High order ImEx method for the shallow water model

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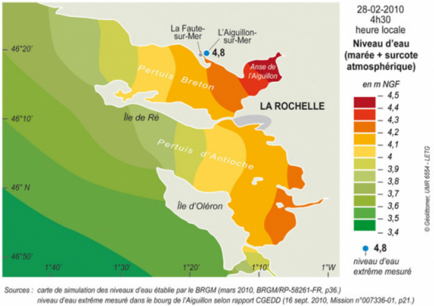
Climate change is expected to increase coastal flooding hazard in years to come.

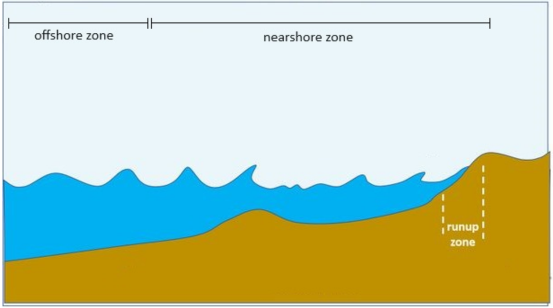
La Faute-sur-mer, western France after Xynthia storm, February 28 2010



Water level simulation map established by BRGM

Recent storm events (Xynthia [2010], Klaus [2009], Lothar [1999]) have shown the very destructive effects that flooding can have in terms of human victims and economic losses.

Objective: Design a multi-scale tool capable of simulating different features of oceanic flows, from large scale linear waves offshore to small scale non-linear flows in coastal areas.



Zone division

Offshore, tidal waves and atmospheric surges are very linear processes under the influence of Earth rotation.

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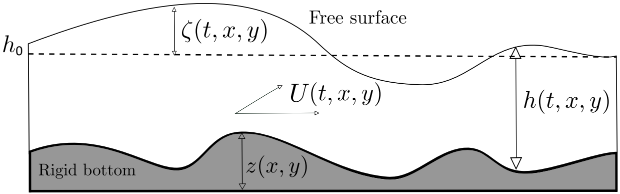

Consider the following rescaling:

$$\tilde{x} = \frac{\mathbf{x}}{l^*}, \quad \tilde{h} = \frac{h}{h^*}, \quad \tilde{U} = \frac{U}{U^*}, \quad \tilde{t} = \frac{t}{t^*}, \quad \tilde{z} = \frac{z}{h^*}$$

System (SW) becomes (after dropping the tildes):

$$\begin{cases} \partial_t h + \nabla.\,(hU) = 0\ , \\ \\ \partial_t(hU) + \nabla.\,(hU \otimes U) + \dfrac{1}{\mathrm{Fr}^2}\nabla\left(\dfrac{h^2}{2}\right) = -\dfrac{h}{\mathrm{Fr}^2}\nabla z\ , \end{cases} \qquad (\mathrm{SW}_{Fr})$$

with the characteristic Froude number

$$\mathrm{Fr} := \frac{U^*}{\sqrt{gh^*}} = \frac{\text{flow velocity}}{\text{velocity of gravity waves}}.$$

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- We focus in particular on the variation of the Froude number Fr that goes from 1 at the coastline to two or three orders less offshore.

$$\left.\begin{array}{l} \blacktriangleright \text{ mean depth } h^* > 3000m \\ \blacktriangleright \text{ current speed } U^* \approx 1m/s \end{array}\right\} \Rightarrow \mathrm{Fr} \approx 10^{-3}$$

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  $\Rightarrow \mathrm{Fr} \approx 10^{-3}$
- Limiting system in the limit $\mathrm{Fr} \to 0$ satisfy the “lake equations".

The system (SW_{Fr}) is a hyperbolic system of conservation laws:

$$\frac{\partial U}{\partial t} + \nabla .\ F(U) = S(U,z)$$

$$U = \begin{pmatrix} h \\ hU \end{pmatrix},\ F(U) = \begin{pmatrix} hU^T \\ hU \otimes U + h^2 \mathbf{I_2}/(2\mathrm{Fr}^2) \end{pmatrix},\ S(U,z) = \begin{pmatrix} 0 \\ -\dfrac{h}{\mathrm{Fr}^2}\nabla z \end{pmatrix}$$

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- D3 high gravity wave speed require small time steps $\Rightarrow$ does not allow long time simulations (the simulation of tide-surge), even with high performance computing. 👎

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  The CPR (centered-potential regularization) ImEx scheme **[Parisot, Vila'16]** is a good first order candidate.

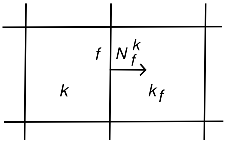
- A fully diagonal segregated method.
- Avoid resolution of large linear systems and limits the number of linear systems to be solved.

Numerical strategy: Cell-centered FV method on uniform Cartesian grid

in cell: $$\psi_k = \frac{1}{|V_k|}\int_{V_k} \psi dx$$

at the face: $2(\psi)_f = \psi_k + \psi_{k_f}$ and $2[\psi]_k^{k_f} = \psi_{k_f} - \psi_k$

parameters: $l_k = \frac{|V_k|}{|\partial V_k|}$ and $\mu_f^k = \frac{|f|}{|\partial V_k|}$



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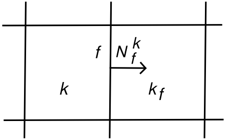

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→ Use an **AUSM** (Advection Upstream Splitting Method) scheme **[Liou, Steffen'93]**

Splitting of the equation into the advection part and the potential forces

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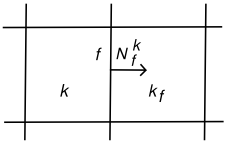

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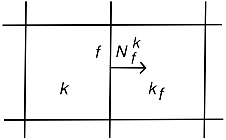

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- Ensure the **dissipation of the discrete kinetic energy**.

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→ Use a **centered discretization** of the potential for any $\mathrm{Fr} > 0$. **[Dellacherie'10]**

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$\rightarrow$ **Regularization** using the potential jump. **[Grenier et al.'13, Parisot and Vila'15]**

$$\mathcal{F}_f^h = (hU)_f - \gamma \Delta t \left(\frac{h}{l}\right)_f \frac{[\phi]_k^{k_f}}{\mathrm{Fr}^2} N_f^k, \qquad (\Delta t : \text{time scale}, \quad \gamma : \text{regu. param.})$$

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Ensure:

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- The **dissipation of the discrete potential energy**.

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Solving a **non-linear** system by using the Newton-Raphson method, transforming the nonlinear system into a series of linear systems.

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Step 2:

→ **velocity: explicit** up-wind scheme with source term.

$$h_k^{n+1} U_k^{n+1} - h_k^n U_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left(U_k^n \left(\mathcal{F}_f^{n+1} . N_f^k\right)_+ - U_{k_f}^n \left(\mathcal{F}_f^{n+1} . N_f^k\right)_- \right) \mu_f^k$$

$$= -\frac{\Delta t}{l_k} h_k^{n+1} \sum_{f \in \mathbb{F}_k} \frac{[\phi^{n+1}]_k^{k_f}}{\mathrm{Fr}^2} N_f^k \mu_f^k$$

Entropy dissipation **[Parisot and Vila '16]**

Let $\gamma \geq 1$ and assume the following CFL-like condition is satisfied

$$\left(|(h^{n+1} U^{n})_f . N_f^k| + (h_k^{n+1})_f \sqrt{\frac{\gamma}{2} \frac{|[\phi^{n+1}]_k^{k_f}|}{\mathrm{Fr}^2}} \right) \Delta t \leq \frac{\min(l_k, l_{k_f}) \min(h_k^{n+1}, h_{k_f}^{n+1})}{2}$$

then the discrete mechanic energy is **decreasing**.

The efficient high order time integration is more challenging in the context of ImEx schemes.

Several Runge-Kutta schemes can be found in the literature:

- **[Dimarco et al.'17, Boscheri et al.'20]** second order schemes based on several multi-stage or step ImEx formulations.
- **[Dimarco et al.'18, Michel-Dansac, Thomann'21]** higher order TVD ImEx schemes.

Large number of linear systems to solve.

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Step 1:

→ **water level: implicit + extrapolation**

$$h_k^{n+1} - h_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \underbrace{\left((h^n U^n)_f . N_f^k - \frac{\Delta t}{2} \left(\frac{h^{n+\frac{1}{2}}}{2l} \right)_f \frac{[\phi^{n+\frac{1}{2}}]_k^{k_f}}{\mathrm{Fr}^2} - \frac{\Delta t}{2} \left(\nabla(\widetilde{h} | \widetilde{U}.N_f^k|^2)^{n+\frac{1}{2}} . N_f^k \right)_f - \frac{\Delta t}{2} \left(\nabla^{\perp} (\widetilde{huv})^{n+\frac{1}{2}} . N_f^k \right)_f \right)}_{\mathcal{F}_f^{n+\frac{1}{2}} . N_f^k} \mu_f^k = 0$$

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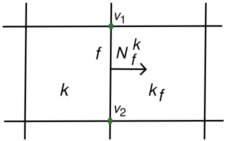
with $\quad (\nabla \phi . N_f^k)_f = \dfrac{\phi_{k_f} - \phi_k}{\overrightarrow{\Delta} . N_f^k} \quad$ and $\quad (\nabla^{\perp} \phi . N_f^k)_f = \dfrac{\phi_{v_1} - \phi_{v_2}}{|f|}$



$$\psi^{n+1/2} := \frac{\psi^n + \psi^{n+1}}{2}, \quad \widetilde{\psi}^{n+1/2} := \frac{\psi^n + \widetilde{\psi}^{n+1}}{2} \quad \text{and} \quad \widetilde{\psi}^{n+1} := 2\psi^n - \psi^{n-1}.$$

Step 2:

→ **velocity: explicit + extrapolation**, up-wind scheme with source term.

$$h_k^{n+1} U_k^{n+1} - h_k^n U_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left(\widetilde{U_f}^{n+\frac{1}{2},+} \left(\mathcal{F}_f^{n+\frac{1}{2}} . N_f^k \right)_+ - \widetilde{U_f}^{n+\frac{1}{2},-} \left(\mathcal{F}_f^{n+\frac{1}{2}} . N_f^k \right)_- \right) \mu_f^k$$

$$= -\frac{\Delta t}{l_k} h_k^{n+\frac{1}{2}} \sum_{f \in \mathbb{F}_k} \frac{[\phi^{n+\frac{1}{2}}]_k^{k_f}}{\mathrm{Fr}^2} N_f^k \mu_f^k$$

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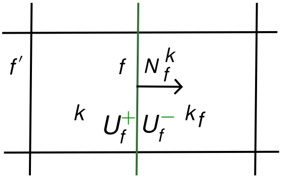
Order 2 in space: MUSCL reconstruction of the velocity (with an unlimited slope).

- Edge f with neighboring cells k and k_f

$$\nabla_{ij}U = \left(\frac{U_{i+1,j}-U_{i,j}}{\Delta x}, \frac{U_{i,j+1}-U_{i,j}}{\Delta y}\right), \qquad \delta_f U = \frac{U_{k_f}-U_k}{\mathrm{dist}(k,k_f)}$$

$$U_f^+ = U_k + \mathrm{dist}(k,f)\times\frac{1}{2}\left(\nabla_k U.N_f^k + \delta_{f'}U\right)$$

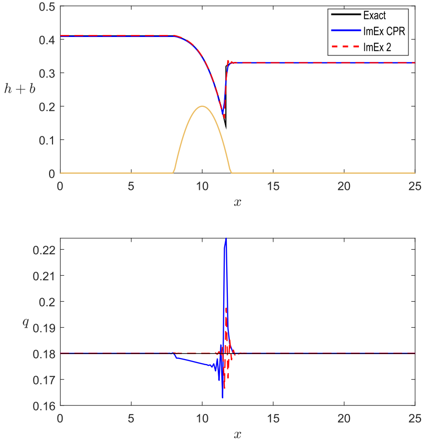
$$U_f^- = U_{k_f} - \mathrm{dist}(f,k_f)\times\frac{1}{2}\left(\nabla_{k_f}U.N_f^k + \delta_f U\right)$$



Transcritical flow

$$\text{Initial data:} \quad h(x,0) + z(x) = 0.33, \quad h(x,0)u(x,0) = 0.$$

$$\text{Boundary conditions:} \quad h(0,t)u(0,t) = 0.18, \quad h(25,t) = 0.33.$$

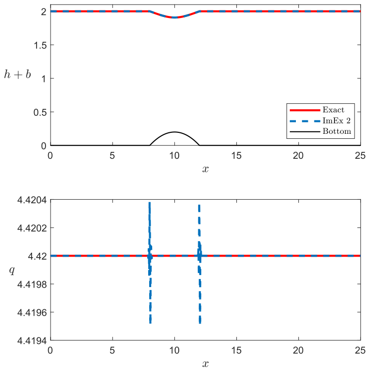


- Good agreement with exact solution. ImEx 2 provides a better solution compared to ImEx CPR. This is clear in the flow discharge figure.

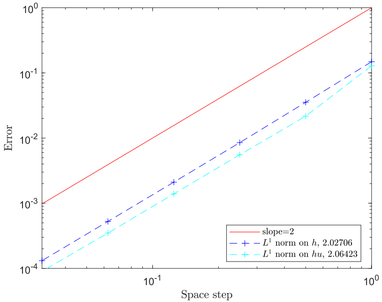
Subcritical flow over a bump

Initial data: $h(x,0) + z(x) = 2, \quad h(x,0)u(x,0) = 0.$

Boundary conditions: $h(0,t)u(0,t) = 4.42, \quad h(25,t) = 2.$

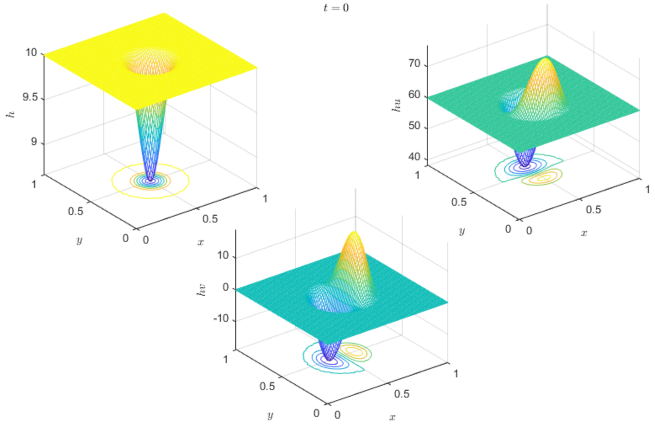


a) Subcritical flow



b) L^1 errors for h and hu

Traveling vortex



Initial condition for the traveling vortex with Fr $=$ 1, computed on the 80 $\times$ 80 grid.

- Exact solution is available. **[Ricchiuto, Bollermann'09]**

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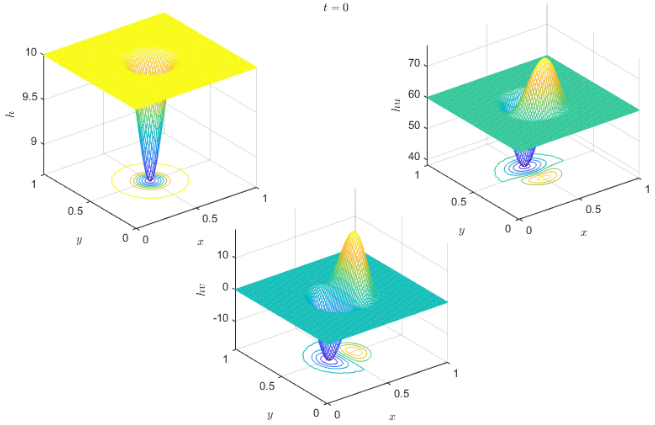

Initial condition for the traveling vortex with $\mathrm{Fr} = 1$, computed on the 80×80 grid.

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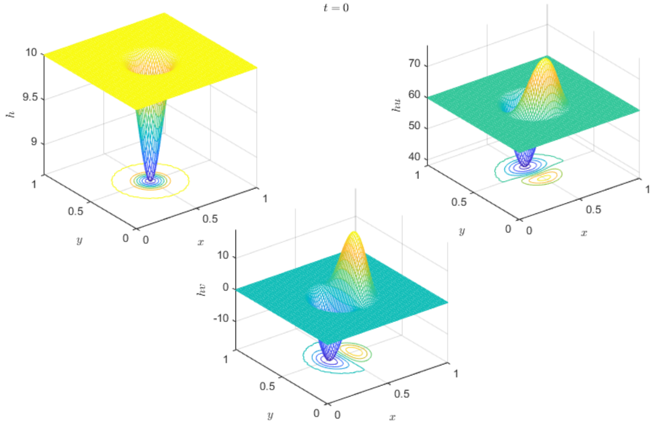

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- $\Omega = [0, 1] \times [0, 1]$. Periodic BC in x, absorbing BC in y.
- A rotating vortex initially located at the domain center $(0.5, 0.5)$ is transported repeatedly from left to right with a period T = 1/6 by an x-directional uniform flow with the advection velocity $U_{ref} = (6, 0)$.

Order of accuracy

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 2, $T = 1/6$

Fr = 1

N	L^1-error in h	EOC	L^1-error in hu	EOC	L^1-error in hv	EOC
40	1.262e-02		2.497e-01		4.027e-01	
80	3.0520e-03	**2.0474**	7.070e-02	**1.8206**	1.259e-01	**1.6771**
160	7.002e-04	**2.1239**	1.827e-02	**1.9515**	3.507e-02	**1.8442**
320	1.677e-04	**2.0614**	4.830e-03	**1.9201**	9.491e-03	**1.8857**

Fr = 0.1

N	L^1-error in h	EOC	L^1-error in hu	EOC	L^1-error in hv	EOC
40	1.106e-04		2.614e-01		4.276e-01	
80	2.988e-05	**1.8889**	7.251e-02	**1.8501**	1.324e-01	**1.6906**
160	7.283e-06	**2.0367**	1.882e-02	**1.9457**	3.658e-02	**1.8565**
320	1.808e-06	**2.0098**	4.964e-03	**1.9227**	9.859e-03	**1.8918**

Order of accuracy:

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 2, $T = 1/6$

Fr = 0.01

N	L^1-error in h	EOC	L^1-error in hu	EOC	L^1-error in hv	EOC
40	1.445e-06		2.617e-01		4.274e-01	
80	3.311e-07	**2.1258**	7.255e-02	**1.8509**	1.324e-01	**1.6901**
160	7.295e-08	**2.1825**	1.884e-02	**1.9450**	3.659e-02	**1.8560**
320	1.756e-08	**2.0547**	4.971e-03	**1.9221**	9.864e-03	**1.8912**

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 0.5, $T = 1/6$

Fr = 0.001

N	L^1-error in h	EOC	L^1-error in hu	EOC	L^1-error in hv	EOC
40	1.282e-08		2.564e-01		4.182e-01	
80	2.850e-09	**2.1697**	7.029e-02	**1.8670**	1.283e-01	**1.7047**
160	7.656e-10	**1.8962**	1.817e-02	**1.9514**	3.527e-02	**1.8629**
320	1.701e-10	**2.1698**	4.784e-03	**1.9255**	9.490e-03	**1.8941**

Grid	Fr = 1		Fr = 0.1		Fr = 0.01		Fr = 0.001	
	ImEx2	Explicit	ImEx2	Explicit	ImEx2	Explicit	ImEx2	Explicit
40×40	8.65	0.62	5.02	2.29	4.83	13.89	5.14	138.44
80×80	38.36	7.54	39.67	31.79	41.59	278.30	41.89	2,052.39
200×200	788.57	41.30	714.20	220.91	715.94	2,133.62	746.72	16,718.47

Table: CPU times in seconds consumed by the ImEx 2 and HLL-MUSCL-RK2 explicit schemes on different grids for different values of Fr. The final time is $T = 1/6$.

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Thank you for your attention!