate of change.

Procedures:

- Recall the concepts learned during previous session.
- Discuss and correct part of homework I.
- Do *Activity 2*. Students will work on this activity individually, and then class discussion will take place. This activity is important because it allows students to give a geometric interpretation of the average rate of change. In this activity the equation and the graph of the function are given.
- Do Activity 3. Students will work on this activity in groups of two. This activity aims to explain how the tangent line can be obtained from the secant line and how the slope of the secant line approaches the slope of the tangent line.
- Do Activity 4. The instructor will use *Autograph* to do this activity which is similar to Activity 3, but it addresses a more complex polynomial function ($y = x^3 - 2x + 2$). Having in mind Activity 3, this activity aims to confirm the formulated conjecture: slope of tangent is the limit of the slope of secant line.
- Finally, discuss the results of the activities (Activities 2, 3, and 4) with students
    - average rate of change = slope of secant line
    - $f'(a)$ = slope of tangent line to f at *a* = instantaneous rate of change at *a*
    - instantaneous rate of change = slope of tangent line (relate it to *Activity* 1)

- Do Exercise B as an application + Assign homework II.

Session -3

Objectives: (50 minutes)

At the end of this session, students are expected to:

- Define the derivative at a point as the slope of the curve at that point
- Discuss and identify the cases where the derivative at a point doesn't not exist.

Procedures:

- Recall the concepts learned during previous session.
- Do *Activity* 5. This activity is to be done individually. The purpose of this activity is to emphasize the concepts learned during last sessions. In this activity, the graph of a function is given (without its algebraic equation).
- Do *Activity* 6. This activity is to be done by the instructor using *Autograph* however all students will be asked to participate.
- Do *Activity* 7. This activity is to be done by the instructor using *Autograph* however all students will be asked to participate. The idea that the function is differentiable at a point if it is locally straight is very important. This activity builds on Activity 6. It is very important because it introduces the left and right derivatives and examines the case where the function is not differentiable.
- Do Activity 8. The purpose of the activity is to discuss some cases whether the function fails to have a derivative at a point (discontinuity, vertical tangent, corner).This activity is be done in groups of three.
- Sum up the main points in the lesson + Assign homework II.

Session - 4

Objectives: (50 minutes)

At the end of this session, students will be able to:

- Recognize the derivative of a function is itself a function
- Make a reasonable plot of the derivative of $y = f(x)$ by calculating the slopes on the graph of f

Procedures

- Solve Activities (9 and 10) as class work .

- Do Activity 11. This activity is to be done in groups of two then corrected using *Autograph*. It is very important because it introduces derivative as a function. In this activity, the graph of a function is given without is algebraic expression
- Do Activity 12. In this activity, some values of the derivative of the function $f\,(x) = 2x^2$ are given in a table. Students are asked to plot $f\,'\,(x)$ then guess its formula .
- Discussion + Assign homework

Session - 5

Objectives: (50 minutes)

At the end of this session, students will be able to:

- Make connection between the sign of the derivative $\boldsymbol{f\,'}$ and the sense of variation of $\boldsymbol{f}$**,** given graphically and through tables of values
    - If $\boldsymbol{f\,' > 0}$ on an interval, the $\boldsymbol{f}$ is increasing over that interval
    - If $\boldsymbol{f\,' < 0}$ on an interval, the $\boldsymbol{f}$ is decreasing over that interval
    - If $\boldsymbol{f\,' = 0}$ on an interval, the $\boldsymbol{f}$ is constant that interval
- Relate roots of $\boldsymbol{f\,'}$ with the maximum and minimum points of $\boldsymbol{f}$

Procedures:

- Recall with students what does it mean for a function to be negative, positive, zero, increasing, decreasing or constant.(Use graphs and equations)
- Do Activity 13. Students solve this activity in groups of two, then corrected using *Autograph*.
- Students solve Activity 14 in a groups of two + use of *Autograph*
- Discussion of the results.
- Do Activities 15 and 16. These activities are important because they include numerical (table form) and symbolic representations of derivative.

Session - 6

Objectives: (50 minutes)

At the end of this session, students will be able to:

- Use the formal definition $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to differentiate functions
- Use the formal definition to calculate the slope of tangent line or instantaneous rate of change

Procedures:

- Recall previous session. Students solve activity 17 in group of three students.
- Students solve Activity 18. The purpose of this activity is to introduce the formal definition of the derivative function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The instructor will use *Autograph* for discussion.
- Students solve Activity 19. This activity is an application of the formal definition of derivative. It is important because it links between different meanings of derivatives (slope of tangent line, instantaneous rate of change and symbolic definition using the limit
- Discussion
- Assign homework

Session 7 (Half the session)

Objective: (20 minutes)

- Find the linearization of a function

Procedure:

- Use *Autograph* to introduce the concept of linearization.
- Do Activity 19 as an individual work followed by class discussion
- Come up with the definition: $f(x) \approx L(x) = f(a) + f'(a)(x - a)$

Appendix I: Quiz

Name: Date:

MTH 101

Quiz

Question I: Find the derivative of the function $f(x) = \sqrt{x+1}$ using the formal definition.

Question II: Find the slope of the curve of $f(x) = 4 - x^2$ at $x = -1$ (using the definition).

Question III. Find the linearization of the function $g(x) = \frac{x}{x+1}$ near the point x= 0, and then approximate g (0.1).

Appendix J: Human research subject protection approval form

LAU
الجامعة اللبنانية الأميركية
Lebanese American University

Committee on Human Subjects in Research (CHSR)

November 11, 2013

Ms. Hana Shatila
School of Arts & Sciences
Lebanese American University

CHSR tracking number: LAU.SOAS.HS1.11/Nov/13
Protocol Title: "Students' Conceptual Understanding of Derivatives in Freshmen Calculus"

Dear Ms. Shatila,

Thank you for submitting to the CHSR the continuing review application for the above named study for review. I have reviewed, with other CHSR members, the above named study and all submitted documents. We hereby grant you approval to conduct the above referenced study. **Kindly use the attached stamped documents.**

Documents Submitted:

CHSR Exempt Application	Received 28 October 2013
Cover Letter	Received 28 October 2013
Email Approvals from instructors: Dr. Leila Issa & Dr. Samer Habre	Received 7 November 2013
Biographical Questionnaire	Received 28 October 2013
Diagnostic Test	Received 28 October 2013
"Epistemology of the notion of Derivatives" Questionnaire	Received 28 October 2013
Observation Log	Received 28 October 2013
Test on Derivatives	Received 28 October 2013
Letter to students	Received 28 October 2013
NIH Training – Hana Shatila	Cert. # 1309495 (dated 24 Oct 2013)

Agenda Item: December 2013
Review Type: Exempt
Action: Approved

Initial Approval:11 November 2013

Expiration Date: NA

If you have any questions concerning this information, please contact CHSR office by email at christine.chalhoub@lau.edu.lb

The CHSR operates in compliance with international guidelines of Good Clinical Practice, the US Federal Regulations (45CFR46) and (21CFR56) of the Food and Drug Administration. LAU CHSR Identifier: FWA00014723 and IRB Registration # IRB00006954 LAUIRB#1

Sincerely,

Constantine Daher, PhD.
CHSR Chair

Committee on Human Subjects in Research
Lebanese American University
11 NOV 2013
APPROVED

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