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Difficulties of Constructing and Formulating Geometric Proofs by Lebanese Middle School Students Learning Math in English

By

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Dedication

To my loving parents
Difficulties of Constructing and Formulating Geometric Proofs by Lebanese Middle School Students Learning Math in English

Hanan Moukhtar Al Masri

Abstract

Developing geometric proofs is an important aspect of the Lebanese mathematics curriculum. Accordingly, exploration of the difficulties that the Lebanese middle school students who learn mathematics using their non-native language (English) is a crucial issue for research. The present study aims at exploring the difficulties, particularly language difficulties that the Lebanese middle school students (grades 6 to 9) who learn mathematics using their non-native language (English) face when constructing and formulating geometric proofs. The study took place at a private school located in Beirut during one academic year. Difficulties were investigated through interviews, tests, and clinical interviews. The interview was with the middle school math coordinator. Two Tests 1 and 2 were specifically prepared based on the Lebanese national curriculum objectives at each of the grade levels 6 to 9 and held at the end of the academic year separated by a duration of one week. Tests 1 and 2 consisted of isomathic problems that varied in terms of proof and language complexity levels. During each of the Tests 1 and 2 at each grade level, randomly selected students were clinically interviewed to provide the researcher with better insightful interpretation of the difficulties faced. Results of the study showed that students’ difficulties when constructing and formulating geometric proofs were related to difficulties in Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT). Yet, these difficulties varied in their level from grade level to another and according to the proof and language complexity levels of the proof tasks.

Keywords: Proof difficulties, Language difficulties, Lebanese middle school students, Lebanese geometry national curriculum.
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CHAPTER ONE

INTRODUCTION

1.1 An Overview

Teaching mathematics using a non-native language is common in many countries. Many research studies have been conducted to investigate the effect of learning mathematics using a second language on students’ performance and achievement. It has been noticed that second language learners’ performance in mathematics is greatly affected by the language of instruction used while learning. According to the National Assessment of Education Progress (NAEP), as cited in Lamberg and Wiest (2007), students who learn mathematics using their mother language achieve better results than students who learn mathematics using a second language. This gap in achievement is due to interactions between students’ ability to read and comprehend the language of instruction, the second language, and to understand the mathematical concepts that are taught using the second language. Cuevas (1984), as cited in Barbu and Beal (2010), asserts that when students learn mathematics using their second language, they have to overcome multi-faceted difficulties. First, they have to understand the language of instruction. Second, they have to understand both mathematical language and mathematical concepts. Thus, learning mathematics using a second language is considered to be cognitively demanding. Moreover, Gorgorió and
Planas (2001) state that students who learn mathematics using a second language face difficulties due to the differences between their spoken language and the language of instruction used in mathematics classrooms. The research work cited agrees, however, that students’ difficulties vary according to their level of mastery of the second language.

In Lebanon, students learn more than one language through their academic life from grades K to 12. At the beginning of their schooling years, Lebanese students, whose native language is the colloquial Arabic, start learning two languages, classical Arabic and one non-native language, most commonly English or French. The number of teaching hours allocated for learning classical Arabic is almost equivalent to that allocated for the foreign language at the Lebanese schools is almost equivalent. That is, Lebanese students, through their school academic life from K to 12, learn classical Arabic and English or French in parallel. Most of the schools in Lebanon adopt using the non-native language (English or French) as a language of instruction and communication in teaching and learning mathematics from grade K to 12. Other Lebanese schools prefer teaching mathematics using a mix of colloquial and classical Arabic during preschool and grades 1, 2, and 3 before shifting to teach mathematics using the second formal school language (English or French) from grade 4 through 12. That is, Lebanese students, from grade 4 till 12, are considered to be learning mathematics using a non-
Few Lebanese schools shift to teach mathematics using the non-native language at grade 7.

In Lebanon, the math textbooks used by Lebanese schools are of different types. Some Lebanese schools prefer using math textbooks that are published in American or European countries depending on the curriculum adopted by the school policy. Other Lebanese schools prefer using math textbooks that are published in Lebanon and abide by the Lebanese curriculum objectives and requirements. Many of these Lebanese math textbooks are usually composed in Arabic or in French languages and then translated into English.

Since the majority of Lebanese schools adopt teaching mathematics using a non-native language (English or French), it is important to investigate the difficulties that Lebanese learners face when learning mathematics. In order to investigate all possible difficulties arising from learning mathematics in a second language, it is important to individually explore these difficulties in each of the mathematical domains and strands and at each grade level. When the difficulties are explored in this manner, results will help in the process of developing strategies or solutions that might decrease these difficulties and their effect on learning mathematics.

Mathematical proof, in particular geometric proof, is a mathematical domain that is viewed as one of the fundamental aspects in mathematics (Balacheff, 2000; Hanna,
2000; Hanna & Barbeau, 2008). In a study conducted by Moore (1994), it is shown that students face difficulties when they are required to construct geometric proofs. One of the main difficulties or obstacles faced by students is the obstacle of language. Knapp (2006) claims that second language learners who are considered as low achievers in mathematics “lack either the language skills or cultural understanding to communicate mathematics” (p. 1). Since language is considered to be an obstacle that students face while communicating mathematical proofs, it is important to explore the language difficulties that mathematics non-native language learners face when constructing and formulating geometric proofs.

According to the Lebanese curriculum (Educational Center for Research and Development [ECRD]) (1997), students are required to construct and formulate formal proofs at the beginning of the middle school, that is, starting grade 7. According to Balacheff “the process of building a valid proof is clearly a complex one: it involves sorting out what is given --the mathematics properties that are already known or can be assumed--from what to be deduced, and then organizing the transformations necessary to infer the second set of properties from the first into a coherent and complete sequence” (Balacheff, 2000, p. 7). In each of these, language plays an important role.

The present study aims at exploring the difficulties, particularly language difficulties that the Lebanese middle school students who learn mathematics using their
non-native language (English), face when constructing and formulating geometric proofs.

1.2 Statement of the Problem

In research, the issue of learning mathematics using a non-native language and its effect on students’ achievement has been a matter of debate for many years. Many studies have been conducted to test the extent to which the language factor affects students’ performance in mathematics. For instance, studies conducted by Essien (2010) and Cirillo, Bruna, and Herbel-Eisenmann (2010) show that second language learners’ proficiency in mathematics is greatly affected and limited by the students’ language proficiency factor in communicating both the second language and the language of mathematics. It is observed that learning mathematics using a non-native language creates some difficulties in the acquisition of mathematical concepts and in communicating mathematical ideas. According to Huang and Normandia (2007), when students lack acquisition of some linguistic features, their ability to construct semantic relations will be affected. Thus, linguistic flexibility has an impact on students’ ability to gain, explain, and express mathematical ideas and understanding.

In addition, proof and proving are important practices in mathematics. Yackel & Hanna (2003) assert that proof is a vital construct in thinking mathematically, for it involves students in logical thinking, reasoning, and understanding. Due to the
importance of proof in mathematics, most mathematics curricula adopt proving and proof as part of their basic goals. According to the National Council of Teachers of Mathematics [NCTM] (2000), students should have the ability “to understand and produce mathematical proofs” (p. 56). As for the Lebanese national curriculum (ECRD, 1997), the first general objective is to “train students to construct arguments,” evaluate them, and “develop critical thinking” (p. 288).

Another important goal of the mathematics curriculum is to enhance students’ abilities to communicate mathematically. According to the ECRD (1997), students should “read, understand and use the mathematical notation and language, present the work orally or in writing, with clarity and rigor, with particular care to writing proof” (p. 302). Moreover, the NCTM (1989) emphasizes in its standards the recommendation that students should be engaged in mathematical arguments through which they need to communicate with each other and write their arguments in order to express their ideas to their peers and teacher. Thus, it is clear that proofs and mathematical language are strongly intertwined.

In teaching practice, it is noticed that students face difficulties in developing and formulating proofs, in particular, geometric proofs. Weber (2001) asserts that students have great problems and face barriers with the task of proof construction. Indeed, when students are formulating and constructing geometric proofs, they are performing multi-
task processes: proof construction and proof communication using their non-native language.

Since the Lebanese curriculum adopts non-native languages (mostly English and French) for teaching mathematics in the middle school, Lebanese students probably face additional difficulties when constructing and formulating geometric proofs. It may be assumed that many of these difficulties are, to a great extent, due to language source. Adegoke and Ibode (2011) state that the language through which ideas and knowledge are transmitted, is considered to be the major vehicle of that transmission. According to Freeman and Crawford (2008), when students learn mathematics using a non-native language, they will face language barriers. Freeman and Crawford (2008) explain that when students find difficulties in understanding what is being mentioned in math classes, they are unable to overcome the language obstacle to understand and master mathematical content and skills. Adegoke and Ibode (2011) agree and add that students’ language competency in a subject affects their achievement.

Therefore, the exploration of the difficulties that learners of mathematics in a non-native language face when writing geometric proof, and in particular language difficulties, is a vital issue for math education researchers.
1.3 Purpose of the Study

In Lebanon, and as previously mentioned, students learn how to develop geometric proofs at the beginning of middle school, using rather a non-native language (English or French) than their native language (colloquial Arabic).

The present study was designed to explore Lebanese middle school students’ difficulties in constructing and formulating formal geometric proofs, in particular, difficulties related to language.

1.4 Research Questions

This study attempted to investigate the following research questions:

- Do Lebanese middle school students face difficulties when constructing geometric proofs? What are these difficulties?
- Is it possible to classify, differentiate or distinguish the difficulties that are due to language and those that are due to proofs’ cognitive complexity?
- Are there any developments or changes in the nature and extent of difficulties through the four grade levels of middle school (grades 6 to 9)?


1.5 Significance of the Study

Senk (1985) states that “only 30% of students in full-year geometry courses that teach proof reach a 75% mastery level in proof writing” (p. 168). According to Weber (2001), the reform of NCTM Standards that took place in 2000 emphasized the necessity of improving proof instruction and led to the production of textbooks that help in developing the abilities of writing proofs. Yet, and from the revised literature, no studies explored the significance and the effectiveness of the improvements made on students’ abilities of writing proofs.

Although some unpublished research has been conducted in Lebanon about proving processes (Nassar, 2010; Tohme, 2005), there is, to the researcher’s knowledge, no published research about the language difficulties that Lebanese students, as being learners of mathematics in a non-native language, face when constructing and formulating proofs. Moreover, the national Lebanese textbooks (ECRD, 2000a; ECRD, 1999a; ECRD, 1998a) don’t provide students with proving tips or guidance that help them develop and build up the notion of proof and proving. It only contains some proofs of the used theorems or properties. Furthermore, the teachers’ guides (ECRD, 2000a; ECRD, 1999a; ECRD, 1998a; ECRD, 2000b; ECRD, 1999b; ECRD, 1998b) of the national Lebanese textbooks don’t provide teachers with methods or strategies that they might use when introducing students to developing and formulating proofs. Thus, it is important to explore the difficulties that Lebanese students face when formulating and
constructing geometric proofs in order to help in the reform of the standards of the
Lebanese mathematics curriculum, textbooks, and teachers’ guides.

Wu (1996), as cited in Varghese (2011a), states that “those who want to know
what mathematics is all about should learn to write proof” (p. 409). Varghese (2011a)
asserts also that the task of proving is a complex task and “involves a range of student
competencies such as identifying assumptions, isolating given properties and structures
and organizing logical arguments” (p. 410). Hence, and as mentioned before,
formulating and writing geometric proofs requires high-level reasoning, logical and
mathematical abilities and proficiency in communicating both the mathematical
language and the non-native language used to learn and to communicate mathematics.
Therefore, it is important to help students overcome the obstacle of language as a barrier
towards achievement in mathematics. The explored difficulties will help math
curriculum developers set strategies and solutions to help students overcome these
difficulties, thus, helping in the development of a generation of Lebanese middle school
students who are fluent in constructing and writing formal geometric proofs using the
non-native language. Moreover, exploring the difficulties faced by students when
constructing and formulating geometric proofs will enable teachers to investigate
teaching strategies that will help in decreasing the levels of difficulties faced by
students.
1.6 Operational Definition of Keywords

Following are the operational definitions of the research keywords.

- **Native language**: The native language used by the majority of the Lebanese people is the colloquial Arabic.

- **Non-native language**: In this study, the term non-native language(s) is considered to be the language of math instruction which is in the current study the English language.

- **Geometric proof**: Geometric proofs are proofs related to geometric figures, concepts, definitions, properties, and theorems.

- **Formal proof**: Formal proofs are proofs that require writing a chain of valid statements and related reasons to prove a geometric fact.

- **Isomathic problems**: Two problems are considered to be isomathic when they present two tasks that are identical in terms of the mathematical premises and conclusions, but are different in terms of text difficulty or in terms of proof task difficulty. Such problems involve the same field of geometric concepts and relationships.

The current chapter presented an introduction about the purpose, significance, research questions, and issues that will be discussed during the research. The next
chapter provides an overview about previously conducted research related to issues of mathematical language and proof.
CHAPTER TWO

LITERATURE REVIEW

The following section includes a brief review of literature about mathematical proof and language.

2.1 Mathematical Proof

2.1.1 Importance of proof in mathematics

All researchers agree to the importance of proof in mathematics but vary in setting definitions of a proof. As asserted by Senk (1985), Harel and Sowder (1998), McCrone and Martin (2004), Arsac (2007), and Stylianides (2007), the learning of proof has been a central goal in the learning of mathematics, especially in the domain of geometry, for many years and in different cultures. The importance of proof is raised due to its vital role in developing mathematical and logical thinking as claimed by Arsac (2007) and Stylianides (2007). Researchers such as Stylianides (2007), Arsac (2007), and McCrone and Martin (2004), believe that proofs are important because they help students gain the ability to seek validity of statements through a systematic method and procedure. It helps students develop a deep understanding of mathematical concepts.
Furthermore, McCrone and Martin (2004) consider proof as a sense-making activity of mathematics.

2.1.2 Definition of mathematical proof

The definition of proof has raised numerous arguments among math educators. Balacheff (2000) wonders if math educators share a common definition of mathematical proof. The definitions derived from the literature about proofs vary in nature. Some defined proofs in terms of the process needed to develop the proof. According to Balacheff (2000), proof is the process through which validity and truth of statements are acquired. When describing the nature and the process of building proofs, Balacheff (2000) states that the process of building a valid proof is clearly a complex one: it involves sorting out what is given --the mathematics properties that are already known or can be assumed--from what to be deduced, and then organizing the transformations necessary to infer the second set of properties from the first into a coherent and complete sequence (p. 7).

Harel and Sowder (2007), Stylianides (2007), and Stylianides and Ball (2008) also define proofs in terms of the process they require and state that proofs are mathematical arguments that have three criteria. First, they use statements that are acceptable and considered to be true by a community without justification. Second, they make use of modes of reasoning. Third, the expressions used to communicate proofs are well-known to the community.
Other definitions of proofs are derived from the components of a proof. For example, Weber and Alcock (2004) and Mariotti (2006) assert that proving a statement requires production of arguments that are logically valid and that relate to the statement needed to be proven. Herbst (2002) states that proofs are a set of steps made up of two columns: one for statement and the other for reasons that show why each statement is true.

Further definitions are set and extracted from the nature of a proof. Heinze, Cheng, Ufer, Lin and Reiss (2008) consider proof as bridging between given conditions and a desired conclusion through the use of hypotheses and mathematical arguments. Besides, Arzarello (2007) considers proof to be a set of ordered statements that are linked by transitions such that one implies the other. These definitions are aligned with those set by the NCTM (1989) which states that proof is “a careful sequence of steps with each step following logically from an assumed or previously proved statement and from previous steps” (p. 144). On the contrary, Reiss and Renkl (2002), as cited in Oner (2008), don’t believe that proofs consist of a sequence of systematic steps. Rather, they argue that proofs require moving continuously “between inductive, explorative and deductive processes” (p. 109).

Other math educators determine the definition of proof as related to its role in mathematics. According to Otte (1994), “proof belongs to the meta-
mathematics and seems to be an exercise in logic and has nothing to say on anything that is not a statement” (p. 299). Additionally, Oner (2008) claims that mathematical statements are said to be true if they are validated through the usage of deductive arguments. Harel and Rabin (2010), Hanna and De Villiers (2008), and Recio and Godino (2002) add that a proof is defined as an argument that one produces to remove any doubts about the validity of an assertion. Besides, and according to Laborde (2000) who researched proof in technological environments (DGS), “proof is the mean for justifying that the new command will provide the expected outcome” (p. 153). Cañadas, Castro, and Gómez (2002) state that proofs have various meanings such as: “explanation, argument, justification, confirmation, verification or validation” (p. 177).

2.1.3 Functions of mathematical proof

Hanna (2000) considers proof to be an essential part of mathematics. When proofs are used to convey understanding, they can make their major contribution in the classroom. McCrone and Martin (2004), Arsac (2007), and De Villiers (2012) claim that traditionally, the main purpose of proof was to verify the truth of a conjecture. According to Laborde (2000), proof is the tool for justifying new commands. Later, Laborde (2000) adds that “proof fulfills thus a twofold role: establishing the validity of a construction for each individual and convincing the other students to accept the construction process” (p. 153).
As the research on proof functions increases, exploration of new functions of proof are recorded. Laborde (2000) claims that when proof is built on theoretical argument, it becomes a tool of understanding. De Villiers (1990), Hanna (2000), Harel and Sowder (2007), and Mejía-Ramos and Inglis (2009) list several functions of proof and proving: (1) verification, (2) explanation, (3) systematization, (4) discovery, (5) communication, (6) construction, (7) exploration, and (8) incorporation. Similarly, Bell (1976), as cited in Clement (2003), recognizes three functions of proof in mathematics: “(1) verification—concerned with establishing truth of a proposition; (2) illumination—concerned with conveying insight into why a proposition is true; and (3) systematization—concerned with organizing propositions into a deductive system” (pp. 167-168).

Finally, De Villiers (2012) adds that intellectual challenge is one of the most important functions of proof.

### 2.1.4 Stages of proof development

Piaget (1928), as cited in Stylianides and Stylianides (2008), recognizes the children’s ability to develop deductive reasoning and produce logical inferences. Children from the ages of 11-12 and above are able to produce what is called “hypothetico-deductive reasoning.” That is, children’s ability to produce deductive reasoning built on assumptions starts around the age of 11-12, which corresponds to the beginning of middle school. Mathematicians developed
models representing the levels of development of the ability to formulate and write geometric proofs. Some models were developed by considering students’ ability of comprehending geometric proofs, while other models considered students’ ability of geometric thinking.

Some of the considered proof development processes consist of two stages. For instance, Jaffe and Quinn (1993) state that students shift from writing heuristic arguments to rigorous arguments.Movshovitz-Hadar (1996) recognizes that students’ proofs develop from informal proofs to formal ones. Martínez (2000), as cited in Cañdas, Castro, and Gómez (2002), asserts that students start the proving process by writing informal arguments and then are able to develop formal arguments. Gutiérrez (2001), as cited in Cañdas, Castro, and Gómez (2002), claims that students’ developmental process of writing proofs starts by writing empirical proofs and then develop to writing deductive proofs.

Balacheff (1988) states that when students are required to produce proofs, the language used is considered to be a tool for producing logical deductions and only a communication tool. Balacheff presents the four levels of proof cognitive development and which were named later as “Balacheff’s taxonomy of proof”. The levels of “Balacheff’s taxonomy” represent the student’s development of the notion of the proof process from inductive arguments towards more deductive and general arguments as their language abilities develop.
The first level is called “naïve empiricism” whereby students achieve the validity of a statement through the trial of several particular cases. The second level is called the “crucial experiment” whereby students arrive at generalization by examining cases that are not considered very particular. The third level is the “generic example” whereby students produce proofs built on properties. The fourth level, “thought experiment” allows students to produce deductions that are logical and built on properties and relationships.

One of the famous theories that describe the levels of development of reasoning in geometry is the Van Hieles’ theory. Aydin and Halat (2009) state that the levels presented by the Van Hieles are hierarchal and continuous.

Level one is the “visualization” level where students use the appearance of geometric figures to recognize and identify them. Yet, students can’t identify the properties these figures at this level. Level two is the “analysis” level where students are able to analyze figures in terms of properties and rules. But, at this level, students can’t find relations between properties because these properties are recognized empirically.
- Level three is the “ordering” level in which students are capable of using properties and rules to produce informal arguments. Moreover, relations between the properties of geometrical figures are realized and students are able to produce logical implications.

- Level four is the “deduction” level where students are able to produce deductive proofs of theorems and recognize the role of definitions and axioms in proofs. In addition, students are able to give reasons for the proofs produced. At this level, the roles of definitions and theorems in a deductive proof are understood.

- Level five is the “rigor” level where “students are able to analyze various deductive systems like establishing theorems in different axiomatic systems, and can compare these systems” (Aydin and Halat, 2009, p. 152).

According to the Van Hieles’ levels, students start developing informal proofs at the third level and shift to construct and write formal proofs at levels four and five. Aydin and Halat (2009) claim that students’ success at levels four and five are related to their success particularly at level three.

Lin and Yang (2007) and Mejia-Ramos, Fuller, Weber, Rhoads and Samkoff (2011) present another model previously developed by Yang and Lin that consists of levels of “reading comprehension of geometry proofs (RCGP)”.

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In their study, Lin and Yang (2007) tested the ability of grade 9 and 10 to comprehend written geometric proofs. This model considers student’s ability to comprehend and analyze written geometric proofs.

- The first level is called “the surface level” where “students acquire basic knowledge regarding the meaning of statements and symbols in the proof” (p. 6). At this level, students’ comprehension is described as epistemic understanding of the elements of a proof which are the premises, conclusions, and properties.

- The second level, termed as “recognizing the elements” allows students to “identify the logical status of the statements that are used either explicitly or implicitly in the proof” (p. 6).

- The third level is labeled “chaining the elements”. At this level, students are able to comprehend ways that connect different statements in a proof through recognizing logical relations that relate the statements to each other. Moreover, students are able to view geometric figures as a reference.

- Finally, the fourth level is called the “encapsulation”. At this point, “students interiorize the proof as a whole by reflecting on how one may apply the proof to other contexts” (p. 6).
The results of the study conducted by Lin and Yang (2007) show that 26% of grade 9 students and 5.8% of grade 10 students were at the surface level. 26% of grade 9 students and 18.5% of grade 10 students were classified as beyond the surface level. 9.8% of grade 9 students and 10.8% of grade 10 students were able to skip the recognizing chaining level. 11.7% of grade 9 students and 18.8% of grade 10 students were beyond the recognizing elements level. Finally, 18.8% of grade 9 students and 36.5% of grade 10 students were beyond the chaining elements level.

One final process we investigated is presented by Simon and Blume (1996), as cited in Varghese (2011b). They state that Van Dormolen recognizes three categories of proof development. In the first category, students focus on particular cases. In the second, students utilize an example as a general embodiment of a concept. In the third category, students utilize arguments that are general and deductive.

2.1.5 Introducing proofs in mathematics education

The best time to start engaging students in the process of proving is a matter of debate. Stylianides and Ball (2008) and Hanna and De Villiers (2008) suggest that engaging students in the process of proving at the elementary level helps students explore mathematical concepts in a meaningful way. Hanna and
De Villiers (2008) explain that students at early ages show a great ability of reasoning and justification. However, Lin and Yang (2007) claim that when students have a good geometrical knowledge, this doesn’t necessarily mean that they are able to comprehend proofs appropriately. Senk (1989) adds that when proofs are informal, students are interested and engaged in the process of proving. But once proofs become formal, students start considering them complex and cognitively demanding.

2.1.6 Difficulties in constructing and formulating proofs

Students’ difficulty in constructing and formulating proofs has become a big issue in math education generating much research in the literature. Weber (2001) categorized students’ difficulties in constructing proofs into two categories. The first category arises from the students’ conception of the nature of mathematical proof. For example, students accept verification of general theories by referring to particular instances. According to Weber (2001), valid proofs can’t be constructed by students with such deficiencies because these students lack the concept of valid proofs. The second difficulty category by Weber (2001) arises from the students’ misunderstanding of either the concept or the theorem. Senk (1985), Morgan (2005), Knapp (2005), and Harel and Sowder (2007) claim that when students don’t understand definitions, theorems, and properties derived appropriately, they will not be able to identify which one the
proof involves. Thus, students will misapply these concepts and theorems. Other researchers, for instance, Recio and Godino, 2002; McCrone and Martin, 2004; Knapp, 2005; Mariotti, 2006; Herbst and Brach, 2006; Hoyles and Healy, 2007; Harel and Sowder, 2007; Arzac, 2007; Yang and Lin, 2008; Hanna and De Villiers, 2008; and Andrew, 2009, state that mathematical proof is a challenge because it requires certifying that something is true and explaining why it is true. Moreover, it requires giving reasons from definitions, theorems, and properties.

Balacheff (2000) briefly presents two sources of difficulties related to proof construction. The first source is related to the content of the statements that are required to be proven. The second source is related to the language that will be used to formulate and produce the proof. Moreover, Balacheff (2000) illustrates that writing mathematics has its specific characters such as symbols and mathematical words that has its own meanings. In addition, Heinze, Cheng, Ufer, Lin and Reiss (2008) and McCrone and Martin (2004), and Andrew (2009), state that one of the important predictors of a proof’s difficulty is the number of arguments that a student has to combine in order to attain validity of the required statement. Hence, and according to Heinze, Cheng, Ufer, Lin and Reiss (2008), McCrone and Martin (2004), and Andrew (2009), single-step proof is less complex than multi-step proof. Moreover, according to Herbst and Brach (2006) proofs that require connection of concepts are more complex than those requiring
a single concept. Cramer, Fisseni, Koepke, Kühlwein, Schröder, and Veldman (2010) assert that the steps of a proof are related to each other on the one hand and they are related to other proofs on the other hand. That is why they require a high level of connection. Gueudet (2008) conducted a study that aimed at carrying out a survey of the results of research on difficulties in proof formulation. The results of the study show that the sources of difficulties in proof production are due to language and proof issues. According to Gueudet (2008), Knapp (2005), Selden and Selden (2003), Yang and Lin (2008), and Robotti (2012), students have difficulty in using signs such as symbols and words, formulating sentences that are syntactically correct and achieving coherence in the overall text articulation.

In a more detailed manner, Moore (1994) investigated difficulties faced by students while constructing formal proofs. The study was conducted on a sample of 16 students at the undergraduate and graduate levels. Students were asked to develop short deductive proofs that require the use of axioms and definitions. Moore (1994) explored seven sources of difficulties that students face:

D1. The students did not know the definitions, that is, they were unable to state the definitions.
D2. The students had little intuitive understanding of the concepts.
D3. The students’ concept images were inadequate for doing the proofs.
D4. The students were unable, or unwilling, to generate and use their own examples.
D5. The students did not know how to use definitions to obtain the overall structure of proofs.
D6. The students were unable to understand and use mathematical language and notation.
D7. The students did not know how to begin proofs (pp. 251-252).

Moore (1994) adds that the performance of students while writing proofs is sometimes influenced and hindered by students’ perception of mathematics and proof.

Other research studies highlighted other sources of difficulties. Stylianides and Ball (2008) and Powers, Craviotto, and Grassel (2010) state that one of the important sources of difficulties that students have when formulating geometric proofs is due to the teachers’ knowledge about proofs and understanding of what constitutes a proof. Harel and Sowder (1998), Lin (2005), Mariotti (2006), Herbst and Brach (2006), Harel and Sowder (2007), and Andrew (2009) state that students consider what is visually true without justification. And thus, they build false conclusions because of visual appearance of geometric figures or particular cases. Moreover, they have the tendency to accept empirical assumptions without justification. Healy and Hoyles (1998, 2000), McCrone and Martin (2004), and Harel and Rabin (2010) find that students don’t know how to start a proof and if they did, they are unable to complete it. Yang and Lin (2008), Andrew (2009), and Powers, Craviotto, and
Grassel (2010) claim that students don’t realize the existence of a relation between statements and conclusions of a proof or statements and conclusions that are not related to the proof. Thus, they have difficulty in developing a coherent sequence of justified statements and inferences.

Sometimes, when students don’t understand the given information or what is required to prove (Lin, 2005; McCrone & Martin, 2004) they will find difficulty to figure out an approach to an appropriate proof. For instance, when what is required to prove is stated clearly such as (show that triangle ABC is isosceles) the proof is easier than those stating (what is the nature of triangle ABC?).

Other studies show that geometric figures are sometimes a source of difficulty or obstacle towards constructing proofs (Senk, 1985; McCrone and Martin, 2004; Knapp, 2005; Herbst & Brach, 2006; Arzac, 2007; Yang & Lin, 2008; and Robotti, 2012). They claim that when geometric figures are labeled with given information, they are easier to understand than those that are not labeled. Moreover, complex and embedded figures are hard to visualize thus, the students are unable to explore the given and relations between elements of a proof.
2.1.7 Types and methods of proving

It is important to distinguish between various forms and kinds of writing proofs. Rav (1999), as cited in Hanna and Barbeau (2008), presents two kinds of proofs. The first kind is a formal proof called “derivation”. This kind of proof “is the syntactical application of rules of logical inference. It consists of a finite string of formulae, to which no meaning need to be assigned; the formulae are either axioms or derived from axioms” (p. 346). The second kind is an informal proof and is called “conceptual proof”. This kind of proof “consists of a rigorous argument acceptable to mathematicians, but it does make appeal to the meaning of the concepts and formulae used” (p. 346). Herbst (2002), Balacheff (2000), and Herbst and Brach (2006) present the two-column proof as a form of writing proofs. The arguments written in the two-column format require the writing of a statement in one column and giving the reason in the second column. The two-column proof is considered a formal way of writing proofs. Gueudet (2008) asserts that the modes of reasoning and proving are strongly intertwined. Herbst (2002) states that giving reasons for every statement in a mathematical proof justifies the statement and gives it relevance.

Furthermore, Weber and Alcock (2004) claim that students who are capable of constructing proofs use arguments that are logically valid to prove statements. Weber and Alcock (2004) define two kinds of proof production. The
first kind is the “syntactic proof production” where the prover doesn’t refer to diagrams or non-formal representations. Rather, the prover has to “unwrap the definitions” and “push symbols”. The second kind of proof production is the “semantic proof production” where “the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws” (p. 210).

2.1.8 Proof schemes

Heinze, Cheng, Ufer, Lin and Reiss (2008) conducted a study that aimed at exploring the competency of developing proofs for students of grades 7 and 8. They classify students’ proofs into “acceptable proof, incomplete proof, improper proof, and intuitive proof” (p. 445). In an acceptable proof, the deductive process used is valid and leads to the required conclusion using symbolic expressions. In an incomplete proof, students have a gap or a logical error. The improper proof type “includes non-deductive approaches, using incorrect geometric properties, and using properties inappropriately” (p. 446). Finally, in an intuitive proof, students depend on visual judgments and intuitions to give responses. Heinze, Cheng, Ufer, Lin and Reiss (2008) summarize that

Constructing an acceptable geometry proof can be seen as a bridging process from given conditions to a wanted conclusion with inferring rules controlled by a coordination process. This includes: 1. To understand the given information and the status of this information, 2. To recognize the
crucial elements (premise, argument, conclusion), which associate to the necessary properties for deduction, 3. Especially in multi-steps proof, to construct intermediary conditions for the next of deduction by hypothetical bridging, and 4. To coordinate the whole process and organize the discourse into an acceptable sequence (p. 445).

One of the most well-known studies that aimed at exploring students’ written mathematical proof schemes is a study conducted by Harel and Sowder (1998). They tested students at different levels, high school and college. Harel and Sowder (1998), as a result of their study, explore and describe a system of proof schemes that consist of 16 subcategories. These proof schemes describe the way that individuals consider to prove or justify mathematical statements.

Mainly, the observed proof schemes are considered to be based on empirical evidences, personal beliefs, intuitions, an authority of a teacher for example, some social conventions, and deductive and logical arguments. Harel and Sowder (1998) state that “a proof scheme consists of what constitutes ascertaining and persuading for that person” (p. 244). Harel and Sowder (1998) present three categories of proof schemes: “external conviction proof schemes”, “empirical proof schemes”, and “analytical proof schemes”. Each of these categories is divided into several subcategories. In “external conviction proof schemes” students tend to remove doubts by “(a) the ritual of the argument presentation—the ritual proof scheme, (b) the word of an authority—the authoritarian proof scheme, or (c) the symbolic form of the argument—the symbolic proof scheme” (pp. 245-246). In “empirical proof schemes” students
validate their conjectures by using sensory experiences or physical facts. Thus, Harel and Sowder (1998) consider two subcategories of “empirical proof schemes”, the “inductive empirical proof scheme” and the “perceptual empirical proof scheme” where students depend in their proof on either examples or perceptions. Finally, in “analytical proof schemes” conjectures are validated by considering set of statements that are logically related and deduced from other set of statements that are considered to be true. These deduced statements involve the use of logical rules of deduction. Students’ proof schemes under this category can be divided into two subcategories, “transformational proof schemes” and “axiomatic proof schemes”. “Axiomatic proof schemes” differ from “transformational proof schemes” by their dependence on a system of axioms and definitions that validate the written statements and conclusions.

2.2 Mathematical Language

The issue of mathematical language and the language of communicating mathematics has been a matter of study and debate for a long time. According to Lampert and Cobb (2003), students are required to do mathematics; that is to be involved in mathematical work through communicating ideas and thoughts and presenting mathematical arguments. All these activities make the use of mathematical language necessary in mathematics classrooms and thus it should be taught and learned in school. When students are learning and communicating mathematics, it is essential to
consider various variables that might affect their fluency of communication. In the literature, it is found that some of the variables that control students’ ability to understand and communicate mathematically are the language of instruction of mathematics and the mathematical language itself.

2.2.1 Learning mathematics in a non-native language

According to Austin and Howson (1979), mathematical education and language are greatly intertwined. The language of learning mathematics affects understanding of mathematics. It has been shown that the countries that adopt teaching and learning mathematics in a language other than their native language face obvious problems (Austin & Howson, 1979; Bernardo & Calleja, 2005; Zakaria & Abd Aziz, 2011; Tan, Lim, Chew, & Kor, 2011). Furthermore, Thompson and Crampton (2008) assert that second language learners have limited opportunity to become fluent in the second language. Thus, if mathematics is learned using a second language, students’ chances of being fluent in communicating mathematically are limited. In addition, and in the context of mathematics, Tan (2011) and Tan, Lim, Chew, and Kor, (2011) claim that students who learn mathematics in a non-native language will learn concepts and words that are unfamiliar to their daily lives, thus facing difficulties in mathematizing real-life situations. One of the reasons behind the difficulties that students face when learning mathematics using a non-native language, as
presented by Zakaria and Abd Aziz (2011) is due to the double effort that students have to exert while learning mathematics. First, students have to understand the language of instruction. Second, students have to grasp new mathematical concepts presented in a non-native language.

2.2.2 Language of mathematics and writing mathematics

Many researchers in the field of mathematical language (Austin & Howson, 1979; Morgan, 1996; Zack, 1999; & Roberts, 2009) point out that mathematics is a language by itself that should be taught using language teaching methodologies. Mathematics, as a language, has its own verbs, grammatical structure and vocabulary. Austin and Howson (1979) claim that students who misunderstand some mathematical terminology will not be able to use them correctly. For example, the geometric term “similar” has its own geometric definition and requires strong understanding of its definition in order to be used correctly but this term has a different/less formal meaning in spoken language. Hence, the informal use of mathematical terms leads to inappropriateness of mathematical texts. According to Austin and Howson (1979), students’ difficulties in using mathematical language arise when they are required to read and write mathematics. For instance, there is a major difference in the use of quantifiers in everyday language versus mathematical language. So, students should be exposed to early introduction of mathematical terms and practice using
them in the right syntax. However, according to Morgan (2005), vocabulary is the main specific aspect of language and mathematical language is defined by its vocabulary. Morgan (2005) explains that a mathematical text is a combination of discrete terms. These terms, in contrast to other types of language, are described as technical; that is, they require conceptual understanding of their meanings.

Huang and Normandia (2007) state that the role of language in the teaching and learning of mathematics is becoming more prominent. Upon exploring the relationships between linguistic skills and mathematical performance, it is recommended that more emphasis be placed on identifying grammatical structure, technical terms and phrases, and discourse analysis in both written and spoken genres of mathematics. Besides, Gueudet (2008) compares learning mathematical language to the learning of foreign languages. When students learn new signs and terms in mathematics, they develop functional uses of these terms and signs built on analogy. For instance, students will first “try to mimic what they read and hear, and consequently, formulate absurd sentences” (Gueudet, 2008, p. 244). Pimm (1987), as cited in Truxaw, Staples and Ewart (2009), illustrates that students must receive help when being exposed to phrases such as “given that the sides of a triangle are…” or “for all x…” Such phrases have a different structure than phrases used in daily language.
Thompson and Crampton (2008) state that writing mathematical statements requires deep understanding of the formality, language, and discipline of mathematics. Hence, writing mathematically demands cognitive understanding of concepts and generating written communication statements. Moreover, Huang and Normandia (2007) have a point of view that is aligned with Thompson and Crampton (2008). They state that meta-cognitive processes and writing are closely related. Thompson and Crampton (2008) add that mathematical writing requires association between procedural knowledge and conceptual understanding. Conceptual understanding is revealed through realizing semantic relations linguistically, the process of relational transitivity, and the lexical items. Gueudet (2008) adds that syntactic knowledge and related languages are “useful to define abstract concepts, to produce formal arguments, and to check semantic knowledge. It is the language of advanced mathematics, required to enter the mathematical community, and to communicate inside this community” (p. 244).

2.2.3 Stages of mathematical language acquisition

Thompson and Crampton (2008) present the stages of language acquisition that second language learners pass through while learning mathematical concepts. Thompson and Crampton (2008) address five stages for learning mathematical language and concepts. Stage one is called “receiving
“mathematics” where teachers communicate verbally with students in order to construct knowledge related to the mathematical topics introduced. Students understand the mathematical language used informally and in particular contexts. In stage one, students’ role is mainly listening while their role of “speaking taking” is considered secondary. Stage two called “replicating mathematics” allow students to create simple sentences and respond to simple questions albeit with many grammatical and pronunciation mistakes. Moreover, in stage two, students start replicating mathematical examples used by the teacher. Thus, the language used by the students gradually becomes more sophisticated. Stage three is termed “applying mathematics”. In this stage, students are engaged in mathematical investigations both independently and cooperatively. Besides, students are able to read mathematical textbooks that include examples which are concrete and semi-concrete. Furthermore, students are capable of communicating and sharing their mathematical thought verbally with others. Thompson and Crampton (2008) state that in stage four, “discussing mathematics”, “mathematics and its language become more formal. Terms and concepts become more understood in different contexts” (p. 25). Moreover, speaking becomes more primary than listening. Both speaking and listening support students’ ability to discuss mathematics. Finally, stage five termed “communicating mathematics” allows students to use mathematical language
appropriately. Besides, mathematical concepts and terms become contextually understood.

Roberts (2009) claims that students’ forms of conversation in mathematics are divided into two stages. First, teachers have students communicate mathematically using informal language to share and present ideas. Second, teachers help students to write and express their thoughts using a formal language of mathematics.

Truxaw, Staples and Ewart (2009) claim that teachers should be aware of the stages of development of students’ academic language. Teachers’ awareness of students’ academic language helps them develop mathematical instructions according to the students’ level of academic language development.

2.2.4 Difficulties of mathematical texts

Numerous studies in literature have shown that one of the hurdles that hinder the understanding of mathematics is language; in particular for students who learn mathematics in their non-native language. Yushau (2009) states that the texts used in mathematics, regardless of the language used, are linguistically dense to great extents. This denseness in text is due to its richness in technical vocabulary and to the specific usage of words. Thus, “processing sentences in such a linguistically dense context, coupled to the logical nature of many
mathematics problems, requires the reader to rely on sentences to convey clear and unambiguous meaning” (Yushau, 2009, p. 916).

There are two factors that affect the mathematical text’s difficulty: semantic and syntactic features of the text. Slavit and Ernst-Slavit (2007) and Morgan (1996) state that vocabulary and word frequency in a sentence are semantic features of a text, while the length of the sentence and its grammatical structure are syntactic features of the text. Thus, any difficulty in mathematical texts arises from either semantic or syntactic features of the text. Szendrei-Radnai and Török (2007) add that mathematical texts are difficult due to the following specific features: multiplicity of meanings that mathematical words have, use of negative statements, consideration of a theorem and its converse, and the level of grammatical complexity of the sentences.

Campbell, Adams, and Davis (2007), Martiniello (2009), Slavit and Ernst-Slavit (2007), Morgan (1996), Zack (1999), and Cirillo, Bruna, and Herbel-Eisenmann (2010) present several lexile difficulties that might blur students’ understanding of mathematical texts and block their ability to write and communicate mathematically. These linguistic difficulties can be due to: the use of pronouns in a sentence which might be incorrectly referred to another noun, the use of compound sentences that is considered cognitively demanding to students and the fact that as the frequency of verbs and words used in a sentence
increases, the more difficult the text becomes. Moreover, Slavit and Ernst-Slavit (2007) contend that using prepositions, comparative structures, logical connectors, and clauses can create difficulties in understanding and writing mathematical texts. Huang and Normandia (2009) explain why writing mathematics is hard and cognitively demanding for students. They illustrate that, when writing a mathematical text, students have to consider the usage of vocabulary that are not commonly used in daily communicated language and the formulation of expressions that contain numbers, symbols, and terms that are logically connected and coherent. Morgan (2005) and Selden and Selden (2003) consider mathematical definitions as a source of difficulty in mathematical texts. Definitions are complex due to the high frequency of words, symbols, and vocabulary contained in them.

Selden and Selden (2003) and Cramer, Fisseni, Koepke, Kühlwein, Schröder, and Veldman (2010) consider mathematical symbols to be an obstacle in writing mathematical texts due to their abstraction. Moreover, Zack (1999), Hanna (2007), and Selden and Selden (2003) assert that the use of “if…then” statements or what is called logical arguments, always used in mathematical texts, is not commonly used in all languages. It thus, causes difficulty in formulating mathematical texts. Hence, and according to Beal, Adams, and
Cohen (2010), the more simplified the language of mathematical texts, the more the students’ ability to comprehend and write mathematically increases.

### 2.3 Mathematical Language and Mathematical Proof

Formulating geometric proofs is considered a highly complex process (Szendrei-Radnai and Török, 2007; Yang and Lin, 2008; Hanna and De Villiers 2008; Stylianides and Ball, 2008; Robotti, 2012). They explain that this complexity is due to the incorporation of student’s knowledge, natural language, symbolic language of mathematics, and geometric figures. Harel and Sowder (1998) and Szendrei-Radnai and Török (2007) claim that even if students have right proofs, it is hard to understand the written proof due to problems in wording and in ability to express themselves. They add that some students use words that don’t match with the proof’s context because of their multiple meanings. Hence, it is vital to recognize the nested relation between mathematical language and proofs in terms of content, structure, and coherence in order to enhance students’ abilities to formulate, write well-developed geometric proofs, and communicate mathematically.

The current chapter presented a summary of research results related to mathematical language and proof. The next chapter presents a detailed explanation of the research design and method of the present study.
CHAPTER THREE

RESEARCH DESIGN AND METHOD

3.1 Research Context

The study took place at a private school located in Beirut. The school provides education from K to grade 12. Most of the students are from Beirut and belong to average socio-economic level families. All students, of both middle and secondary grades, are girls. The implemented curriculum and textbooks are based on the National Lebanese Curriculum.

The study used qualitative methods to collect and analyze data. The following techniques were used to collect data: interview with the math coordinator, test analysis, and clinical interviews with selected students. The middle school math coordinator was interviewed about the difficulties faced by students learning geometric proofs. Geometric proofs written by middle school students were analyzed. These proofs were collected from students’ responses on constructed tests which included problems developed specifically for the study. Furthermore, four selected students (selection details to be discussed under “Instruments” in the upcoming section), one from each class involved in the study, took the tests in a clinical interview setting for a more
focused exploration of difficulties in constructing and formulating geometric proofs. They were solving the same problems as their classmates.

### 3.2 Participants

The participants are students of the middle school (grades 6, 7, 8, and 9) whose age ranges between 11 and 15. The majority of the students have studied mathematics in their early elementary level (grades 1, 2, & 3) using classical Arabic which is close to, but more formal than their native spoken language (colloquial Arabic). They then studied mathematics in their upper elementary school (grades 4, 5 & 6) in English. All the participants are girls because the school provides education in the middle and secondary school for girls only. In grade 7, there are two sections, A and B, consisting of 13 and 14 students respectively. Their age ranges between 12 and 13. In grade 8 there are 16 students. Their age ranges between 13 and 14. Finally, in grade 9 there are 16 students. Their age ranges between 14 and 15. All participants are Lebanese students whose native language is colloquial Arabic; they study mathematics in English which is their second formally learned language (after classical Arabic). As a result of piloting the tests (details will be clarified in the piloting section), students of grade 6 were also involved in the study as participants. In grade 6, there are two sections, A and B, consisting of 22 students each. Their age ranges between 11 and 12. (See Table 1)
The teachers of the middle school math are two female teachers, T₁ and T₂, each having a teaching diploma in math and teaching experience of more than ten years. In particular, teacher T₁, the researcher, has a bachelor’s degree in mathematics and is working towards a master degree in math education while teacher T₂ has a bachelor’s degree in math education for secondary level. Both teachers T₁ and T₂ learned mathematics in colloquial and classical Arabic throughout grades K to 5 of their school education years and then studied mathematics using English. Teacher T₁, the researcher, teaches grade 6 sections A and B, grade 7 section B, and grade 8. Teacher T₂ is the math coordinator of the elementary and middle grade levels at the school and teaches grade 7 section A and grade 9.

Table 1

*Students across Grade Levels and Sections*

<table>
<thead>
<tr>
<th>Grade Level</th>
<th>Section</th>
<th>Number of Students</th>
<th>Age Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade 6</td>
<td>A</td>
<td>22</td>
<td>11-12</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>Grade 7</td>
<td>A</td>
<td>14</td>
<td>12-13</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Grade 8</td>
<td>A</td>
<td>16</td>
<td>13-14</td>
</tr>
<tr>
<td>Grade 9</td>
<td>A</td>
<td>16</td>
<td>14-15</td>
</tr>
</tbody>
</table>
3.3 Procedures and Data Collection Methods

Several steps were carried out to accomplish the aim of this research.

1. Interview with a math teacher/coordinator:

An interview was conducted with T2, the math coordinator of the middle school. The interview was semi-structured (see Appendix A) and aimed to explore students’ difficulties from a global perspective because the coordinator of the middle school is experienced in a variety of difficulties faced by both teachers and students.

2. Analysis of curricula and textbooks’ content pertaining to proof:

It is important to discuss the notion of proof as addressed in both the Lebanese national curriculum and textbooks, adopted by the participating school. The analysis included a general overview of the general objectives related to proof, specific objectives related to proving at each of the grade levels at the middle school, and instructions and guidelines about proofs that are provided for both teachers and students in both teachers’ guides and students’ mathematics textbooks.

3. Construction of tests:

Eight tests were prepared, two for each grade level, with different levels of complexity in both language and proof. The tests were constructed in a way that the analysis of students’ work would help in exploring their
difficulties and possibly the origin of those difficulties, be it language or proof complexity (see Appendices B, C, D & E). More details about the tests are presented in the section titled “Instruments”. Rubrics were prepared and used in the preparation of the tests in order to help in determining the text complexity level and the levels of cognitive demands of proof tasks that were addressed in the tests (see Appendices F & G).

4. Judging of tests:

The two tests for each grade level contain isomathic problems. Isomathic problems will be defined as two problems that have the same premises (given data or information), involve the same geometrical and mathematical concepts and can be solved using the same procedures; they are, however, different in the level of language complexity or proof task cognitive demands.

The eight tests were subject to a judging process to determine their level of complexity of both language and proof tasks. The judges of the tests’ proof complexity were a committee of four math teachers of the middle school and grade 6, the math coordinator, and three teachers who are experienced in teaching middle school and grade 6. The judges of the tests for language complexity were a committee of four English teachers of the middle school and grade 6. Judging and evaluating the levels of the
administered tests by a committee of teachers provided the tests with more validity and accuracy.

5. Piloting of tests:

Four of the eight tests were piloted (the tests of grade 6 and 9 didn’t undergo the process of piloting. Details are provided in the “Instruments” section). Students from another branch of the same school took the tests (one section from each grade level). The piloting sample consisted of 12 grade-7 students who are girls and whose age ranges between 11 and 12, and 20 grade-8 students who are girls and whose ages range between 12 and 13. In the second branch of the school, the students use the same textbooks, same curriculum, and are usually assessed using unified tests. The researcher didn’t have the chance of piloting the tests of grade 9 due to lack of time availability as the students of grade nine have to sit for Lebanese official exams. Typical conditions that were to be provided for the original sample of participating students were also provided to the pilot sample. Piloting the tests helped in raising the level of validity of the instruments used. Piloting the tests provided the researcher with the following information:

- Clarity of the instructions set in the tests from similar students’ perspective.
- Accuracy of the time needed for students to complete the tests.
From the piloting results, needed edits took place before working with the original research participants.

6. Administration of tests:

Participants in each grade level were asked to sit for the first test which requires fifty minutes to be completed (it was expected to last for thirty minutes before piloting). Before starting with the test, students were instructed to write and represent their thoughts about the given proof tasks even if they weren’t able to find the solution. Students weren’t allowed to ask any questions during the test. After a week, students in each grade level were asked to sit for the second test which required fifty minutes to be completed (it was expected to last for thirty minutes before piloting). The instructions and conditions given to the students in both tests were identical. The researcher decided to separate administration of the two tests by a week in order to decrease the possibility that students remember and connect the two tests.

Because the tests were held by the end of the academic year, the researcher decided to postpone having students of grade 9 sit for the test. That was due to the fact that grade 9 students at the end of the academic year are exposed to all types of problems by their teachers because they are prepared for all possible types of questions that might be addressed in the
Lebanese official exams and that would affect the results of the study. So, the researcher decided to have the students of grade 9 of the next year, who were grade-8 students of the year of administration of the other tests, sit for the test at the beginning of their school academic year.

7. Clinical interviews:

During each session within which a class was completing a test, the researcher was holding a clinical interview with one of the students of the class. The time required to complete each clinical interview was around fifty minutes. Students to be clinically interviewed were selected using the following process: One student from each grade level/section was randomly selected. Names of the students of each class were written on papers, placed in a hat, mixed thoroughly, and one name was randomly selected. The selected student from each grade level was clinically interviewed when completing both Tests 1 and 2; which means that a total of four students were clinically interviewed while each was completing the tests. The students will be referred to as S6 for the student of grade 6, S7 for the student of grade 7, S8 for the student of grade 8, and S9 for the student of grade 9. The students’ clinical interviews are referred to as: 6C1 and 6C2 for the grade-6 student, 7C1 and 7C2 for grade 7, 8C1 and 8C2 for grade 8, and 9C1 and 9C2 for grade 9. The clinical interviews were videotaped and transcribed for later
data analysis. Analysis of the transcript of the clinical interviews helped in exploring various difficulties of constructing proof, in particular, difficulties related to language.

3.4 Instruments

The following instruments were used to help in exploring students’ difficulties when constructing and formulating geometric proofs:

3.4.1 Interview

A semi-structured interview with teacher T₂, the coordinator, was conducted. Teacher T₂ was asked about the following: difficulties that students face while learning how to develop geometric proofs, the language difficulties that students face when constructing, formulating and communicating geometric proofs, and the strategies followed to help students overcome those difficulties (For more details see Appendix A.)

3.4.2 Tests

The two tests for each grade level contain isomathic problems. Isomathic problems will be defined as two problems that have the same premises (given data or information), involve the same geometrical and mathematical concepts
and can be solved using the same procedures; but they are different in the level of language complexity or proof task cognitive demands.

Students in each grade level were required to sit for two tests (see Appendices B, C, D & E) separated by a duration of a week. The tests are referred to as: 6T1 and 6T2 for grade 6, 7T1 and 7T2 for grade 7, 8T1 and 8T2 for grade 8, and 9T1 and 9T2 for grade 9. Each test consists of two problems. That is, the study required preparation of four problems for each grade level. The first problem, which is referred to as P1, requires an above-average cognitive level of proof development. Moreover, the tasks of P1 are easy to read and require simple language and below-average language abilities to be answered. The second problem, which is referred to as P2, requires a below-average cognitive level of proof development, but uses complex language structure. Besides, the tasks of P2 are not easy to read and require above-average language abilities to be answered. The third problem, which is referred to as P3, is isomathic to P1 but with simplified level of proof complexity. The fourth problem, which is referred to as P4, is isomathic to P2 but with simplified level of text complexity.

Therefore, for each grade level, the first test consisted of two problems: P1 and P4 while the second test consisted of two problems: P2 and P3 as shown in Table 2.
### Design of and Relations between Problems in the Tests

<table>
<thead>
<tr>
<th>Test</th>
<th>Problem</th>
<th>Characteristics</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1</td>
<td>P1</td>
<td>Simple language and complex proof</td>
<td>The proof tasks of P1 will be simplified in P3</td>
</tr>
<tr>
<td></td>
<td>P4</td>
<td>Simple language and simple proof</td>
<td>Isomathic to P2</td>
</tr>
<tr>
<td>Test 2</td>
<td>P2</td>
<td>Complex language and simple proof</td>
<td>The text of the problem of P2 will be simplified in P4</td>
</tr>
<tr>
<td></td>
<td>P3</td>
<td>Simple language and simple proof</td>
<td>Isomathic to P1</td>
</tr>
</tbody>
</table>

The cognitive level of the addressed proof tasks of all tests was determined according to a rubric of proof complexity level (see Appendix F). This rubric is developed according to the proof complexity criteria provided by Heinze, Cheng, Ufer, Lin and Reiss (2008), Moore (1994), Weber (2001) and others mentioned in the literature review section. The language complexity level of all the tests was determined according to the rubric of text complexity level (see Appendix G). This rubric was developed by the researcher who is a math
teacher at the middle school, with the help of an English teacher at the middle school. Elements of the rubric were compiled from various sources. Furthermore, the eight tests match the curriculum requirements for the corresponding grade level. Each test requires fifty minutes to be completed. Using the tests as instruments in the conducted study helped in exploring the difficulties that students face when constructing and formulating geometric proofs through text analysis.

3.4.2.1 Test preparation

Each of the Tests 1 and 2 for each grade level required preparation of four problems P1, P2, P3, and P4 as mentioned before. Problems P1 and P3 are isomathic problems and P2 and P4 are isomathic problems. The problems preparation required the researcher to think of problems that can be written using different levels text complexity (P2 and P4) and to think of problems that can be edited in terms of proof complexity. Thus, the researcher had to develop a problem P1 which is considered to be complex in terms of proof level and then simplify it to develop its isomathic problem P3. Furthermore, the researcher had to develop a problem P2 which is considered to have high level of text complexity and then simplify its text to develop problem P2 by replacing the lengthy sentences by short ones and decreasing the use of pronouns
and clauses. Figure 1 presents a comparison for the changes that occurred between the proof tasks of each of P1 and P3 for grade 6 as an example, while Figure 2 presents a comparison for the changes that occurred between the texts of each of P2 and P4 for grade 6 as an example. Other comparisons between problems P1 and P3 and between problems P2 and P4 for grades 7, 8, and 9 are presented in Appendix H.

3.4.2.2 Test judging

After the tests have been developed, they were submitted to the judges to evaluate their level of text and proof complexities and compare it with the researcher’s evaluation of the problems’ difficulty. Problems P1, P2, P3, and P4 for each grade level were given to the judges with several documents: 1) The text and proof difficulty rubrics (see Appendices F & G), 2) The mathematical concepts that the students covered during the academic year (see Appendix I), and 3) The list of definitions, properties, and theorems that are expected to be used by the students while solving the given problems (see Appendix J). Results of the evaluation of the problems P1, P2, P3, and P4 given by the judges are shown in Table 3. It is important to note that problems P1 and P3 of grade 8 were changed after the judge’s disagreement about considering P1 to be too hard. For more details of the judging results see Appendix K.
It is worth to mention that only Tests 1 and 2 of grade 9 didn’t undergo the judging process because they were prepared by both the researcher and the math coordinator of the middle school.
Figure 1.

Comparison of Problems P1 and P3 for Grade 6

P1: (Complex proof, simple language)
In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE].
$\hat{D}BE = 30^\circ$. AB = 3cm.

P3 (Iso P1): (Simplified proof, simple language)
In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE].
$\hat{D}BE = 30^\circ$. AB = 3cm.

What is the length of [BC]? Justify your answer.

a) Show that (AD) is the perpendicular bisector of [BC].
b) Show that AB = AC. Justify your answer.
c) Calculate the measure of $\hat{ABC}$. Show your work. Justify your answer.
d) Show that triangle ABC is equilateral. Justify your answer.
e) Deduce the length of [BC].
### Comparison of Problems P2 and P4 for Grade 6

<table>
<thead>
<tr>
<th>P2: (Complex language, simple proof)</th>
<th>P4 (Iso P2): (Simplified language, simple proof)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider an isosceles triangle MNP whose main vertex is M and the length of its equal sides is 4cm. Let $\hat{NMP} = 120^\circ$.</td>
<td>Triangle MNP is isosceles at M. NM = MP = 4cm. $\hat{NMP} = 120^\circ$.</td>
</tr>
<tr>
<td><strong>Change 1</strong></td>
<td><strong>Change 1</strong></td>
</tr>
<tr>
<td>1. Draw triangle MNP.</td>
<td>1. Draw triangle MNP.</td>
</tr>
<tr>
<td>2. <strong>Compute</strong> the measures of $\hat{MNP}$ and $\hat{MPN}$. Show your work and justify your answer.</td>
<td>2. <strong>Calculate</strong> the measures of $\hat{MNP}$ and $\hat{MPN}$. Show your work. Justify your answer.</td>
</tr>
<tr>
<td><strong>Change 2</strong></td>
<td><strong>Change 2</strong></td>
</tr>
<tr>
<td>3. Let R be the foot of the perpendicular drawn from M to [NP] and (d) be the parallel to (MR) drawn through P and cutting (NM) at Q.</td>
<td>3. Through M, draw (x) perpendicular to [NP]. (x) cuts [NP] at R. Through P, draw (d) parallel to (x). (d) cuts (NM) at Q.</td>
</tr>
<tr>
<td><strong>Change 3</strong></td>
<td><strong>Change 3</strong></td>
</tr>
<tr>
<td>a) <strong>How are (NP) and (d) related?</strong> Justify your answer.</td>
<td>a) <strong>What is the relative position of (NP) and (d)?</strong> Justify your answer.</td>
</tr>
<tr>
<td>b) Name a height of triangle NPQ. Justify your answer.</td>
<td>b) Name a height of triangle NPQ. Justify your answer.</td>
</tr>
</tbody>
</table>
Table 3

*Text and Proof Complexity Rubrics Results*

<table>
<thead>
<tr>
<th>Problem #</th>
<th>Grade 6</th>
<th>Grade 7</th>
<th>Grade 8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Text of</td>
<td>Text of</td>
<td>Proof</td>
</tr>
<tr>
<td>Source of Complexity</td>
<td>Problem</td>
<td>Possible</td>
<td>Task</td>
</tr>
<tr>
<td>Justifications</td>
<td>Justifications</td>
<td>Justifications</td>
<td></td>
</tr>
<tr>
<td>Judging Score</td>
<td>/ 21</td>
<td>/ 21</td>
<td>/ 21</td>
</tr>
<tr>
<td>P1</td>
<td>8</td>
<td>10</td>
<td>17</td>
</tr>
<tr>
<td>P2</td>
<td>16</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>P3</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>P4</td>
<td>11</td>
<td>14</td>
<td>10</td>
</tr>
</tbody>
</table>
3.4.2.3 Test piloting

Tests 1 and 2 for each of grades 7 and 8 were piloted. As mentioned before, the sample of participating students who sat for the pilot test are students of another branch of the same school. The numbers of students in grade 7 and 8 who sat for the pilot tests are 12 and 20 respectively.

Piloting the tests helped in improving the following aspects of each of the tests: 1. the duration of test which was expected to need thirty minutes and was modified to fifty minutes. 2. The clarity and readability of the questions presented in each test. Moreover, the researcher tried to analyze the piloted tests of grades 7 and 8, and noticed that there are some difficulties that cannot be explored except from proofs written by beginners who are the sixth graders. It was then decided to extend the sample of participants to grade 6 students of the participating school. It is important to mention that the school under study introduces geometric proofs at the beginning of grade 6.

Adding sixth graders to the participants at the end of the academic year didn’t give the researcher the chance to pilot the tests of grade 6.
3.4.3 Clinical interview

Clinical interviews took place with a selection of four students from the participants, one from each grade level. The clinical interviews, in each grade level, took place while the other participating students were performing the tests. The interviews were individual without any interference from the interviewer in the solution, or expression of approval or disapproval of the proof formulated or written by students. The interview included questions such as, “Why did you decide to start with this step?”; “Why did you stop?” and “What led you to that conclusion?” The interviews were videotaped and transcribed for later analysis. The clinical interview helped to closely explore and discuss with students the kinds of difficulties they face when solving a proof task.

3.5 Data Analysis Framework

The purpose of the study was to investigate the difficulties that Lebanese middle school students face when writing and formulating geometric proofs using their non-native language. The proof problems addressed in each of Tests 1 and 2 for each grade level varied in complexity of both proof and language (concerning both, semantic and syntactic features). Thus, to analyze the students’ responses to each of the tests, criteria for proof writing and mathematical text writing abilities were set.
Research in the field of math education is rich with studies investigating the difficulties that students face when formulating and writing geometric proofs. Some studies investigated students’ difficulties in proof formulation such as Senk (1985), Harel and Sowder (1998), Weber (2001), and Gueudet (2008), Herbst and Brach, (2006) Duval, (2007), Yang and Lin, (2008), Andrew, (2009), and Moore, (1998) while others investigated students’ difficulties in mathematical language such as Moore (1998), Balacheff (2000), Gueudet (2008), Morgan, (1996), Robotti, (2012), and Yang and Lin, (2008). The current study aims to investigate a combination of language and proof difficulties when formulating and writing geometric proofs. The analysis requires taking into consideration the difficulties explored in previous research and other difficulties that were identified when testing the Lebanese participants’ ability to formulate and write geometric proofs in a non-native language.

In order to have a more reliable data analysis framework, it was essential to adopt a framework examined in other research processes. However, it is crucial to make some adaptations to have the selected framework fit the current study.

3.5.1 Data analysis framework development

Andrew (2009) has set a list of faulty proofs written by students at both graduate and undergraduate levels that he collected from classroom situations, and from research articles. He constructed a Proof Error Evaluation Tool
(PEET). This tool helps in analyzing and evaluating students’ written proofs. The developed PEET tool sheds light on two main themes that contribute to formulating proofs, proof structure (S) and conceptual understanding (U). Under these two themes, Andrew (2009) developed sixteen headings, nine belonging to the proof structure theme (S), such as “only gave an example to establish the truth of a mathematical statement” (p. 462), and “made a false assumption somewhere in the proof” (p. 461), and seven belonging to the conceptual understanding theme (U) such as “wrote a statement that wasn’t justified, explained or verified” (p. 462). Andrew (2009) claims that the PEET helps in labeling students’ errors and providing them with a description of the errors done, hence helping students understand their mistakes and communicate with other students what errors to avoid.

It is worth mentioning that the PEET was developed to evaluate students’ errors when writing mathematical proofs in calculus, and not particularly geometric proofs. In the current research study, Andrew’s PEET is adopted with edits. One of the main edits that the current research had to consider is eliminating those errors that don’t contribute to geometric proofs and adding some errors that are recognized as remarkable errors when writing geometric proofs by the researcher and extracted from other research work. Therefore, the test analysis will basically focus on exploring lists of errors committed by
students. Later, the researcher will try to extract, from the explored errors, possible proof and language difficulties that Lebanese middle school students have when writing geometric proofs using a non-native language.

Table 4 provides a list of the adopted criteria from the PEET. Table 5 provides a list of other errors derived from the literature. Moreover, Table 6 provides a list of all errors that will be considered when analyzing students’ written responses to the tests administered in this research. These errors are abbreviated to facilitate tabulating and coding them later.

As a summary, the analysis of Tests 1 and 2 requires considering a classification of errors committed by students according to Table 3 for further interpretations of these errors and analysis of the possible reflected difficulties.
### Table 4

*Classification of Students’ Errors according to Andrew (2009, pp.461-462)*

<table>
<thead>
<tr>
<th>Type of Error</th>
<th>Error Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proof Structure Errors (S)</td>
<td>“The approach taken in proving a statement will not work”.</td>
</tr>
<tr>
<td></td>
<td>“Made a false assumption somewhere in the proof”.</td>
</tr>
<tr>
<td></td>
<td>“Didn’t proceed through the proof in a linear fashion, and ideas were not in logical order”.</td>
</tr>
<tr>
<td></td>
<td>“The proof contained extraneous details or steps that didn’t really contribute to the proof”.</td>
</tr>
<tr>
<td></td>
<td>“The length of the proof was unnecessarily long and thus extremely difficult to follow”.</td>
</tr>
<tr>
<td>Conceptual Understanding Errors (U)</td>
<td>“Wrote a statement that wasn’t justified, explained or verified”.</td>
</tr>
<tr>
<td></td>
<td>“Wrote a statement or paragraph that was ambiguous, confusing, and/or unnecessarily complex”.</td>
</tr>
<tr>
<td></td>
<td>“Did not sufficiently justify a crucial step in a proof”.</td>
</tr>
<tr>
<td></td>
<td>“Made a false statement or incorrect computation in the proof”.</td>
</tr>
<tr>
<td></td>
<td>“Incorrectly claimed that one statement implied or equaled another statement”.</td>
</tr>
</tbody>
</table>
Table 5

Classification of Students’ Errors Extracted from Research Resources

<table>
<thead>
<tr>
<th>Error Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did not start a proof.</td>
</tr>
<tr>
<td>Started what could be a proof plan but did not complete the proof.</td>
</tr>
<tr>
<td>Drew a wrong figure.</td>
</tr>
<tr>
<td>Did not differentiate between the premises of a proof and their inferences and</td>
</tr>
<tr>
<td>considered an inference as if it were given.</td>
</tr>
<tr>
<td>Proved statements other than the required ones.</td>
</tr>
<tr>
<td>Used irrelevant definition, theorem, or property to justify a statement.</td>
</tr>
<tr>
<td>Used a premise that is not given.</td>
</tr>
<tr>
<td>Used mathematical symbols incorrectly.</td>
</tr>
<tr>
<td>Used mathematical terms, words, or vocabulary incorrectly.</td>
</tr>
<tr>
<td>Wrote statement of definitions, properties, or theorems incorrectly.</td>
</tr>
<tr>
<td>Used logical connectors incorrectly.</td>
</tr>
</tbody>
</table>
Table 6

Adopted Classification of Students’ Errors for Data Analysis

<table>
<thead>
<tr>
<th>Code</th>
<th>Error Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>DWF</td>
<td>Drew a Wrong Figure.</td>
</tr>
<tr>
<td>SOR</td>
<td>Proved Statements Other than the Required ones.</td>
</tr>
<tr>
<td>NSP</td>
<td>Did Not Start a Proof.</td>
</tr>
<tr>
<td>SPNC</td>
<td>Started what could be a Proof plan but did Not Complete the proof.</td>
</tr>
<tr>
<td>ANW</td>
<td>“The Approach taken in proving a statement will Not Work”.</td>
</tr>
<tr>
<td>NL</td>
<td>“Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.</td>
</tr>
<tr>
<td>ED</td>
<td>“The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.</td>
</tr>
<tr>
<td>UL</td>
<td>“The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.</td>
</tr>
<tr>
<td>ISIE</td>
<td>“Incorrectly claimed that one Statement Implied or Equaled another statement”.</td>
</tr>
<tr>
<td>PNG</td>
<td>Used a Premise that is Not Given.</td>
</tr>
<tr>
<td>NDPI</td>
<td>Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.</td>
</tr>
<tr>
<td>IDTP</td>
<td>Used Irrelevant Definition, Theorem, or Property to Justify a Statement.</td>
</tr>
<tr>
<td>FA</td>
<td>“Made a False Assumption somewhere in the proof”.</td>
</tr>
<tr>
<td>FSC</td>
<td>“Made a False Statement or incorrect Computation in the proof”.</td>
</tr>
<tr>
<td>NJEV</td>
<td>“Wrote a statement that wasn’t Justified, Explained or Verified”.</td>
</tr>
<tr>
<td>NJCS</td>
<td>“Did not sufficiently Justify a Crucial Step in a proof”.</td>
</tr>
<tr>
<td>SA</td>
<td>“Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex”.</td>
</tr>
<tr>
<td>MSI</td>
<td>Used Mathematical Symbols Incorrectly.</td>
</tr>
<tr>
<td>VI</td>
<td>Used Mathematical terms, words, or Vocabulary Incorrectly.</td>
</tr>
<tr>
<td>DPTI</td>
<td>Wrote statement of Definitions, Properties, or Theorems Incorrectly.</td>
</tr>
<tr>
<td>LCI</td>
<td>Used Logical Connectors Incorrectly.</td>
</tr>
</tbody>
</table>
3.5.2 Examples on students’ errors

The following section provides examples for each of the presented errors. These examples are captured from sample students’ responses to proofs of Tests 1 and 2. These examples will be referred to using codes such as B6T1P1(3) which means that the provided example is a student response to a question shown in appendix B, for grade 6, Test 1, Problem 1, part 3.

1- Drew a Wrong Figure (DWF) “D8T2P2(1)”:

The student didn’t locate points E and F appropriately. E and F are the feet of the perpendiculars drawn from B and C respectively. She switched the points.

![](image)

2- Proved Statements Other than the Required ones (SOR)

“D8T1P4(4b)”:

The student has to calculate the length of [EI] and [CI]. The student didn’t calculate the length of any of the segments. Rather, she proved them equal. That is, she compared their lengths rather than calculating their measures.
3- Did Not Start a Proof (NSP) “B6T1P1”:

The student has to find the length of [BC]. She wrote that she wasn’t able to find any clues that help in developing the proof. Moreover, she didn’t code the figure or label it with the given properties. So, she wasn’t able to start the proof.

4- Started what could be a Proof plan but did Not Complete the proof (SPNC) “C7T1P1(b)”: 

The student has to show that lines (AD) and (BC) are parallel. The student started the proof by considering two triangles and trying to prove them congruent in order to reach two suitable equal angles, but wasn’t able to complete it as it is clear in the example.
5- “The Approach taken in proving a statement will Not Work”(ANW)

“E9T2P3(3b)”: 

The student has to calculate the length of [AM]. The plan set to calculate AM will not work because she wrote AM in terms of segments whose lengths are not given.

\[
AM = AB + MN
\]

6- “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”(NL) “B6T2P2(2)”: 

The student has to calculate the measures of \( \overline{MNP} \) and \( \overline{MPN} \). She wrote statements that are not organized logically. She calculated the measures of the angles at two different steps; however, she should calculate their measures in one step because both are the base angles of the isosceles triangle MNP.
7- “The proof contained Extraneous Details or steps that didn’t really contribute to the proof” (ED) “B6T2P3(c)”:

The student is required to calculate the measure of $\overline{ABC}$. She calculated the measure of the required angle. However, she added extra detail to the proof which is “$AB = 3\text{cm (given)}$”. Knowing the length of $[AB]$ would not help in calculating the measure of $\overline{ABC}$.

\[
\text{In triangle } \triangle ABC: \\
\overline{AB} = 3\text{cm (given)} \\
\angle ABC = 180^\circ = 60^\circ \quad \text{(sum of angles in a triangle is } 180^\circ) \\
\]

8- “The length of the proof was Unnecessarily Long and thus extremely difficult to follow” (UL) “C7T1P1(b)”:

To prove that lines $(AD)$ and $(BC)$ are parallel, the student set a lengthy proof plan that contains unnecessary statements such as: “$JB = DI$ (proved), $(AB) \parallel (DC)$ (given) so $\overline{AB} \parallel \overline{BC}$ (alternate interior angles of $(AB) \parallel (DC)$.” She proved some of the statements that were already proved in previous parts of the problem.
9- “Incorrectly claimed that one Statement Implied or Equaled another statement” (ISIE) “B6T2P3(a)”:  

The student is required to show that \((AD)\) is the perpendicular bisector of \([BC]\). She claimed that if \((AD)\) passes through the midpoint of \([BC]\) this implies that \((AD)\) is the perpendicular bisector of \([BC]\), thus neglecting the necessity of the perpendicularity.

\[
(AD) \text{ is the perpendicular bisector of } [BC].
\]

It has 2 equal sides, it is at the half of \([BC]\) and also it is the midpoint of \([BC]\).

10- Used a Premise that is Not Given(PNG) “C7T2P3(a)”:  

The student was proving that triangles IAB and JCD are equal. She mentioned that the segments \([AB]\) and \([DC]\) are equal in length. This statement is not given. She rather considered it because of either an empirical assumption or the use of ruler to measure the lengths of the segments. Moreover, she didn’t justify why these segments are equal.

Consider triangles IAB and JCD:

\[
\begin{align*}
\triangle IAB & = \triangle JCD \quad \text{(given)} \\
\angle ABI & = \angle DCB \quad \text{(alternate exterior angles of } (AB) \parallel (CD)). \\
AB & = DC \quad \text{hence, } \triangle IAB = \triangle JCD \text{ by } \text{S.A.S}\quad \text{proven by homologous elements.}
\end{align*}
\]
11- Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given (NDPI)

“D8T1P1”:

The student wrote that AM is equal to ME and justified the statement as a given property. However, the equality of AM and ME is an inference from the premise “E is the symmetric of A with respect to M”.

The student wrote:

- Quadrilateral ACBE
- AM = ME (given)
- MH = MC

The quadrilateral ACBE is a parallelogram (A quadrilateral having diagonals bisect each other at their midpoint is a parallelogram).

12- Used Irrelevant Definition, Theorem, or Property to Justify a Statement (IDTP) “C7T2P3(a)”: The student had to show that BI = DJ. She stated that both JB and ID are equal to JI. Then, she concluded that DJ = BI and justified her statement by using the substitution property. She used the substitution property however she didn’t substitute any of the two equal quantities. If she was to substitute, she should have written that JB = ID.

The student wrote:

- DJ = JB (Def. of midpoint)
- DI = ID (Def. of midpoint)
- DJ = BI (substitution)
13- “Made a False Assumption somewhere in the proof” (FA) “D8T1P1”:

The student was proving that [AD] and [BM] are equal and parallel. The student built her proof on a false assumption which is the symmetry of M and B with respect to H. This false assumption led to writing a false statement which is the equality of BH and HM.

\[ \text{In Quadrilateral } ABMH: \]
\[ AH = HM \text{ (def of symmetry)} \]
\[ BH = HM \text{ (def)} \]

14- “Made a False Statement or incorrect Computation in the proof” (FSC) “B6T2P2(3a)”: 

The student had to find how the lines (NP) and (d) are related. She wrote that line (NP) is the perpendicular bisector of (d). This is a false statement because a line doesn’t have a perpendicular bisector.

(NP) is the perpendicular bisector of (d).

15- “Wrote a statement that wasn’t Justified, Explained or Verified” (NJEV) “B6T2P3(d)”: 

The student was required to prove that triangle ABC is an equilateral triangle. To justify her answer, she listed the measures of the angles of triangle ABC without justifying how she got each measure.

\[ \text{First, is an equilateral triangle:} \]
\[ \theta = 60^\circ \text{ } \text{and } \theta = 60^\circ \]
\[ \text{So, } \angle B = 60^\circ \text{ and } \angle P = 60^\circ \]
\[ \text{and } 60^\circ, 60^\circ, 60^\circ = \text{equilateral triangle} \]
16- “Did not sufficiently Justify a Crucial Step in a proof’ (NJCS)

“B6T1P1”:

The student was trying to prove that triangle ABC is an equilateral triangle in order to deduce the length of its sides. The student missed proving that the three angles of triangle ABC are equal. She was satisfied by proving a pair of equal angles to deduce that the sides of the triangle are equal.

\[
\begin{align*}
&\text{In \ triangle } \triangle \text{ACD}, \\
&\measuredangle \text{CAD} = 30^\circ \text{ (proved)} \quad \measuredangle \text{ABC} = 90^\circ \text{ (given)} \\
&\text{So, } \measuredangle \text{ACD} = 180^\circ - (30 + 90^\circ) \quad \text{sum of angles in a triangle is } 180^\circ \\
&180^\circ - 120^\circ = 60^\circ \\
&\text{Hence, } \measuredangle \text{ACD} \text{ is } 60^\circ \text{ (def of isosceles triangle)} \\
&\text{So, } AB = BC = 3\text{cm} \quad \text{proven.}
\end{align*}
\]

17- “Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex” (SA “B6T1P4(3a)”:)

The student proved that lines (d) and (NP) are perpendicular. She wasn’t able to present an appropriate statement of the property that justifies her statements of the proof and her conclusion. The statement of the property doesn’t reflect clear mathematical text. The appropriate statement is: if two lines are parallel, then any line perpendicular to one of them is perpendicular to the other.

\[
\text{(NP) and (d) are perpendicular since: (x) \parallel (d) \text{ and } (x) \perp (NP)}
\]

\[
\text{So, they’re perpendicular. (IF \parallel \text{ lines are parallel to the same line, and they are parallel, then any line would be } \perp \text{ to it).}
\]
18- Used Mathematical Symbols Incorrectly (MSI) “B6T1P1”:

While writing the proof, the student used the symbol of angles to name a triangle. She wrote “in triangle BDE” and used the symbol of angles to name the triangles.

\[
\text{In triangle BDE:}
\]
\[\hat{D}OE = \hat{D}E = 30^\circ \text{ (given)}\]
\[\hat{C}DE = \hat{B}DC = 30^\circ \text{ (vertically opposite angles)}\]
\[\text{So, } \hat{D}OE = 180^\circ - (30^\circ + 30^\circ) = 180^\circ - 60^\circ = 120^\circ \]
\[\text{So, } \hat{BDE} = 180^\circ - 120^\circ = 60^\circ \text{ (sum of angles in a triangle is } 180^\circ)\]

So, BDE is an equilateral triangle (B P of an equilateral triangle) because its measure are \( 90^\circ, 30^\circ, 60^\circ \).

19- Used Mathematical terms, words, or Vocabulary Incorrectly (VI) “C7T1P4(e)”:

The student had to compare the lengths of [MF] and [FR]. She used the term “equidistant” instead of “equal”. The term equidistant is incorrectly used.

\[\text{They are equidistant, any point on the perpendicular bisector is equidistant to the end point of the semi-line.}\]

20- Wrote statement of Definitions, Properties, or Theorems Incorrectly (DPTI) “D8T1P4(a)”:

The student proved that quadrilateral BRAE is a rectangle by proving that the diagonals of this quadrilateral bisect each other and
that one of its angles measures 90°. But she presented the property in a wrong way. It should be written as “a quadrilateral with one of its angles measuring 90° and diagonals bisecting each other is a rectangle”.

\[
\text{hence, } \text{BRAC is a rectangle \ (in a rectangle, a \ angle \ go \ and \ diagonals \ intersect \ each \ other \ is \ a \ rectangle.}
\]

21- Used Logical Connectors Incorrectly (LCI) “C7T1P1(a)”:

The student used the logical connector “so” inappropriately. The logical connector “so” is used to conclude a statement from a previous statement or set of statements. She wrote in her first statement that lines (AB) and (CD) are given to be parallel then she concluded in the last step, using the logical connector “so”, that lines (AB) and (CD) were parallel. This indicates that she is not able to use logical connectors appropriately.

Show that AI = JC.

Consider triangles DJC and AIB:

\[
\begin{align*}
(AB) \parallel (CD) \ (\text{given})
\end{align*}
\]

\[
I \text{ and } J \text{ are the midpoints of } CD \text{ and } ET \text{ respectively (given),}
\]

\[
\text{Bjl = jCD (given)}
\]

\[
\text{So, } (AB) \parallel (CD) \text{ and transversal } (DB) \ (\text{given})
\]
3.5.3 The adopted data analysis framework

The use of the framework adopted helps in exploring the errors committed by the students when constructing and formulating geometric proofs. Since the purpose of the research is exploring the difficulties that the students have when constructing and formulating geometric proofs, it is vital to look for the difficulties that are reflected through these committed errors. Errors explored are sorted under six groups of proof difficulty sources: Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Understanding-Applying Mathematical Concepts (UAMC), Conducting Deductive Reasoning (CDR), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT).

The framework identifies 21 possible errors committed by students when constructing and formulating geometric proofs. Some of the errors reflect more than one proof difficulty. Following, is a list of the identified six groups of proof difficulties and the possible kinds of errors that might reflect each:

1- Difficulty in Understanding the Notion of Proof (UNP):

   a) “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order” (NL).
b) “The proof contained Extraneous Details or steps that didn’t really contribute to the proof” (ED).

c) Used a Premise that is Not Given (PNG).

d) Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given (NDPI).

e) “Wrote a statement that wasn’t Justified, Explained or Verified” (NJEV).

f) “Did not sufficiently Justify a Crucial Step in a proof” (NJCS).

2- Difficulty in Setting Proof Plans (SPP):

a) Did Not Start a Proof (NSP).

b) Started what could be a Proof plan but did Not Complete the proof (SPNC).

c) “The Approach taken in proving a statement will Not Work” (ANW).

d) “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order” (NL).

e) “The proof contained Extraneous Details or steps that didn’t really contribute to the proof” (ED).

f) “The length of the proof was Unnecessarily Long and thus extremely difficult to follow” (UL).
3- Difficulty in Conducting Deductive Reasoning (CDR):

   a) “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order” (NL).
   b) “The proof contained Extraneous Details or steps that didn’t really contribute to the proof” (ED).
   c) “Incorrectly claimed that one Statement Implied or Equaled another statement” (ISIE).
   d) Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given (NDPI).
   e) “Made a False Statement or incorrect Computation in the proof” (FSC).

4- Difficulty in Understanding-Applying of Mathematical Concepts (UAMC):

   a) Proved Statements Other than the Required ones (SOR).
   b) “Incorrectly claimed that one Statement Implied or Equaled another statement” (ISIE).
   c) Used Irrelevant Definition, Theorem, or Property to Justify a Statement (IDTP).
   d) “Made a False Assumption somewhere in the proof” (FA).
5- Difficulty in Comprehending Mathematical Texts (CMT):
   a) Drew a Wrong Figure (DWF).
   b) Proved Statements Other than the Required ones (SOR).
   c) Did Not Start a Proof (NSP).

6- Difficulty in Writing Mathematical Texts (WMT):
   a) “Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex” (SA).
   b) Used Mathematical Symbols Incorrectly (MSI).
   c) Used Mathematical terms, words, or Vocabulary Incorrectly (VI).
   d) Wrote statement of Definitions, Properties, or Theorems Incorrectly (DTPI).
   e) Used Logical Connectors Incorrectly (LCI).

Figure 3 shows a diagram that relates the errors presented in the framework to the proof difficulties that they reflect.
Figure 3

Students’ Errors and Proof Difficulties

Understanding the Notion of Proof (UNP)

Conducting Deductive Reasoning (CDR)

Understanding-Applying Mathematical Concepts (UAMC)

Comprehending Mathematical Texts (CMT)

Setting Proof Plans (SPP)

Writing Mathematical Texts (WMT)
3.6 Data Analysis Method and Procedure

The data collected using the three instruments mentioned above was analyzed as follows:

3.6.1 Analysis of curriculum, textbooks, and teachers’ guides

The analysis of the Lebanese middle school mathematics curriculum, Lebanese national textbooks, and their teachers’ guides aimed to explore the following aspects of mathematical proofs and communication:

1) Investigating the introduction to geometric proofs: when does it occur and what language are students supposed to use?

2) Exploring whether the curriculum documents include a special section for language difficulties.

3) Exploring what forms of proofs students are expected to develop and what strategies help in implementing these forms.

4) Exploring and comparing the intended and implemented cognitive levels of proof in the curriculum documents.

5) Examining if the curriculum allocates a section that is related to development of the notion of proof regardless of the mathematical content.
3.6.2 Analysis of the interview

The transcript of the interview conducted with the math coordinator of the middle school (see Appendix L) was analyzed according to the adopted framework (see Appendix M) to identify the student difficulties that are identified by the teacher while teaching how to construct and formulate geometric proof at each grade level. The difficulties explored were categorized according to the difficulties identified in the framework.

3.6.3 Analysis of the students’ tests

The Tests 1 and 2 for each grade level were analyzed according to the adopted data analysis framework. Below, the steps followed for the tests’ analysis.

1- Exploration of errors:

Errors committed by students in each of the Tests 1 and 2 of each grade level were detected and classified according to the error codes provided in Table 6. The percentage of the students committing each kind of error in each of the problems P1, P2, P3, and P4 was calculated. However, the detailed discussion of the tests’ analysis will be for grades 6 and 7 only. The researcher decided to present a detailed discussion of the tests for grade 6 because at grade 6,
students start constructing and formulating proofs and face the notion of proof for the first time. We assume that the way they learn about proving the first time affects their subsequent conceptions and skills. As for grade 7, the researcher presented a detailed analysis of the tests to explore the how would the abilities of the six graders to construct and formulate geometric proofs develop. Results of the test analysis for grades 8 and 9 were presented in Appendix N and Appendix O and just used for later comparison of proof development abilities across grade levels.

2- Identification of difficulties:

The explored errors committed by the students in each of the Tests 1 and 2 at each grade level were sorted according to the difficulties they reflect and had a detailed analysis. This helps in answering the research question “do Lebanese middle school students face difficulties when constructing geometric proofs? What are these difficulties?”

3- Comparison of errors across isomathic problems:

Identified errors committed and difficulties in each pair of the isomathic problems were compared, for each type of error, in order to explore the existence of any changes in the error types and student
percentages between the isomathic problems. This helped in determining the source of difficulties that caused these errors according to the variation between the complexities of the two isomathic problems (language or proof complexity). Moreover, it helped in answering the research questions “is it possible to classify, differentiate or distinguish the difficulties that are due to language and those that are due to proofs’ cognitive complexity?”

4- Comparison of errors across grade levels 6 to 9:

The percentages of the students committing each kind of error and their reflected difficulties across grade levels were compared in order to answer the research question “are there any developments or changes in the nature and extent of difficulties through the four grade levels of middle school (grades 6 to 9)?”

3.6.4 Analysis of the clinical interviews

The clinical interviews that were conducted with the students S6, S7, S8, and S9 while sitting for Tests 1 and 2 were transcribed. The clinical interviews 6C1, 6C2, 7C1, and 7C2 were the only analyzed interviews due to the same reasons mentioned before. The clinical interviews’ analysis provided the researcher with a clearer view of the difficulties faced by students when
constructing and formulating geometric proofs. It helped in giving explanations for the errors committed and difficulties identified.

3.7 Validity and Reliability

Validity and reliability of instruments and results was attained through several steps. The first step was creating the proof and language complexity rubrics to identify the problems’ complexity levels (see Appendices F & G). The second step was having the test go under a judging process. The judges who approved both content and complexity levels of the developed tests were specialists in math and English language (see Appendix K). The third step was asking math teacher who has a master degree in math education to review the analysis of the students’ tests after being analyzed by the researcher and approve the errors identified. Finally, throughout the research, a math education specialist was consulted to approve the construction of the tests for each grade level, the error analysis, and difficulties’ identification. She was always providing the researcher with needed suggestions, corrections, and improvements while constructing and analyzing the tests.

The current chapter presented a detailed explanation of the research design and method used to collect and analyze data. The next chapter presents analysis and interpretation of the collected data.
CHAPTER FOUR
ANALYSIS AND FINDINGS

The following chapter presents analysis, concerning the teaching and learning of geometric proof, of the following: Lebanese national geometry curriculum and textbooks for grades 6 to 9, the interview with the math coordinator of the intermediate level at the participating school, students’ responses to Tests 1 and 2, and the clinical interviews conducted with a sample of students while solving Tests 1 and 2.

4.1 Lebanese Geometry National Curriculum and Textbooks for Grades 6 to 9

According to the NCTM (2000), proof and reasoning should be part of the mathematics curriculum at all school levels. Since the school under study adopts the Lebanese curriculum and textbooks, it is important to discuss the notion of proof as addressed in both the Lebanese national curriculum and textbooks.

Aligned with the NCTM recommendations, the Lebanese mathematics general objectives recognize proof and reasoning as one of the fundamental objectives of mathematics. This is mainly observed in the general objectives of the mathematics middle school curriculum that highlights and focuses on proofs. According to the ECRD (1997), middle school students should be able to complete proofs that are simple and to
identify proofs that are false. Moreover, students should use the correct mathematical language and notation to write a proof that is clear and rigorous.

However, the specific objectives of geometry in the Lebanese mathematics curriculum don’t fully illustrate what is mentioned in the general objectives. Though the objectives mention that students are required to use the properties and theorems related to plane figures in formulating proofs, the form and the level of complexity of proofs required are not mentioned. In addition, there is no specific objective that requires addressing proving as a skill in its own right, regardless of the geometric concepts, properties or theorems. Thus, the specific objectives of geometry in the Lebanese mathematics curriculum at the middle school level focus less on proof than on the properties and theorems.

As for the Lebanese national textbooks (ECRD, 2000a; ECRD, 1999a; ECRD, 1998a), they provide students with clear definitions, properties and theorems related to the geometric concepts taught. In some lessons or chapters, the textbooks include a section that provides students with directions and guidelines related to particular proofs. As for the exercises and the problems in the textbooks, the term “proof” is mentioned frequently where most of the exercises and problems require students to prove certain statements. Moreover, it is worth mentioning that the textbooks address students using other varieties of terms that indicate the need for proving such as “show that”, “verify
that”, and “justify”. However, the geometry chapters don’t include proving guidelines or sample forms of proofs that students might follow.

The teachers’ guide of the Lebanese national textbooks (ECRD, 2000b; ECRD, 1999b; ECRD, 1998b), fail to provide the teachers with strategies needed for introducing proofs especially at the beginning of middle school. At this early stage, teachers need to introduce the notion of proof, and the principles of writing formal proofs to students. The pedagogical guide leaves it to the teacher to investigate appropriate strategies that would enhance students’ ability to construct and formulate geometric proofs.

4.2 Interview with the Math Coordinator

The purpose of conducting an interview with the math coordinator of the middle school was to investigate, from a teacher’s perspective, the main difficulties that Lebanese middle school students face when constructing and formulating geometric proofs. The transcript of the interview (see Appendix L) was analyzed based on the research questions and the adopted framework (see Appendix M). The results obtained provide sufficient information from which two major components come to mind. The first component is the types of proof errors committed by middle school students. The second component is the possible sources of those proof errors.
4.2.1 The types of proof errors committed by Lebanese middle school students

The math coordinator believes that students, when constructing and formulating geometric proofs, commit a lot of errors.

- First, students try to extract premises from the geometric figures. They build their proofs on empirical assumptions that are not supported with evidence from theorems, properties, or even definitions. Thus, their proofs are not valid.

- Second, students justify and support their proof statements with an irrelevant definition, theorem, or property. They try to justify their statements without paying attention to alignment and relation between the written statement and the used definition, theorem, or property used.

- Third, students write proofs that are incomplete. When validating a proof statement, they don’t make sure that they presented or proved all its required conditions to be mathematically valid and acceptable. Hence, their proofs have gaps.

- Fourth, students commit errors when using the “if … then” statements. They are not able to identify or differentiate between the essential or the sufficient conditions that should be satisfied and the implied conclusions.
Moreover, sometimes they try to reverse these statements without recognizing that most of the “if … then” statements are not reversible.

- Finally, students do commit errors regarding the appropriate use of mathematical symbols. The coordinator stated that she doesn’t give much importance to such errors because she believes that students commit them because of carelessness or of being in a hurry. The most important thing to her is that they don’t have errors or problems when reading and comprehending a mathematical text that contains mathematical symbols.

4.2.2 The possible sources of the proof errors committed by Lebanese middle school students

The math coordinator claims that errors committed by students when constructing and formulating geometric proofs are due to several reasons or sources.

- First, when students justify a statement with the inappropriate definition, property, or theorem, this reflects their inability to comprehend the statement of each of the definition, property, and theorem. They are not able to identify the conditions under which each of these justifications is used.

- Second, some mathematical misconceptions that students have developed about the definitions of some basic geometrical elements are also a source
of errors. When the definition is clear and well understood, students are able to use it and thus differentiate between a definition and other properties and theorems.

- Third, students’ inattentiveness to the importance of proof and the deductive nature of math makes them unaware of the gap created when they do not justify every statement of their proofs. The coordinator prefers that the Lebanese curriculum address proofs as a topic in its own right, regardless of geometry as a context. This, she says, will help students develop the notion of proof and value the necessity of supporting any statement with appropriate justifications.

- Finally, proofs require high-order cognitive abilities to be developed. This is true because proofs require analysis, synthesis, and evaluation of the learnt information. According to the math coordinator, introducing formal geometric proofs should be postponed to the beginning of the secondary level.

4.2.3 Conclusion to the interview with the math coordinator

In summary, difficulties extracted from the coordinator’s interview transcript (see Appendix L) are sorted such that they give justification to some of the six groups of proof and language difficulties identified in the adopted framework.
1- Understanding the Notion of Proof (UNP):

Middle school students have difficulties related to the understanding of the notion of proof due to several aspects. The first aspect is the strategies used to introduce proving as a mathematical process that is related to analytical and critical thinking. They way and the approach that is used to teach students taught how to develop proofs absolutely affect their ability to understand the notion of proof. The second aspect is the time frame allocated to introducing the notion of proof regardless of the mathematical context and to the lack of prior introduction to proofs. The third aspect is the students’ inattentiveness to the importance of proof, and to the lack of interest in it. The fourth aspect is the students’ tendency to introduce premises that are not valid and that are built on students’ empirical assumptions.

2- Conducting Deductive Reasoning (CDR):

Middle school students have difficulties related to deriving the appropriate conclusions from the given premises. They are not capable of identifying premises and conclusions from the “if - then” statements.
3- Comprehending Mathematical Texts (CMT):

Middle school students have difficulties in comprehending texts regarding mathematical definitions, theorems, and properties. This is reflected through their inability to extract the necessary and sufficient conditions that allow the usage of appropriate definitions, theorems, and properties related to the proof at hand.

4- Writing Mathematical Texts (WMT):

Middle school students have difficulties in writing mathematical texts. These difficulties are best reflected through the students’ failure to rewrite the appropriate statement of the definitions, theorems, and properties accurately.

4.3 Tests Analysis

As presented in chapter three, two tests were administered to students at each grade level from 6 to 9. The data collected from the tests of each grade level (6 to 9) were summarized in tables representing the percentages of the errors that students committed in each of the isomathic problems P1, P2, P3, and P4, based on the framework developed in Appendix M. It is worth to remind that problems P1 and P3 are said to be isomathic because they address the same mathematical concepts but vary in terms of proof complexity, while problems P2 and P4 are said to be isomathic because they vary in terms of language complexity.
The current research provides detailed analysis of Tests 1 and 2 for each of grades 6 and 7. As for grades 8 and 9, summary of the results will be presented as comparison across the middle school grade levels. The researcher decided to present a detailed discussion of the tests for grade 6 because at grade 6, students start constructing and formulating proofs and face the notion of proof for the first time. We assume that the way they learn about proving the first time affects their subsequent conceptions and skills. As for grade 7, the researcher presented a detailed analysis of the tests to explore how the abilities of the six graders to construct and formulate geometric proofs develop. For detailed information about grades 8 and 9 (see Appendices N& O).

Tables 7 to 18 show the percentages of the errors committed by grade 6 and 7 students in P1, P3, P2, and P4 respectively according to nature of the errors committed and the difficulty they reflect.

### 4.3.1 Analysis of grade 6 Tests 1 and 2

#### 4.3.1.1 Understanding the Notion of Proof (UNP) at grade 6

Table 7 represents the percentages of grade 6 students who committed errors that reflect difficulty regarding the understanding of the notion of proof (UNP) in problems P1, P3, P2, and P4.

In problem P1, which addresses complex proof tasks, the percentages of the students who had difficulty in Understanding the
**Notion of Proof** (UNP) varied between 4.55% and 47.73%. The major error was **writing proofs that contain gaps**. 47.73% of the grade 6 students **did not justify a crucial step of their proof modules** (NJCS). This shows the students’ inability to think of necessary statements of a proof and include all elements needed to verify a complex proof task. 40.91% of the grade 6 students **wrote statements that are not justified or verified** thus lacking validity (NJEV). 31.82% of the grade 6 students **did not differentiate between the premises given and their inferences** (NDPI). **Understanding the notion of proof** requires identification and differentiation between premises and inferences. 27.27% of the students **used premises that are not given** (PNG), thus building their proofs on empirical assumptions. They might have considered these premises because they appeared to be true in the geometric figures used, which shows lack of awareness that mathematical proofs should be built on valid statements. The two types of errors that were least committed by grade 6 students are ED (4.55%) and NL (6.82%). Only 4.55% of the students **developed proofs that contain extra details that do not contribute to the proof** (ED), and 6.82% of the students **constructed proofs that contain statements written in an illogical order** (NL). Since the highest percentage of students who have committed a type of errors under this
difficulty is 47.73%, we may safely infer that at least 47.73% of the grade 6 students have difficulties related to the understanding of the notion of the proof at Problem P1 level of complexity.

In problem P3, which is considered to address the same mathematical concepts addressed in P1, and to prove the same resulting statement, but with simplified intermediary proof tasks, the percentages of students committing errors reflecting difficulty in Understanding the Notion of Proof (UNP) ranged between 1.36% and 23.64%. It is clear that the percentages of the students committing most of the errors related to difficulty in Understanding the Notion of Proof became less upon reducing the complexity of the proof tasks such as writing proofs that contain statements that are not connected logically (NL), writing or considering premises that are not given (PNG), not differentiating between the premises and their inferences (NDPI), writing statements that are not justified (NJEV), and writing proofs that lack justification of a crucial step (NJCS). We notice that the percentage of students who wrote proofs that contain statements that are not connected logically (NL) dropped to almost its quarter (from 6.82% to 1.36%). Also, the percentage of the students who used premises that are not given (PNG) dropped to almost its third (from 27.27% to 10.45%). Maybe they felt
more comfortable to reconsider the given premises and to think more about them and try to use them instead of considering not given ones. The percentage of students who did not differentiate between premises of a proof and their inferences (NDPI) dropped to around its fifth (from 31.82% to 6.82%). The percentage of the students who wrote statements that are not justified (NJEV) dropped to around its half (from 40.91% to 23.64%). The percentage of the students who wrote proofs that lack justification of a crucial step (NJCS) dropped to less than its half (from 47.73% to 19.55%). However, the percentages of the students committing errors related to writing proofs that contain extra details (ED) approximately did not change. The student’s tendency to include extra details in their proofs is not necessarily related to the complexity of the proof task. However, when the proof tasks are simplified, students find it easier to justify and validate all statements and inferences.

Problems P2 and P4, which are considered to be isomathic, address the same simple proof tasks but with different levels of mathematical text complexity. The mathematical text in P2 is more complex than that in P4. It is noticed that the percentages of students committing errors related to the Understanding of the Notion of the Proof (UNP) are very close in problems P2 and P4. 3.03% and 3.79% of the
grade 6 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 2.27% of the grade 6 students in each of the problems P2 and P4 wrote proofs that contain extra details (ED). 0.76% and 0% of the students in problems P2 and P4 respectively used premises that are not given (PNG). In each of the problems P2 and P4, 12.88% of the grade 6 students did not justify a crucial step in their proofs (NJCS) and thus having a gap. This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and in some of them it is considered to be minimal. In problem P2, 2.27% of the students did not differentiate between the premises and their inferences (NDPI). However, 0.76% of them did not differentiate between the premises and their inferences (NDPI) in problem P4. In problem P2, 31.82% of the students wrote statements that are not justified or verified (NJVE). However, this percentage dropped to 22.73% in P4. These changes in the percentages of the students who committed such errors could be due the students’ inability to comprehend mathematical texts and when this factor is released in problem P4, the percentage of students committing these errors dropped to almost its fifth (NDPI) and to its two thirds (NJEV) in P4.
The highest percentages of students having committed errors reflecting the difficulty of *Understanding the Notion of Proof* (UNP) in each of problems P1, P2, P3 and P4 are respectively: 47.73%, 31.82%, 23.64%, and 22.73%. It is noticed that the highest percentage of the students who showed difficulty in *Understanding the Notion of the Proof* (UNP) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P2 (simple proof tasks; complex mathematical language). The third highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P4 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who *did not justify a crucial step in their proofs* (NJCS), had the highest percentage in P1 (47.73%) and dropped to around its half in P3 and to around its quarter in P2 and P4, and the percentage of the students who *did not justify or verify their statements* (NJEV) dropped from 40.91% in P1 to around its half in P3 and to around its two thirds in P2 and P4. Errors related to *using premises that are not given* (PNG), *writing statements of a proof in an illogical order* (NL), *writing extra details in a proof* (ED), the
differentiation between the premises of proof and their inferences (NDPI) are also affected by the variation of the proof and language complexity but not at similar rates from problem P1 to P3 and then from P2 to P4. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.
Table 7

*Grade 6 Errors Related to Understanding the Notion of Proof (UNP)*

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NL</td>
<td>6.82%</td>
<td>1.36%</td>
<td>3.03%</td>
<td>3.79%</td>
</tr>
<tr>
<td>ED</td>
<td>4.55%</td>
<td>5.45%</td>
<td>2.27%</td>
<td>2.27%</td>
</tr>
<tr>
<td>PNG</td>
<td>27.27%</td>
<td>10.45%</td>
<td>0.76%</td>
<td>0.00%</td>
</tr>
<tr>
<td>NDPI</td>
<td>31.82%</td>
<td>6.82%</td>
<td>2.27%</td>
<td>0.76%</td>
</tr>
<tr>
<td>NJEV</td>
<td>40.91%</td>
<td>23.64%</td>
<td>31.82%</td>
<td>22.73%</td>
</tr>
<tr>
<td>NJCS</td>
<td>47.73%</td>
<td>19.55%</td>
<td>12.88%</td>
<td>12.88%</td>
</tr>
<tr>
<td>LD (UNP)</td>
<td>47.73%</td>
<td>23.64%</td>
<td>31.82%</td>
<td>22.73%</td>
</tr>
</tbody>
</table>

NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
PNG: Used a Premise that is Not Given.
NDPI: Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.
NJEV: “Wrote a statement that wasn’t Justified, Explained or Verified”.
NJCS: “Did not sufficiently Justify a Crucial Step in a proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.1.2 Setting Proof Plans (SPP) at grade 6

Table 8 represents the percentages of grade 6 students who committed errors that reflect difficulty related to Setting Proof Plans (SPP) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Setting Proof Plans (SPP) varied between 2.27% and 11.36%. While 4.55% of the grade 6 students did not start a proof (NSP), maybe due to the complexity of the proof task in P1, 11.36% of the students started what could be a proof plan but did not complete it (SNPC) and 9.09% of the students decided on an approach that will not work to prove a statement (ANW). The major error, committed by 11.36% of the students, was starting what could be a proof plan but not completing it (SPNC). Students who started what could be a proof plan but were not able to complete it (SPNC) have a problem in setting proof plans that require multi-step proofs. It is noticed that 6.82% of the grade 6 students wrote the statements of a proof in an illogical order (NL). The two types of errors that were least committed by grade 6 students are UL (2.27%) and ED (4.55%). Since the highest percentage of students who have committed a type of error under this difficulty is 11.36%, we may safely
infer that at least 11.36% of the grade 6 students have difficulties related to Setting Proof Plans (SPP) at Problem P1 level of complexity.

In problem P3, which is considered to address the same mathematical concepts addressed in P1, and to prove the same resulting statement, but with simplified intermediary proof tasks, the percentages of students committing errors reflecting difficulty in Setting Proof Plans (SPP) ranged between 0.91% and 5.45%. Students’ tendency to commit errors such as starting what could be a proof plan and not completing it (SPNC), deciding on approaches that will not work to prove a statement (ANW), and writing statements of a proof in an illogical order (NL) was affected by the complexity level of the proof task. The percentage of the students who started what could be a proof plan but did not complete it (SPNC) dropped from 11.36% to around its eleventh (0.91%); the percentage of the students who set proving approaches that will not work (ANW) dropped from 9.09% to around its half (4.09%), and the percentage of the students wrote the statements of a proof in an illogical order (NL) dropped from 6.82% to around its quarter in P3. However, the percentages of the students who committed the errors (NSP), (ED), and (UL) were not affected by varying the complexity level of the proofs. As mentioned before, this could be due to the nature of the error itself.
Students, who did not start a proof, write lengthy proofs, and add extra details tend to commit such errors regardless of the proof complexity level.

Problems P2 and P4, which are considered to be isomathic, address the same simple proof tasks but with different levels of mathematical text complexity. It is noticed that some of the percentages of students committing errors related to Setting Proof Plans (SPP) were affected by varying the language complexity level between problems P2 and P4, and that other percentages were not affected. The percentage of students who did not start a proof (NSP) dropped from 9.09% to around its two thirds (6.06%). This shows the effect of language complexity of the mathematical texts on students’ ability to develop a proof plan. Due to the language complexity, some of the students were not able to either draw the figure or to comprehend the text of the proof task, thus not able to develop a proof. However, the percentages of the students, regarding other types of errors, were almost not affected by the change of language complexity level. 2.27% of the grade 6 students in each of the problems P2 and P4 wrote proofs that contain extra details (ED). 1.52% and 2.27% of the students in problems P2 and P4 respectively started what could be a proof plan but did not complete it (SPNC). 0% and 0.76% of the
students in problems P2 and P4 respectively were not able to set the appropriate approach to prove a statement (ANW). 3.03% and 4% of the grade 6 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 0% and 0.76% of the grade 6 students in problems P2 and P4 respectively wrote proofs that are unnecessarily lengthy (UL). This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and considered to be minimal.

The highest percentages of students having committed errors reflecting the difficulty in Setting Proof Plans (SPP) in each of problems P1, P2, P3 and P4 are respectively: 11.36%, 9.09%, 5.45%, and 6.06%. It is noticed that the highest percentage of the students who showed difficulty in Setting Proof Plans (SPP) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P2 (simple proof tasks; complex mathematical language). The third highest percentage was in problem P4 (simple proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the
students who started what could be a proof plan but did not complete it (SPNC) dropped from 11.36% in P1 to around its eleventh in P3 and P2 and to around its fifth in P4, and the percentage of the students who developed an inappropriate approach to proof a statement (ANW) had the highest percentage in P1 (9.09%) and dropped to around its half in P3 and to its lowest minimal percentages in P2 and P4. The errors related to ability to start a proof (NSP), the logical order of the statements of a proof (NL), to writing extra details in a proof (ED), and to developing unnecessarily lengthy proofs (UL) are also affected by the variation of the proof and language complexity but not at similar rates from problem P1 to P3 and then from P2 to P4. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.
Table 8

*Grade 6 Errors Related to Setting Proof Plans (SPP)*

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NSP</td>
<td>4.55%</td>
<td>3.64%</td>
<td>9.09%</td>
<td>6.06%</td>
</tr>
<tr>
<td>SPNC</td>
<td>11.36%</td>
<td>0.91%</td>
<td>1.52%</td>
<td>2.27%</td>
</tr>
<tr>
<td>ANW</td>
<td>9.09%</td>
<td>4.09%</td>
<td>0.00%</td>
<td>0.76%</td>
</tr>
<tr>
<td>NL</td>
<td>6.82%</td>
<td>1.36%</td>
<td>3.03%</td>
<td>4%</td>
</tr>
<tr>
<td>ED</td>
<td>4.55%</td>
<td>5.45%</td>
<td>2.27%</td>
<td>2.27%</td>
</tr>
<tr>
<td>UL</td>
<td>2.27%</td>
<td>2.73%</td>
<td>0.00%</td>
<td>0.76%</td>
</tr>
<tr>
<td>LD (SPP)</td>
<td>11.36%</td>
<td>5.45%</td>
<td>9.09%</td>
<td>6.06%</td>
</tr>
</tbody>
</table>

NSP: Did Not Start a Proof.
SPNC: Started what could be a Proof plan but did Not Complete the proof.
ANW: “The Approach taken in proving a statement will Not Work”.
NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
UL: “The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.1.3 Conducting Deductive Reasoning (CDR) at grade 6

Table 9 represents the percentages of grade 6 students who committed errors that reflect difficulty in Conducting Deductive Reasoning (CDR) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Conducting Deductive Reasoning (CDR) varied between 2.27% and 31.82%. The major errors were incorrectly claiming that a statement implied or equaled another statement and not differentiating between the premises given and their inferences. 31.82% of the grade 6 students incorrectly claimed that a statement implied or equaled another statement (ISIE), and did not differentiate between the premises given and their inferences (NDPI). This shows the students’ inability to derive appropriate inferences. It is well known that conducting deductive reasoning necessarily requires identification and differentiation between premises and inferences. 25% of the grade 6 students made false computations or wrote false statements (FSC). This shows their inability to find valid inferences. The three types of errors that were least committed by grade 6 students are NL (6.82%), ED (4.55%), and UL (2.27%). Only 6.82% of the students constructed proofs that contain statements written in an illogical order (NL). 4.55% of the students
developed proofs that contain extra details that do not contribute to the proof (ED) and only a minimal percentage of 2.27% of the students developed proofs that are unnecessarily lengthy (UL). Since the highest percentage of students who have committed a type of errors under this difficulty is 31.82%, we may safely infer that at least 31.82% of the grade 6 students have difficulties related to Conducting Deductive Reasoning (CDR) at Problem P1 level of complexity.

In problem P3, the percentages of students committing errors reflecting difficulty in Conducting Deductive Reasoning (CDR) ranged between 1.36% and 11.36%. It is clear that the percentages of the students committing some of the errors related to difficulty in conducting deductive reasoning became less upon reducing the complexity of the proof tasks such as incorrectly claiming that a statement equaled or implied another statement (ISIE), not differentiating between the premises and their inferences (NDPI), and writing proofs that contain false statements and/or computations (FSC), and writing proofs that contain statements written in an illogical order (NL). We notice that the percentage of students who incorrectly claimed that a statement equaled or implied another statement (ISIE) dropped to around its third (from 31.82% to 11.36%), and percentages of students who did not differentiate
between premises of a proof and their inferences (NDPI) and wrote false statements or made false computations (FSC) dropped to around its fifth (from 31.82% to 6.82%) and to its third (from 25% to 7.73%) respectively. Furthermore, the percentage of the students who wrote the statements of a proof in an illogical order (NL) dropped to around its quarter (from 6.82% to 1.36%) in P3. However, the percentages of the students committing errors related to writing proofs that contain extra details (ED), and developing unnecessarily lengthy proofs (UL) approximately did not change. This difference in the changes in the percentage of students committing errors is due to the nature of the error itself. The student’s tendency to include extra details in their proofs, or to develop lengthy proofs is not necessarily related to the complexity of the proof task. However, when the proof tasks are simplified, students will feel more comfortable to think of the validity of their inferences, and to reconsider their computations and conclusions.

In problems P2 and P4, it is noticed that the percentages of students committing errors related to Conducting Deductive Reasoning (CDR) are very close in problems P2 and P4. 3.03% and 4% of the grade 6 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 2.27% of the grade 6
students in each of the problems P2 and P4 wrote proofs that contain extra details (ED). 0% and 0.76% of the students in problems P2 and P4 wrote unnecessarily lengthy proofs (UL). This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and considered to be minimal. In problem P2, 2.27% of the students did not differentiate between the premises and their inferences (NDPI). However, 0.76% of them did not differentiate between the premises and their inferences (NDPI) in problem P4. This change in the percentage of the students who committed such error could be due the students’ inability to comprehend mathematical texts and when this factor is released in problem P4, the percentage of students committing this error dropped to almost its half. In problem P2, 1.52% of the students incorrectly claimed that a statement equaled or implied another statement (ISIE). However, this percentage changed to 3.03% in P4. Furthermore, in problem P2, 28.03% of the students wrote proofs that contain false statements or computations (FSC). Nevertheless, this percentage changed to 35.61% in P4. Having higher percentage of students committing such errors in problem P4 may be due to the fact that the complex text addressed in problem P2 may have prevented students from constructing and formulating a proof.
altogether thus may have prevented them from writing any statements. As for problem P4, and because of the simple language used, students were able to construct and formulate proofs but with inappropriate derivation of inferences and conclusions.

The highest percentages of students having committed errors reflecting the difficulty in Conducting Deductive Reasoning (CDR) in each of problems P1, P2, P3 and P4 are respectively: 31.82%, 28.03%, 11.36%, and 35.61%. It is noticed that the highest percentage of the students who showed difficulty in Conducting Deductive Reasoning (CDR) occurred in problem P4 (simple proof tasks; simple mathematical language). The next highest percentage was in problem P1 (complex proof tasks; simple mathematical language). The third highest percentage was in problem P2 (simple proof tasks; complex mathematical language). Finally, the fourth highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who wrote false statements or made false computations (FSC) had its highest percentage in P4 and dropped, at different rates, to lower values in problems P1, P2, and P3, and that the percentage of the students who incorrectly claimed that a
statement equaled or implied another one (ISIE) and did not differentiate between the premises of proof and their inferences (NDPI) had the highest percentages in P1 (31.82%) and dropped to around its third and fifth in P3 respectively, and became minimal in P2 and P4. Errors related to the logical order of the statements of a proof (NL), to writing extra details in a proof (ED), and to writing unnecessarily lengthy proofs (UL) were almost not affected by varying either the proof or the language complexity level. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.
Table 9

*Grade 6 Errors Related to Conducting Deductive Reasoning (CDR)*

<table>
<thead>
<tr>
<th>Conducting Deductive Reasoning (CDR)</th>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NL</td>
<td>6.82%</td>
<td>1.36%</td>
<td>3.03%</td>
<td>4%</td>
<td></td>
</tr>
<tr>
<td>ED</td>
<td>4.55%</td>
<td>5.45%</td>
<td>2.27%</td>
<td>2.27%</td>
<td></td>
</tr>
<tr>
<td>UL</td>
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<td>2.73%</td>
<td>0.00%</td>
<td>0.76%</td>
<td></td>
</tr>
<tr>
<td>ISIE</td>
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<td>1.52%</td>
<td>3.03%</td>
<td></td>
</tr>
<tr>
<td>NDPI</td>
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<td>2.27%</td>
<td>0.76%</td>
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</tr>
<tr>
<td>FSC</td>
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<td>28.03%</td>
<td>35.61%</td>
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</tr>
<tr>
<td>LD (CDR)</td>
<td>31.82%</td>
<td>11.36%</td>
<td>28.03%</td>
<td>35.61%</td>
<td></td>
</tr>
</tbody>
</table>

NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
UL: “The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.
ISIE: “Incorrectly claimed that one Statement Implied or Equaled another statement”.
NDPI: Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.
FSC: “Made a False Statement or incorrect Computation in the proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.1.4 Understanding-Applying Mathematical Concepts (UAMC) at grade 6

Table 10 represents the percentages of grade 6 students who committed errors that reflect difficulty in Understanding-Applying Mathematical Concepts (UAMC) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Understanding-Applying Mathematical Concepts (UAMC) varied between 0% and 31.82%. The major error was incorrectly claiming that a statement implied or equaled another one. 31.82% of the grade 6 students were not able to derive appropriate implications (ISIE). The students’ inability to appropriately derive inferences reflects their misunderstanding of some mathematical concepts. 25% of the students used inappropriate definition, theorem, or property to justify a statement (IDTP), and derived false statements and/or made false computations (FSC). The students’ inability to identify or recognize the existence of a false conclusion or computation, that might create some contradictions in the statements of the proof module, shows that they have issues related to appropriate Understanding-Applying of Mathematical Concepts (UAMC). It is noticed that 4.55% of the students made false assumptions (FA), and nobody (0%) of the students proved statements other than the
required ones (SOR). Since the highest percentage of students who have committed a type of errors under this difficulty is 31.82%, we may safely infer that at least 31.82% of the grade 6 students have difficulties related to the Understanding-Applying of Mathematical Concepts (UAMC) at Problem P1 level of complexity.

In problem P3, the isomathic problem to P1, the percentages of students committing errors reflecting difficulty in Understanding-Applying Mathematical Concepts (UAMC) ranged between 0.45% and 11.36%. It is clear that the percentages of the students committing some of the errors related to difficulty in understanding-applying mathematical concepts became less upon reducing the complexity of the proof tasks. We notice that the percentage of students who incorrectly claimed that a statement equaled or implied another one (ISIE) dropped to almost its third in P3 (from 31.82% to 11.36%), and that the percentage of the students who were not able to identify the appropriate definition, theorem, or property to justify a statement (IDTP) also dropped to around its third in P3 (from 25% to 7.27%). Furthermore, the percentage of the students who made false computations or statements (FSC) dropped to around its third in P3 (from 25% to 7.73%). Also, the percentage of the students who made false assumptions (FA) dropped to around its ninth in
P3 (from 4.55% to 0.45%). Knowing that errors that reflect misunderstanding of mathematical concepts are usually not affected by the complexity of the proof tasks, we may consider that the variation in the percentages of the students who committed errors related to the Understanding-Applying of Mathematical Concepts (UAMC) may be affected by a psychological aspect. When students find complexity in the proof tasks, they rush answers and become confused and thus, commit more errors. However, when the proof complexity is reduced, students are more comfortable to think about the addressed proof tasks and thus, commit fewer errors. Students’ tendency to prove statements other than the required ones (SOR) was not affected to variation of the proof complexity level and maintained its corresponding minimal percentage of students committing such error.

In problems P2 and P4, which are considered to be isomathic, the percentages of the grade 6 students who committed errors related to the Understanding-Applying of Mathematical Concepts (UAMC) are almost the same. Though the mathematical text in P2 is more complex than that in P4, the percentages of the students committing errors related to the Understanding-Applying of Mathematical Concepts (UAMC) were almost equivalent in problems P2 and P4. 0% and 0.76% of the grade 6
students in problems P2 and P4 respectively proved statements other than the required ones (SOR). 1.52% and 3.03% of the students in problems P2 and P4 respectively incorrectly claimed that a statement implied or equaled another statement (ISIE). 11.36% and 10.61% of the students in problems P2 and P4 respectively used the inappropriate definition, theorem, or property to justify a statement (IDTP). In each of the problems P2 and P4, 0% of the students made false assumptions (FA). Moreover, 28.03% and 35.61% of the students in problems P2 and P4 respectively made false computation or wrote a false statement (FSC). This shows that the language complexity level of the mathematical text in each of the isomatric problems P2 and P4 did not affect the students’ tendency to commit most of the errors related to difficulty in Understanding-Applying Mathematical Concepts (UAMC). Furthermore, the major error that reflected students’ difficulty in Understanding-Applying Mathematical Concepts (UAMC) was writing proofs that contain false statements or false computations (FSC).

The highest percentages of students having committed errors reflecting the difficulty of Understanding-Applying Mathematical Concepts (UAMC) in each of problems P1, P2, P3 and P4 are respectively: 31.82%, 28.03%, 11.36%, and 35.61%. It is noticed that the
highest percentage of the students who showed difficulty in

*Understanding-Applying Mathematical Concepts* (UAMC) occurred in problem P4 (simple proof tasks; simple mathematical language). The next highest percentage was in problem P1 (complex proof tasks; simple mathematical language). The third highest percentages were in problems P2 (simple proof tasks; complex mathematical language). Finally, the fourth highest percentage was in problem P3 (simple proof tasks; simple mathematical language). The variation of the complexity level of both proof and language in problems P1, P2, P3, and P4, led to affecting the percentage of the students who did not identify the appropriate definition, theorem, or property needed to justify a statement (IDTP) which dropped from its highest percentage in P1 (25%) to around its third in P3 and to around its half in P2 and P4, and the percentage of the students who incorrectly claimed that a statement equaled or implied another one (ISIE) dropped from its highest percentage (31.82%) in P1 to around its third in P3, and to become minimal in P2 and P4. However, the percentage of the students who made false computations or statements (FSC) was only affected by the variation of proof complexity between P1 and P3 dropped from 25% in P1 to 7.73% in P3. Errors related to proving statements other than the required ones (SOR) and to making false
assumptions (FA) were not affected by varying either the proof or the language complexity level.

Table 10

Grade 6 Errors Related to Understanding-Applying Mathematical Concepts (UAMC)

<table>
<thead>
<tr>
<th>Understanding-Applying Mathematical Concepts (UAMC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problems</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>Error</td>
</tr>
<tr>
<td>SOR</td>
</tr>
<tr>
<td>ISIE</td>
</tr>
<tr>
<td>IDTP</td>
</tr>
<tr>
<td>FA</td>
</tr>
<tr>
<td>FSC</td>
</tr>
<tr>
<td>LD (UMC)</td>
</tr>
</tbody>
</table>

SOR: Proved Statements Other than the Required ones.
ISIE: “Incorrectly claimed that one Statement Implied or Equaled another statement”.
IDTP: Used Irrelevant Definition, Theorem, or Property to Justify a Statement.
FA: “Made a False Assumption somewhere in the proof”.
FSC: “Made a False Statement or incorrect Computation in the proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.1.5 Comprehending Mathematical Texts (CMT) at grade 6

Table 11 represents the percentages of grade 6 students who committed errors that reflect difficulty in Comprehending Mathematical Texts (CMT) in problems P1, P3, P2, and P4.

In problem P1, the percentages of the students who had difficulty in Comprehending Mathematical Texts (CMT) were 4.55% and 0%. The only error was not starting a proof. 4.55% of the grade 6 students did not start a proof (NSP). This might show the students’ inability to comprehend the mathematical text of the proof task or that of the premises. Nevertheless, 0% of the students proved statements other than the required ones (SOR). Since the highest percentage of students who have committed a type of errors under this difficulty is 4.55%, we may safely infer that at least 4.55% of the grade 6 students have difficulties related to Comprehending Mathematical Texts at Problem P1 level of complexity.

In problem P3, the percentages of students committing errors reflecting difficulty in Comprehending Mathematical Texts (CMT) were 3.64% and 0.45%. It is clear that the percentages of the students committing some of the errors related to difficulty in comprehending
mathematical texts almost did not change upon reducing the complexity of the proof tasks.

In problem P2, the percentages of the grade 6 students committing errors related to difficulty in *Comprehending Mathematical Texts* (CMT) were: 37.5%, 9.09%, and 0%. The major error committed by grade 6 students was *drawing a wrong figure* (DWF). 37.5% of the students *did not draw a correct figure* may be due to their inability to comprehend the text of either the premises. Errors detected in the drawn figures reflected the students’ inability to comprehend mathematical texts that are considered to be complex in problem P2. 9.09% of the students *did not start a proof* (NSP). 0% of the students *proved statements other than the required ones* (SOR). Since the highest percentage of students who have committed a type of errors under this difficulty is 37.5%, we may safely infer that at least 37.5% of the grade 6 students have difficulties related to *Comprehending Mathematical Texts* (CMT) at Problem P2 level of complexity.

In problem P4, it is noticed that the percentages of the grade 6 students committing errors related to *Comprehending Mathematical Texts* (CMT) almost did not change upon reducing the complexity level of the mathematical texts and were as follows: DWF (47%), NSP (6.06%), and
SOR (0.76%). This constancy in results shows that the grade 6 students have, regardless of the text complexity level, difficulties in comprehending mathematical texts.

The highest percentages of students having committed errors reflecting the difficulty of Comprehending Mathematical Texts (CMT) in each of problems P1, P2, P3 and P4 are respectively: 4.55%, 37.5%, 3.64%, and 47%. It is noticed that the highest percentage of the students who showed difficulty in Comprehending Mathematical Texts (CMT) occurred in problem P4 (simple proof tasks; simple mathematical language). The next highest percentage was in problem P2 (simple proof tasks; complex mathematical language). The third highest percentage was in problem P1 (complex proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who did not start a proof (NSP) had the highest percentage in P2 (9.09%) and dropped to around its half in P1 and P3 and to its two thirds in P4, and the percentage of the students who drew a wrong figure (DWF) ranged between 37.5% and 47% in P2 and P4 respectively. However, the percentage of the students
who proved statements other than the required ones (SOR) was almost
the same in problems P1, P2, P3, and P4 and thus, not affected by the
variation of both proof and language complexities.

Table 11

*Grade 6 Errors Related to Comprehending Mathematical Texts (CMT)*

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>DWF</td>
<td>N/A</td>
<td>N/A</td>
<td>37.50%</td>
<td>47%</td>
</tr>
<tr>
<td>SOR</td>
<td>0%</td>
<td>0.45%</td>
<td>0.00%</td>
<td>0.76%</td>
</tr>
<tr>
<td>NSP</td>
<td>4.55%</td>
<td>3.64%</td>
<td>9.09%</td>
<td>6.06%</td>
</tr>
<tr>
<td>LD (CMT)</td>
<td>4.55%</td>
<td>3.64%</td>
<td>37.50%</td>
<td>46.59%</td>
</tr>
</tbody>
</table>

DWF: Drew a Wrong Figure.
SOR: Proved Statements Other than the Required ones.
NSP: Did Not Start a Proof.
N/A: Not applicable. The problem doesn’t require testing this error.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.1.6 Writing Mathematical Texts (WMT) at grade 6

Table 12 represents the percentages of grade 6 students who committed errors that reflect difficulty in Writing Mathematical Texts (WMT) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Writing Mathematical Texts (WMT) varied between 2.27% and 11.36%. The major error was writing mathematical symbols incorrectly. 11.36% of the grade 6 students did not write mathematical symbols correctly (MSI). This shows that 11.36% of the grade 6 students were not capable of differentiating between various mathematical symbols and identifying the ones needed to address the required mathematical meaning. 6.82% of the students wrote ambiguous statements or paragraphs (SA). 4.55% of the students used inappropriate mathematical vocabulary (VI). 2.27% of the students did not use logical connectors correctly (LCI) and wrote an inappropriate text of the definitions, theorems, or properties (DTPI).

Since the highest percentage of students who have committed a type of errors under this difficulty is 11.36%, we may safely infer that at least 11.36% of the grade 6 students have difficulties related to Writing Mathematical Texts (WMT) at Problem P1 level of complexity.
In problem P3, the percentages of students committing errors reflecting difficulty in Writing Mathematical Texts (WMT) ranged between 1.36% and 3.64%. It is clear that the percentages of the students committing some of the errors related to difficulty in writing mathematical texts changed upon reducing the complexity of the proof tasks. The percentage of the students who used mathematical symbols incorrectly (MSI) dropped to around its quarter in P3 (from 11.36% to 3.64%), and the percentages of the students who wrote ambiguous statements (SA) and used mathematical vocabulary incorrectly (VI) each dropped to around its half in P3 (from 6.82% to 3.64%) and (4.55% to 2%) respectively. However, the percentages of the students who wrote inappropriate statement of definitions, theorems, or properties (DTPI) and used logical connectors inappropriately (LCI) did not change upon varying the proof complexity. This shows that the students’ ability to write mathematical texts correctly is not affected by the proof tasks complexity level.

In problem P2, the percentages of the grade 6 students committing errors related to difficulty in Writing Mathematical Texts (WMT) varied between 0.76% and 7.58%. The major error committed by grade 6 students was writing inappropriate statements of the definitions,
theorems, and properties (DTPI). 7.58% of the students wrote the statements of the definitions, theorems, or properties incorrectly. 5.3% of the students wrote mathematical symbols incorrectly (MSI). As for the other errors related to difficulty in Writing Mathematical Texts (WMT), the percentages of the students who committed these errors were minimal. 3.03% of the students wrote statements that were ambiguous (SA), 2.27% of the students used mathematical vocabulary incorrectly (VI), and 0.76% of the students had difficulty in using logical connectors in a mathematical text (LCI). These low percentages do not necessarily imply that the grade 6 students are capable of writing mathematical texts correctly. Rather, this could be due to the fact that the grade 6 students did not write a proof, justify their proof statements, or use logical connectors in their proofs. However, and since the highest percentage of students who have committed a type of errors under this difficulty is 7.58%, we may safely infer that at least 7.58% of the grade 6 students have difficulties related to writing mathematical texts at Problem P2 level of complexity.

In problem P4, which is considered to be isomorphic to P2, it is noticed that the percentages of students committing some errors related to Writing Mathematical Texts (WMT) are very close in problems P2 and
P4 and were not affected by varying the level of the text complexity. 5.3% and 6.82% of the students in problems P2 and P4 respectively did not use mathematical symbols correctly (MSI). 7.58% of the students in each of the problems P2 and P4 wrote inappropriate statements of the definitions, theorems, or properties (DTPI). This shows that the students’ tendency to commit such errors is not affected. Rather, it is almost constant. Hence, the students’ ability to Write Mathematical Texts (WMT) is not affected by the complexity of the mathematical text.

However, in P4, the percentage of the students who wrote ambiguous statements (SA), used mathematical vocabulary incorrectly (VI), and used logical connectors incorrectly (LCI) doubled upon reducing the text complexity. This raise in percentages might be due to the fact that upon reducing the mathematical text complexity, students were able to develop proofs and thus commit more errors.

The highest percentages of students having committed errors reflecting the difficulty of Writing Mathematical Texts (WMT) in each of problems P1, P2, P3 and P4 are respectively: 11.36%, 7.58%, 3.64%, and 7.58%. It is noticed that the highest percentage of the students who showed difficulty in Writing Mathematical Texts (WMT) occurred in problem P1 (complex proof tasks; simple mathematical language). The
next highest percentage was in problems P2 (simple proof tasks; complex mathematical language) and P4 (simple proof tasks; simple mathematical language). The third highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who used mathematical symbols incorrectly (MSI) had the highest percentage in P1 (11.36%) and dropped to around its quarter in P3 and to around its half in P2 and P4, and the percentage of the students who wrote inappropriate statement of the definitions, theorems, and properties (DTPI) dropped from 7.58% in P2 and P4 to around its fifth in P1 and P3. Furthermore, the percentage of the students who wrote ambiguous statements of a proof (SA) dropped from 6.82% in P1 to around its half in P3 and P4 and to become negligible in P2. This might be due to the difference in the nature of the statements of the definitions, theorems, and properties addressed in each pair of the isomathic problems. Errors related to writing appropriate mathematical terms and vocabulary (VI), and to using logical connectors appropriately (LCI) were not affected by varying either the proof or the language complexity level.
Table 12

Grade 6 Errors Related to Writing Mathematical Texts (WMT)

<table>
<thead>
<tr>
<th>Writing Mathematical Texts (WMT)</th>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td></td>
</tr>
<tr>
<td>SA</td>
<td>6.82%</td>
<td>3.64%</td>
<td>0.76%</td>
<td>2.27%</td>
<td></td>
</tr>
<tr>
<td>MSI</td>
<td>11.36%</td>
<td>3.64%</td>
<td>5.30%</td>
<td>6.82%</td>
<td></td>
</tr>
<tr>
<td>VI</td>
<td>4.55%</td>
<td>2%</td>
<td>2.27%</td>
<td>4.55%</td>
<td></td>
</tr>
<tr>
<td>DTPI</td>
<td>2.27%</td>
<td>1.36%</td>
<td>7.58%</td>
<td>7.58%</td>
<td></td>
</tr>
<tr>
<td>LCI</td>
<td>2.27%</td>
<td>1.82%</td>
<td>3.03%</td>
<td>6.82%</td>
<td></td>
</tr>
<tr>
<td>LD (WMT)</td>
<td>11.36%</td>
<td>3.64%</td>
<td>7.58%</td>
<td>7.58%</td>
<td></td>
</tr>
</tbody>
</table>

SA: “Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex”.
MSI: Used Mathematical Symbols Incorrectly.
VI: Used Mathematical terms, words, or Vocabulary Incorrectly.
DTPI: Wrote statement of Definitions, Properties, or Theorems Incorrectly.
LCI: Used Logical Connectors Incorrectly.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.2 Analysis of grade 7 Tests 1 and 2

4.3.2.1 Understanding the Notion of Proof (UNP) at grade 7

Table 13 represents the percentages of grade 7 students who committed errors that reflect difficulty regarding the Understanding of the Notion of Proof (UNP) in problems P1, P3, P2, and P4.

In problem P1, which addresses complex proof tasks, the percentages of students who had difficulty in Understanding the Notion of Proof (UNP) varied between 1.85% and 25.93%. The major error was writing proofs that contain gaps. 25.93% of the grade 7 students did not justify a crucial step of their proof modules (NJCS). This shows the students’ inability to think of necessary statements of a proof and include all elements needed to verify a complex proof task. 14.81% of the grade 7 students did not differentiate between the premises given and their inferences (NDPI). Understanding the notion of proof requires identification and differentiation between premises and inferences. 14.81% of the grade 7 students wrote statements that are not justified or verified thus lacking validity (NJEV). 12.96% of the grade 7 students used premises that are not given (PNG), thus building their proofs on empirical assumptions. They might have considered these premises
because they appeared to be true in the geometric figures used, which shows lack of awareness that mathematical proofs should be built on valid statements. The two types of errors that were least committed by grade 7 students are ED (7.41%) and NL (1.85%). Only 7.41% of the students developed proofs that contain extra details that do not contribute to the proof (ED) and only a minimal percentage of 1.85% constructed proofs that contain statements written in an illogical order (NL). Since the highest percentage of students who have committed a type of errors under this difficulty is 25.93%, we may safely infer that at least 25.93% of the grade 7 students have difficulties related to the understanding of the notion of the proof at Problem P1 level of complexity.

In problem P3, which is considered to address the same mathematical concepts addressed in P1, and to prove the same resulting statement, but with simplified intermediary proof tasks, the percentages of students committing errors reflecting difficulty in Understanding the Notion of Proof (UNP) ranged between 1.06% and 9.52%. It is clear that the percentages of the students committing some of the errors related to difficulty in understanding the notion of proof became less upon reducing the complexity of the proof tasks such as writing or considering premises that are not given (PNG), not differentiating between the premises and
their inferences (NDPI), writing statements that are not justified (NJEV), and writing proofs that lack justification of a crucial step (NJCS). We notice that the percentage of students who used premises that are not given (PNG) dropped to almost its quarter (from 12.96% to 3.17%). Maybe they felt more comfortable to reconsider the given premises and to think more about them and try to use them instead of considering not given ones. The percentage of students who did not differentiate between premises of a proof and their inferences (NDPI) dropped to around its fifth (from 14.81% to 2.65%). The percentage of the students who wrote statements that are not justified (NJEV) dropped to around its half (from 14.81% to 6.88%). The percentage of the students who wrote proofs that lack justification of a crucial step (NJCS) dropped to around its third (from 25.93% to 9.52%). However, the percentages of the students committing errors related to writing proofs that contain statements written in an illogical order (NL) and to writing proofs that contain extra details (ED) approximately did not change. This difference in the changes in the percentage of students committing errors is due to the nature of the error itself. The student’s tendency to write statements in an illogical order or to include extra details in their proofs is not necessarily related to the complexity of the proof task. However, when the proof tasks are
simplified, students find it easier to justify and validate all statements and inferences.

Problems P2 and P4, which are considered to be isomathic, address the same simple proof tasks but with different levels of mathematical text complexity. The mathematical text in P2 is more complex than that in P4. It is noticed that the percentages of students committing errors related to the Understanding of the Notion of the Proof (UNP) are very close in problems P2 and P4. 1.06% and 0% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 5.29% and 5.82% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain extra details (ED). 2.65% of the students in each of problems P2 and P4 used premises that are not given (PNG). In problems P2 and P4, 8.99% and 9.52% of the grade 7 students respectively did not justify a crucial step in their proofs (NJCS) and thus having a gap. This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and considered to be minimal. In problem P2, 7.41% of the students did not differentiate between the premises and their inferences (NDPI). However, 4.76% of them did not differentiate between the
premises and their inferences (NDPI) in problem P4. This change in the percentage of the students who committed such error could be due the students’ inability to comprehend mathematical texts and when this factor is released in problem P4, the percentage of students committing this error dropped to almost its half. In problem P2, 6.88% of the students wrote statements that are not justified or verified (NJVE). However, this percentage changed to 12.17% in P4. Having higher percentage of students committing such error in problem P4 may be due to the fact that the complex text addressed in problem P2 may have prevented students from constructing and formulating a proof altogether thus may have prevented them from writing any statements. As for problem P4, and because of the simple language used, students were able to construct and formulate proofs but lacking justification of statements.

The highest percentages of students having committed errors reflecting the difficulty of Understanding the Notion of Proof (UNP) in each of problems P1, P2, P3 and P4 are respectively: 25.93%, 8.99%, 9.52%, and 12.17%. It is noticed that the highest percentage of the students who showed difficulty in Understanding the Notion of the Proof (UNP) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P4
(simple proof tasks; simple mathematical language). The third highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P2 (simple proof tasks; complex mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who did not justify a crucial step in their proofs (NJCS) had the highest percentage in P1 (25.93%) and dropped to around its third in P3, P2, and P4 and the percentage of the students who used premises that are not given (PNG) dropped from 12.96% in P1 to around its quarter in P3, P2, and P4. Errors related to the logical order of the statements of a proof (NL) and to writing extra details in a proof (ED) were not affected by varying either the proof or the language complexity level. Errors related to the differentiation between the premises of proof and their inferences (NDPI) and to the justification of the proof statements (NJEV) are also affected by the variation of the proof and language complexity but not at similar rates from problem P1 to P3 and then from P2 to P4. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.
Table 13

*Grade 7 Errors Related to Understanding the Notion of Proof (UNP)*

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NL</td>
<td>1.85%</td>
<td>1.06%</td>
<td>1.06%</td>
<td>0%</td>
</tr>
<tr>
<td>ED</td>
<td>7.41%</td>
<td>7.41%</td>
<td>5.29%</td>
<td>5.82%</td>
</tr>
<tr>
<td>PNG</td>
<td>12.96%</td>
<td>3.17%</td>
<td>2.65%</td>
<td>2.65%</td>
</tr>
<tr>
<td>NDPI</td>
<td>14.81%</td>
<td>2.65%</td>
<td>7.41%</td>
<td>4.76%</td>
</tr>
<tr>
<td>NJEV</td>
<td>14.81%</td>
<td>6.88%</td>
<td>6.88%</td>
<td>12.17%</td>
</tr>
<tr>
<td>NJCS</td>
<td>25.93%</td>
<td>9.52%</td>
<td>8.99%</td>
<td>9.52%</td>
</tr>
<tr>
<td>LD (UNP)</td>
<td>25.93%</td>
<td>9.52%</td>
<td>8.99%</td>
<td>12.17%</td>
</tr>
</tbody>
</table>

NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
PNG: Used a Premise that is Not Given.
NDPI: Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.
NJEV: “Wrote a statement that wasn’t Justified, Explained or Verified”.
NJCS: “Did not sufficiently Justify a Crucial Step in a proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.2.2 Setting Proof Plans (SPP) at grade 7

Table 14 represents the percentages of grade 7 students who committed errors that reflect difficulty related to Setting Proof Plans (SPP) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Setting Proof Plans (SPP) varied between 1.85% and 16.67%. While 5.56% of the grade 7 students did not start a proof (NSP), maybe due to the complexity of the proof task in P1, 11.11% of the students started what could be a proof plan but did not complete it (SNPC) and 16.67% of the students decided on an approach that will not work to prove a statement (ANW). The major error, committed by 16.67% of the students, was taking approaches to prove a statement that will not work (ANW). This shows the low capability of the students to determine an appropriate approach to prove a complex proof task. Students who started what could be a proof plan but were not able to complete it (SPNC) have a problem in setting proof plans that require multi-step proofs. It is noticed that 7.41% of the grade 7 students developed proofs that contain extraneous details (ED). This shows the students’ inability to decide on the necessary elements and proof modules for an appropriate proof plan. The two types of errors that were least committed by grade 7 students are...
UL (1.85%) and NL (1.85%). Since the highest percentage of students who have committed a type of error under this difficulty is 16.67%, we may safely infer that at least 16.67% of the grade 7 students have difficulties related to Setting Proof Plans (SPP) at Problem P1 level of complexity.

In problem P3, which is considered to address the same mathematical concepts addressed in P1, and to prove the same resulting statement, but with simplified intermediary proof tasks, the percentages of students committing errors reflecting difficulty in Setting Proof Plans (SPP) ranged between 1.06% and 7.94%. Students’ tendency to commit errors such as starting what could be a proof plan and not completing it (SPNC) and deciding on approaches that will not work to prove a statement (ANW) was affected by the complexity level of the proof task. The percentage of the students who started what could be a proof plan but did not complete it (SPNC) dropped from 11.11% to around its quarter (3.17%), and the percentage of the students who set proving approaches that will not work (ANW) dropped from 16.67% to around its third (5.82%). However, the percentage of the students who did not start a proof (NSP) became more, at a small rate of change, upon varying the complexity level of the proof from a complex one to a simple proof task.
It changed from 5.56% to 7.94%. Having higher percentage of students committing such error in P3 could be due to the bigger number of proof tasks that are addressed in P3 and the variety of mathematical concepts that are addressed in each. The percentages of the students who committed the errors (NL), (ED), and (UL) were not affected by varying the complexity level of the proofs. As mentioned before, this could be due to the nature of the error itself. Students who write lengthy proofs, add extra details, and write statements that are not arranged in a logical order tend to commit such errors regardless of the proof complexity level.

In problem P2 which addresses simple proof tasks using a complex mathematical language, the percentages of the grade 7 students committing errors related to difficulty in Setting Proof Plans (SPP) ranged between 1.06% and 10.58%. The major error committed was not starting a proof (NSP). Taking into consideration the factor of the complex language and the simple proof tasks addressed in P2, we recognized that language complexity prevented 10.58% of the grade 7 students from starting a proof (NSP) and 6.35% from determining the appropriate approach for proving a statement (ANW). This showed the students’ inability to comprehend the text of the premises and the proof
tasks and be able to develop appropriate proof plans. The other types of errors committed by students and that reflect difficulty in *Setting Proof plans* (SPP) such as SPNC (3.17%), NL (1.06%), ED (5.29%), and UL (3.7%) were considered to have minimal percentages. Thus, they are not altered by the complexity level of language addressed in problem P2.

Problems P2 and P4, which are considered to be isomathic, address the same simple proof tasks but with different levels of mathematical text complexity. It is noticed that some of the percentages of students committing errors related to *Setting Proof Plans* (SPP) were affected by varying the language complexity level between problems P2 and P4, and that other percentages were not affected. The percentage of students who *did not start a proof* (NSP) dropped from 10.58% to around its half (6.35%). This shows the effect of language complexity of the mathematical texts on students’ ability to develop a proof plan. Due to the language complexity, some of the students were not able to either draw the figure or to comprehend the text of the proof task, thus not able to develop a proof. However, the percentages of the students, regarding other types of errors, were almost not affected by the change of language complexity level. 3.17% of the grade 7 students in each of the problems P2 and P4 *started what could be a proof plan but did not complete*
6.35% and 5.82% of the students in problems P2 and P4 respectively were not able to set the appropriate approach to prove a statement (ANW). 1.06% and 0% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 5.29% and 5.82% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain extra details (ED). 3.7% and 3.17% of the students in problems P2 and P4 respectively wrote proofs that are unnecessarily lengthy (UL). This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and considered to be minimal.

The highest percentages of students having committed errors reflecting the difficulty in Setting Proof Plans (SPP) in each of problems P1, P2, P3 and P4 are respectively: 16.67%, 10.58%, 7.94%, and 6.35%. It is noticed that the highest percentage of the students who showed difficulty in Setting Proof Plans (SPP) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P2 (simple proof tasks; complex mathematical language). The third highest percentage was in problem P3 (simple proof tasks; simple mathematical language). Finally, the fourth highest percentage
was in problem P4 (simple proof tasks; simple mathematical language).

Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who were developed an inappropriate approach to proof a statement (ANW) had the highest percentage in P1 (16.67%) and dropped to around its third in P3, P2, and P4 and that the percentage of the students who started what could be a proof plan but did not complete it (SPNC) dropped from 11.11% in P1 to around its third in P3, P2, and P4.

The error related to ability to start a proof (NSP) is also affected by the variation of the proof and language complexity but not at similar rates from problem P1 to P3 and then from P2 to P4. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.

Errors related to the logical order of the statements of a proof (NL), to writing extra details in a proof (ED), and to developing unnecessarily lengthy proofs (UL) were not affected by varying either the proof or the language complexity level.
Table 14

*Grade 7 Errors Related to Setting Proof Plans (SPP)*

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NSP</td>
<td>5.56%</td>
<td>7.94%</td>
<td>10.58%</td>
<td>6.35%</td>
</tr>
<tr>
<td>SPNC</td>
<td>11.11%</td>
<td>3.17%</td>
<td>3.17%</td>
<td>3.17%</td>
</tr>
<tr>
<td>ANW</td>
<td>16.67%</td>
<td>5.82%</td>
<td>6.35%</td>
<td>5.82%</td>
</tr>
<tr>
<td>NL</td>
<td>1.85%</td>
<td>1.06%</td>
<td>1.06%</td>
<td>0%</td>
</tr>
<tr>
<td>ED</td>
<td>7.41%</td>
<td>7.41%</td>
<td>5.29%</td>
<td>5.82%</td>
</tr>
<tr>
<td>UL</td>
<td>1.85%</td>
<td>1.59%</td>
<td>3.70%</td>
<td>3.17%</td>
</tr>
<tr>
<td>DL (SPP)</td>
<td>16.67%</td>
<td>7.94%</td>
<td>10.58%</td>
<td>6.35%</td>
</tr>
</tbody>
</table>

NSP: Did Not Start a Proof.
SPNC: Started what could be a Proof plan but did Not Complete the proof.
ANW: “The Approach taken in proving a statement will Not Work”.
NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
UL: “The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.2.3 Conducting Deductive Reasoning (CDR) at grade 7

Table 15 represents the percentages of grade 7 students who committed errors that reflect difficulty in *Conducting Deductive Reasoning* (CDR) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in *Conducting Deductive Reasoning* (CDR) varied between 1.85% and 22.22%. The major error was *incorrectly claiming that a statement implied or equaled another statement*. 22.22% of the grade 7 students *incorrectly claimed that a statement implied or equaled another statement* (ISIE). This shows the students’ inability to derive appropriate inferences. 14.81% of the grade 7 students *did not differentiate between the premises given and their inferences* (NDPI). *Conducting deductive reasoning* necessarily requires identification and differentiation between premises and inferences. 14.81% of the grade 7 students *made false computations or wrote false statements* (FSC). This shows their inability to find valid inferences. The three types of errors that were least committed by grade 7 students are ED (7.41%), NL (1.85%), and UL (1.85%). Only 7.41% of the students *developed proofs that contain extra details that do not contribute to the proof* (ED) and only a minimal percentage of 1.85% *constructed proofs that contain statements written in*...
an illogical order (NL) and that are unnecessarily lengthy (UL). Since the highest percentage of students who have committed a type of errors under this difficulty is 22.22%, we may safely infer that at least 22.22% of the grade 7 students have difficulties related to conducting deductive reasoning at Problem P1 level of complexity.

In problem P3, the percentages of students committing errors reflecting difficulty in Conducting Deductive Reasoning (CDR) ranged between 1.06% and 7.41%. It is clear that the percentages of the students committing some of the errors related to difficulty in conducting deductive reasoning became less upon reducing the complexity of the proof tasks such as incorrectly claiming that a statement equaled or implied another statement (ISIE), not differentiating between the premises and their inferences (NDPI), and writing proofs that contain false statements and/or computations (FSC). We notice that the percentage of students who incorrectly claimed that a statement equaled or implied another statement (ISIE) dropped to around its third (from 22.22% to 6.88%), and percentages of students who did not differentiate between premises of a proof and their inferences (NDPI) and wrote false statements or made false computations (FSC) dropped to around its fifth (from 14.81% to 2.65%) and (from 14.81% to 3.7%) respectively.
However, the percentages of the students committing errors related to writing proofs that contain statements written in an illogical order (NL), writing proofs that contain extra details (ED), and developing unnecessarily lengthy proofs (UL) approximately did not change. This difference in the changes in the percentage of students committing errors is due to the nature of the error itself. The student’s tendency to write statements in an illogical order, to include extra details in their proofs, or to develop lengthy proofs is not necessarily related to the complexity of the proof task. However, when the proof tasks are simplified, students will feel more comfortable to think of the validity of their inferences, and to reconsider their computations and conclusions.

In problems P2 and P4, it is noticed that the percentages of students committing errors related to Conducting Deductive Reasoning (CDR) are very close in problems P2 and P4. 1.06% and 0% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain statements written in an illogical order (NL). 5.29% and 5.82% of the grade 7 students in problems P2 and P4 respectively wrote proofs that contain extra details (ED). 3.7% and 3.17% of the students in problems P2 and P4 wrote unnecessarily lengthy proofs (UL). 6.88% and 5.29% of the students in problems P2 and P4 respectively wrote false statements or
made false computations (FSC). This shows that despite of the lower mathematical text complexity, students’ tendency to commit such errors is not affected. Rather, it is almost constant and considered to be minimal. In problem P2, 7.41% of the students did not differentiate between the premises and their inferences (NDPI). However, 4.76% of them did not differentiate between the premises and their inferences (NDPI) in problem P4. This change in the percentage of the students who committed such error could be due the students’ inability to comprehend mathematical texts and when this factor is released in problem P4, the percentage of students committing this error dropped to almost its half. In problem P2, 6.35% of the students incorrectly claimed that a statement equaled or implied another statement (ISIE). However, this percentage changed to 8.99% in P4. Having higher percentage of students committing such error in problem P4 may be due to the fact that the complex text addressed in problem P2 may have prevented students from constructing and formulating a proof altogether thus may have prevented them from writing any statements. As for problem P4, and because of the simple language used, students were able to construct and formulate proofs but with inappropriate derivation of inferences and conclusions.
The highest percentages of students having committed errors reflecting the difficulty in Conducting Deductive Reasoning (CDR) in each of problems P1, P2, P3 and P4 are respectively: 22.22%, 7.41%, 7.41%, and 8.99%. It is noticed that the highest percentage of the students who showed difficulty in Conducting Deductive Reasoning (CDR) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P4 (simple proof tasks; simple mathematical language). The third highest percentage was in problems P2 and P3 (simple proof tasks; complex mathematical language) and (simple proof tasks; simple mathematical language) respectively. Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who incorrectly claimed that a statement equaled or implied another one (ISIE) had the highest percentage in P1 (22.22%) and dropped to around its third in P3, P2, and P4. Errors related to the logical order of the statements of a proof (NL), to writing extra details in a proof (ED), and to writing unnecessarily lengthy proofs (UL) were not affected by varying either the proof or the language complexity level. Errors related to the differentiation between the premises of proof and their inferences (NDPI) and to writing false statements or making
false computations (FSC) are also affected by the variation of the proof and language complexity but not at similar rates from problem P1 to P3 and then from P2 to P4. This could be due to the nature of the mathematical concepts that are addressed in each of P1 and P3, and P2 and P4.
Table 15

*Grade 7 Errors Related to Conducting Deductive Reasoning (CDR)*

<table>
<thead>
<tr>
<th>Conducting Deductive Reasoning (CDR)</th>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>NL</td>
<td>1.85%</td>
<td>1.06%</td>
<td>1.06%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>ED</td>
<td>7.41%</td>
<td>7.41%</td>
<td>5.29%</td>
<td>5.82%</td>
<td></td>
</tr>
<tr>
<td>UL</td>
<td>1.85%</td>
<td>1.59%</td>
<td>3.70%</td>
<td>3.17%</td>
<td></td>
</tr>
<tr>
<td>ISIE</td>
<td>22.22%</td>
<td>6.88%</td>
<td>6.35%</td>
<td>8.99%</td>
<td></td>
</tr>
<tr>
<td>NDPI</td>
<td>14.81%</td>
<td>2.65%</td>
<td>7.41%</td>
<td>4.76%</td>
<td></td>
</tr>
<tr>
<td>FSC</td>
<td>14.81%</td>
<td>3.70%</td>
<td>6.88%</td>
<td>5.29%</td>
<td></td>
</tr>
<tr>
<td>LD (CDR)</td>
<td>22.22%</td>
<td>7.41%</td>
<td>7.41%</td>
<td>8.99%</td>
<td></td>
</tr>
</tbody>
</table>

NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
ED: “The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.
UL: “The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.
ISIE: “Incorrectly claimed that one Statement Implied or Equaled another statement”.
NDPI: Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.
FSC: “Made a False Statement or incorrect Computation in the proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.2.4 Understanding-Applying Mathematical Concepts (UAMC) at grade 7

Table 16 represents the percentages of grade 7 students who committed errors that reflect difficulty in Understanding-Applying Mathematical Concepts (UAMC) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in Understanding-Applying Mathematical Concepts (UAMC) varied between 0% and 27.78%. The major error was using irrelevant definition, theorem, or property to justify a statement. 27.78% of the grade 7 students were not able to determine the appropriate definition, theorem, or property that they should use to justify a statement (IDTP). The students’ inability to appropriately match statements and reasons reflects their misunderstanding of some mathematical concepts. 22.22% of the students incorrectly claimed that a statement implied or equaled another one (ISIE). This also shows that these students did not grasp correctly the mathematical concepts used. 14.81% of the grade 7 students derived false statements and/or made false computations (FSC). The students’ inability to identify or recognize the existence of a false conclusion or computation, that might create some contradictions in the statements of the proof module, shows that they have issues related to appropriate
Understanding-Applying Mathematical Concepts (UAMC). It is noticed that nobody (0%) of the grade 7 students proved statements other than the required ones (SOR) or made false assumptions (FA). Since the highest percentage of students who have committed a type of errors under this difficulty is 27.78%, we may safely infer that at least 27.78% of the grade 7 students have difficulties related to the Understanding-Applying Mathematical Concepts (UAMC) at Problem P1 level of complexity.

In problem P3, the isomathic problem to P1, the percentages of students committing errors reflecting difficulty in Understanding-Applying Mathematical Concepts (UAMC) ranged between 1.06% and 10.05%. It is clear that the percentages of the students committing some of the errors related to difficulty in understanding-applying mathematical concepts became less upon reducing the complexity of the proof tasks. We notice that the percentage of students who incorrectly claimed that a statement equaled or implied another one (ISIE) dropped to almost its third in P3 (from 22.222% to 6.88%), and that the percentage of the students who were not able to identify the appropriate definition, theorem, or property to justify a statement (IDTP) also dropped to around its third in P3 (from 27.78% to 10.05%). Furthermore, the percentage of the students who made false computations or statements (FSC) dropped
to around its fifth in P3 (from 14.81% to 3.7%). Knowing that errors that reflect misunderstanding of mathematical concepts are usually not affected by the complexity of the proof tasks, we may consider that the variation in the percentages of the students who committed errors related to the Understanding-Applying of Mathematical Concepts (UAMC) may be affected by a psychological aspect. When students find complexity in the proof tasks, they rush answers and become confused and thus, commit more errors. However, when the proof complexity is reduced, students are more comfortable to think about the addressed proof tasks and thus, commit fewer errors. Two errors related to the Understanding-Applying of Mathematical Concepts (UAMC) were not affected by the variation of the proof complexity level and maintained their corresponding minimal percentages of students committing such errors. 0% and 2.12% of the students in problems P1 and P3 respectively proved statements other than the required ones (SOR). 0% and 1.06% of the students in problems P1 and P3 respectively made false assumptions (FA). This shows that the proof complexity level does not affect the students' ability to commit such errors.

In problems P2 and P4, which are considered to be isomathic, the percentages of the grade 7 students who committed errors related to the
Understanding-Applying of Mathematical Concepts (UAMC) are almost the same. Though the mathematical text in P2 is more complex than that in P4, the percentages of the students committing errors related to the understanding-applying of Mathematical Concepts (UAMC) were almost minimal and equivalent in problems P2 and P4. 0.53% of the grade 7 students in each of the problems P2 and P4 proved statements other than the required ones (SOR). 6.35% and 8.99% of the students in problems P2 and P4 respectively incorrectly claimed that a statement implied or equaled another statement (ISIE). 8.99% and 7.94% of the students in problems P2 and P4 respectively used the inappropriate definition, theorem, or property to justify a statement (IDTP). In each of the problems P2 and P4, 0.53% of the students made false assumptions (FA). 6.88% and 5.29% of the students in problems P2 and P4 respectively made false computation or wrote a false statement (FSC).

This shows that the language complexity level of the mathematical text in each of the isomathic problems P2 and P4 did not affect the students’ tendency to commit errors related to difficulty in Understanding-Applying Mathematical Concepts (UAMC).

The highest percentages of students having committed errors reflecting the difficulty of Understanding-Applying Mathematical
Concepts (UAMC) in each of problems P1, P2, P3 and P4 are respectively: 27.78%, 8.99%, 10.05%, and 8.99%. It is noticed that the highest percentage of the students who showed difficulty in Understanding-Applying Mathematical Concepts (UAMC) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P3 (simple proof tasks; simple mathematical language). The third highest percentages were in problems P2 (simple proof tasks; complex mathematical language) and P4 (simple proof tasks; simple mathematical language). The variation of the complexity level of both proof and language in problems P1, P2, P3, and P4, led to affecting the percentage of the students who did not identify the appropriate definition, theorem, or property needed to justify a statement (IDTP) which dropped from its highest percentage in P1 (27.78%) to around its third in P3, P2, and P4 and the percentage of the students who incorrectly claimed that a statement equaled or implied another one (ISIE) dropped from its highest percentage (22.22%) in P1 to around its third in P3, P2, and P4. Furthermore, the percentage of the students who made false computations or statements (FSC) also dropped from its highest percentage (14.81%) in P1 to around its fifth in P3, P2, and P4. Errors related to proving statements other than the required ones (SOR)
and to making false assumptions (FA) were not affected by varying either the proof or the language complexity level.

Table 16

*Grade 7 Errors Related to Understanding-Applying Mathematical Concepts (UAMC)*

<p>| Understanding-Applying Mathematical Concepts (UAMC) |</p>
<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>SOR</td>
<td>0%</td>
<td>2.12%</td>
<td>0.53%</td>
<td>0.53%</td>
</tr>
<tr>
<td>ISIE</td>
<td>22.22%</td>
<td>6.88%</td>
<td>6.35%</td>
<td>8.99%</td>
</tr>
<tr>
<td>IDTP</td>
<td>27.78%</td>
<td>10.05%</td>
<td>8.99%</td>
<td>7.94%</td>
</tr>
<tr>
<td>FA</td>
<td>0%</td>
<td>1.06%</td>
<td>0.53%</td>
<td>0.53%</td>
</tr>
<tr>
<td>FSC</td>
<td>14.81%</td>
<td>3.70%</td>
<td>6.88%</td>
<td>5.29%</td>
</tr>
<tr>
<td>DL (UMC)</td>
<td>27.78%</td>
<td>10.05%</td>
<td>8.99%</td>
<td>8.99%</td>
</tr>
</tbody>
</table>

SOR: Proved Statements Other than the Required ones.
ISIE: “Incorrectly claimed that one Statement Implied or Equaled another statement”.
IDTP: Used Irrelevant Definition, Theorem, or Property to Justify a Statement.
FA: “Made a False Assumption somewhere in the proof”.
FSC: “Made a False Statement or incorrect Computation in the proof”.
PS: Percentage of Students.
LD: Level of the difficulty.
4.3.2.5 Comprehending Mathematical Texts (CMT) at grade 7

Table 17 represents the percentages of grade 7 students who committed errors that reflect difficulty in Comprehending Mathematical Texts (CMT) in problems P1, P3, P2, and P4.

In problem P1, the percentages of the students who had difficulty in Comprehending Mathematical Texts (CMT) were 5.56% and 0%. The only error was not starting a proof. 5.56% of the grade 7 students did not start a proof (NSP). This might show the students’ inability to comprehend the mathematical text of the proof task or that of the premises. Nevertheless, 0% of the students proved statements other than the required ones (SOR). Since the highest percentage of students who have committed a type of errors under this difficulty is 5.56%, we may safely infer that at least 5.56% of the grade 7 students have difficulties related to comprehending mathematical texts at Problem P1 level of complexity.

In problem P3, the percentages of students committing errors reflecting difficulty in Comprehending Mathematical Texts (CMT) were 2.12% and 7.94%. It is clear that the percentages of the students committing some of the errors related to difficulty in comprehending
mathematical texts became more upon reducing the complexity of the proof tasks. We notice that the percentage of the students who did not start a proof (NSP) became 7.94% (from 5.56% to 7.94%), and the percentage of the students who proved statements other than the required ones (SOR) became 2.12% (from 0% to 2.12%). This elevation in the percentages might be due to the difference in the number of proof tasks between P1 to P3. The more the number of proof tasks is, the more is the possibility of committing errors.

In problem P2, the percentages of the grade 7 students committing errors related to difficulty in Comprehending Mathematical Texts (CMT) were: 10.58%, 3.7%, and 0.53%. The major error committed by grade 7 students was not starting a proof (NSP). 10.58% of the students did not start a proof maybe due to their inability to comprehend the text of either the premises or the proof tasks. 3.7% of the students drew wrong figures (DWF). Errors detected in the drawn figures reflected the students’ inability to comprehend mathematical texts that are considered to be complex in problem P2. 0.53% of the students, which is considered to be a minimal value, proved statements other than the required ones (SOR). Since the highest percentage of students who have committed a type of errors under this difficulty is 10.58%, we may safely infer that at least
10.58% of the grade 7 students have difficulties related to
Comprehending Mathematical Texts (CMT) at Problem P2 level of complexity.

In problem P4, the percentages of the grade 7 students committing errors related to Comprehending Mathematical Texts (CMT) became less upon reducing the complexity level of the mathematical texts. The percentage of the students who did not start a proof (NSP) dropped from 10.58% in P2 to around its half in P4, and the percentage of the students who drew a wrong figure (DWF) dropped from 3.7% to 0%. This shows that reducing the complexity level of the mathematical texts lessens the students’ tendency to commit errors related to Comprehending Mathematical Texts (CMT). Nevertheless, the percentage of the students who proved statements other than the required ones did not change upon varying the complexity level of the mathematical texts.

The highest percentages of students having committed errors reflecting the difficulty of Comprehending Mathematical Texts (CMT) in each of problems P1, P2, P3 and P4 are respectively: 5.56%, 10.58%, 7.94%, and 6.35%. It is noticed that the highest percentage of the students who showed difficulty in Comprehending Mathematical Texts (CMT) occurred in problem P2 (simple proof tasks; complex mathematical
language). The next highest percentage was in problem P3 (simple proof tasks; simple mathematical language). The third highest percentage was in problem P4 (simple proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P1 (complex proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who did not start a proof (NSP) had the highest percentage in P2 (10.58%) and dropped to around its half in P1 and P4 and to its two thirds in P3, and the percentage of the students who drew a wrong figure (DWF) dropped from 3.7% in P2 to disappear completely in P4. However, the percentage of the students who proved statements other than the required ones (SOR) was almost the same in problems P1, P2, P3, and P4 and thus, not affected by the variation of both proof and language complexities.
### Table 17

**Grade 7 Errors Related to Comprehending Mathematical Texts (CMT)**

<table>
<thead>
<tr>
<th>Problems</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td>PS</td>
</tr>
<tr>
<td>DWF</td>
<td>N/A</td>
<td>N/A</td>
<td>3.70%</td>
<td>0%</td>
</tr>
<tr>
<td>SOR</td>
<td>0%</td>
<td>2.12%</td>
<td>0.53%</td>
<td>0.53%</td>
</tr>
<tr>
<td>NSP</td>
<td>5.56%</td>
<td>7.94%</td>
<td>10.58%</td>
<td>6.35%</td>
</tr>
<tr>
<td>LD (CMT)</td>
<td>5.56%</td>
<td>7.94%</td>
<td>10.58%</td>
<td>6.35%</td>
</tr>
</tbody>
</table>

DWF: Drew a Wrong Figure.  
SOR: Proved Statements Other than the Required ones.  
NSP: Did Not Start a Proof.  
N/A: Not applicable. The problem doesn’t require testing this error.  
PS: Percentage of Students.  
LD: Level of the difficulty.

#### 4.3.2.6 Writing Mathematical Texts (WMT) at grade 7

Table 18 represents the percentages of grade 7 students who committed errors that reflect difficulty in *Writing Mathematical Texts* (WMT) in problems P1, P3, P2, and P4.

In problem P1, the percentages of students who had difficulty in *Writing Mathematical Texts* (WMT) varied between 0% and 9.26%. The major error was *writing mathematical symbols incorrectly*. 9.26% of the
grade 7 students did not write mathematical symbols correctly (MSI).
This shows that 9.26% of the grade 7 students were not capable of
differentiating between various mathematical symbols and identifying the
ones needed to address the required mathematical meaning. As for the
other errors that reflect difficulty in Writing Mathematical Texts (WMT),
the percentages of the students committing these errors were of minimal
values and thus, show that grade 7 students do not have difficulty in
writing mathematical texts or those students did not write any
mathematical text. 1.85% of the grade 7 students used inappropriate
mathematical vocabulary (VI), and did not use logical connectors
correctly (LCI). Furthermore, 0% of the students wrote ambiguous
statements or paragraphs (SA) or inappropriate text of the definitions,
theorems, or properties (DTPI). Since the highest percentage of students
who have committed a type of errors under this difficulty is 9.26%, we
may safely infer that at least 9.26% of the grade 7 students have
difficulties related to writing mathematical texts at Problem P1 level of
complexity.

In problem P3, the percentages of students committing errors
reflecting difficulty in Writing Mathematical Texts (WMT) ranged
between 0% and 8.47%. It is clear that the percentages of the students
committing some of the errors related to difficulty in writing mathematical texts did not change upon reducing the complexity of the proof tasks. This shows that the students’ ability to write mathematical texts correctly is not affected by the proof tasks complexity level.

In problem P2, the percentages of the grade 7 students committing errors related to difficulty in Writing Mathematical Texts (WMT) varied between 0% and 8.99%. The major error committed by grade 7 students was writing inappropriate statements of the definitions, theorems, and properties (DTPI). 8.99% of the students wrote the statements of the definitions, theorems, or properties incorrectly. 5.82% of the students wrote mathematical symbols incorrectly (MSI). As for the other errors related to difficulty in Writing Mathematical Texts (WMT), the percentages of the students who committed these errors were minimal. 2.12% of the students wrote statements that were ambiguous (SA) and used mathematical vocabulary incorrectly (VI). Furthermore, 0% of the students had difficulty in using logical connectors in a mathematical text (LCI). These low percentages do not necessarily imply that the grade 7 students are capable of writing mathematical texts correctly. Rather, this could be due to the fact that the grade 7 students did not write a proof, justify their proof statements, or use logical connectors in their proofs.
However, and since the highest percentage of students who have committed a type of errors under this difficulty is 8.99%, we may safely infer that at least 8.99% of the grade 7 students have difficulties related to writing mathematical texts at Problem P2 level of complexity.

In problem P4, which are considered to be isomathic to P2, it is noticed that the percentages of students committing errors related to Writing Mathematical Texts (WMT) are very close in problems P2 and P4 and were not affected by varying the level of the text complexity. This shows that the students’ tendency to commit such errors is not affected. Rather, it is almost constant. Hence, the students’ ability to Write Mathematical Texts (WMT) is not affected by the complexity of the mathematical text.

The highest percentages of students having committed errors reflecting the difficulty of Writing Mathematical Texts (WMT) in each of problems P1, P2, P3 and P4 are respectively: 9.26%, 8.99%, 8.47%, and 7.94%. It is noticed that the highest percentage of the students who showed difficulty in Writing Mathematical Texts (WMT) occurred in problem P1 (complex proof tasks; simple mathematical language). The next highest percentage was in problem P2 (simple proof tasks; complex mathematical language). The third highest percentage was in problem P3
(simple proof tasks; simple mathematical language). Finally, the fourth highest percentage was in problem P4 (simple proof tasks; simple mathematical language). Upon varying the complexity level of both proof and language in problems P1, P2, P3, and P4, it is recognized that the percentage of the students who used mathematical symbols incorrectly (MSI) had the highest percentage in P1 and P3 (9.26% and 8.47%), and dropped to around its two thirds in P2 and P4, and the percentage of the students who wrote inappropriate statement of the definitions, theorems, and properties (DTPI) dropped from 8.99% and 7.94% in P2 and P4 respectively to diminish completely in P1 and P3. This might be due to the difference in the nature of the statements of the definitions, theorems, and properties addressed in each pair of the isomathic problems. Errors related to writing ambiguous statements (SA), writing appropriate mathematical terms and vocabulary (VI), and to using logical connectors appropriately (LCI) were not affected by varying either the proof or the language complexity level.
Table 18

*Grade 7 Errors Related to Writing Mathematical Texts (WMT)*

<table>
<thead>
<tr>
<th>Error</th>
<th>P1</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>SA</td>
<td>0%</td>
<td>0.53%</td>
<td>2.12%</td>
<td>1.59%</td>
</tr>
<tr>
<td>MSI</td>
<td>9.26%</td>
<td>8.47%</td>
<td>5.82%</td>
<td>5.29%</td>
</tr>
<tr>
<td>VI</td>
<td>1.85%</td>
<td>0%</td>
<td>2.12%</td>
<td>0.53%</td>
</tr>
<tr>
<td>DPTI</td>
<td>0%</td>
<td>0%</td>
<td>8.99%</td>
<td>7.94%</td>
</tr>
<tr>
<td>LCI</td>
<td>1.85%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

LD (WMT) | 9.26% | 8.47% | 8.99% | 5.29%

SA: “Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex”. MSI: Used Mathematical Symbols Incorrectly. VI: Used Mathematical terms, words, or Vocabulary Incorrectly. DPTI: Wrote statement of Definitions, Properties, or Theorems Incorrectly. LCI: Used Logical Connectors Incorrectly. PS: Percentage of Students. LD: Level of the difficulty.

**4.3.3 Comparison across grade levels**

It is important to compare the level of difficulty that the Lebanese middle school students have in each of the problems P1 (complex proof tasks; simple mathematical language) and P2 (simple proof tasks; complex mathematical language). The level of difficulty for each of the six difficulties identified in the
framework is compared across the middle school levels as well as the parameters affecting the students’ ability to construct and formulate geometric proofs at each level, as shown in Table 19, Chart 1, Table 20, and Chart 2.

Table 19

*Difficulty Level across Grades 6 to 9 in Problem P1*

<table>
<thead>
<tr>
<th>Problem</th>
<th>P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>LD</td>
<td>LD6</td>
</tr>
<tr>
<td>UNP</td>
<td>47.73%</td>
</tr>
<tr>
<td>SPP</td>
<td>11.36%</td>
</tr>
<tr>
<td>CDR</td>
<td>31.82%</td>
</tr>
<tr>
<td>UAMC</td>
<td>31.82%</td>
</tr>
<tr>
<td>CMT</td>
<td>4.55%</td>
</tr>
<tr>
<td>WMT</td>
<td>11.36%</td>
</tr>
<tr>
<td>HLD</td>
<td>47.73%</td>
</tr>
</tbody>
</table>

LD: Level of difficulty.
UNP: Understanding the notion of proof.
SPP: Setting proof plans.
CDR: Conducting deductive reasoning.
UAMC: Understanding-Applying mathematical concepts.
CMT: Comprehending mathematical texts.
WMT: Writing mathematical texts.
HLD: Highest level of difficulty.
Chart 1

*Difficulty Level across Grades 6 to 9 in Problem P1*

**Difficulty Level across Grades 6 to 9 in Problem P1**

![Bar chart showing percentage of students by grade and problem difficulty.](chart)
Table 20

*Difficulty Level across Grades 6 to 9 in Problem P2*

<table>
<thead>
<tr>
<th>Problem</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LD</td>
</tr>
<tr>
<td>UNP</td>
<td>31.82%</td>
</tr>
<tr>
<td>SPP</td>
<td>9.09%</td>
</tr>
<tr>
<td>CDR</td>
<td>28.03%</td>
</tr>
<tr>
<td>UAMC</td>
<td>28.03%</td>
</tr>
<tr>
<td>CMT</td>
<td>37.50%</td>
</tr>
<tr>
<td>WMT</td>
<td>7.58%</td>
</tr>
<tr>
<td>HLD</td>
<td>37.50%</td>
</tr>
</tbody>
</table>

LD: Level of difficulty.
UNP: Understanding the notion of proof.
SPP: Setting proof plans.
CDR: Conducting deductive reasoning.
UAMC: Understanding-Applying mathematical concepts.
CMT: Comprehending mathematical texts.
WMT: Writing mathematical texts.
HLD: Highest level of difficulty.
Chart 2

Difficulty Level across Grades 6 to 9 in Problem P2
4.3.3.1 Understanding the Notion of Proof (UNP) across grades 6 to 9

In problem P1, the level of difficulty related to Understanding the Notion of Proof (UNP) varied from 47.73% to 4.17% across grades 6 to 9, as shown in Table 19 and Chart 1. The highest level of difficulty in understanding the notion of proof was in grade 6 (47.73%). The second level of difficulty was in grade 7 (25.93%). The third level of difficulty was in grade 8 (18.25%). Finally, the least level of difficulty was in grade 9 (4.17%). This shows that the students’ difficulty in Understanding the Notion of Proof (UNP), at the problem P1 level of complexity, is affected and reduced as students grow and experience more proving across grade levels. This might be due to two factors, the actual practice of developing proofs, and the fact that the students’ cognitive abilities and logical reasoning have developed as they grew up and hence, they understand better what a mathematical proof requires.

In problem P2, the level of difficulty related to Understanding the Notion of Proof (UNP) varied from 31.82% to 8.99% across grades 6 to 9, as shown in Table 20 and Chart 2. The highest level of difficulty of understanding the notion of proof was in grade 6 (31.82%). The least level of difficulty was in grade 7 (8.99%). The second level of difficulty was in grade 8 (15%). Finally, the third level of difficulty was in grade 9
(12.5%). It is clear that the level of difficulty in P2, at each grade level, is less than that in P1. This is due to the simpler proof tasks addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1 except for grade 6; it maintained the highest level of difficulty in understanding the notion of proof. As for the other grade levels, the complexity of language affected the students’ ability to show understanding of the notion of proof.

As a conclusion, irrespective of the proof or language complexity levels, among the Lebanese middle school students, grade 6 students have the highest level of difficulty of Understanding the Notion of Proof (UNP).

4.3.3.2 Setting Proof Plans (SPP) across grades 6 to 9

In problem P1, the level of difficulty related to Setting Proof Plans (SPP) varied from 25% to 11.36% across grades 6 to 9, as shown in Table 19 and Chart 1. The least level of difficulty was in grade 6 (11.36%). The third level of difficulty was in grade 7 (16.67%). The highest level of difficulty of setting proof plans was in grade 8 (25%). Finally, the second level of difficulty was in grade 9 (20.83%). This shows that the students’ difficulty in Setting Proof Plans (SPP), at the
problem P1 level of complexity, is affected and became more as students grow across grade levels. This might be due to the fact that, in grade 6 when proof is just introduced, problems are more direct and proofs require just direct application of properties and theorems, with not many intermediary steps. Thus proving at that grade level does not require much planning. As we progress through grade levels, problems become more complex and require more planning to reach a proof. Complex proof tasks at higher levels require constructing and formulating many proof modules than at lower grade levels. Moreover, students at grades 8 and 9 become more careful about not writing any proof before making sure that it is appropriate thus they might not even start writing a proof. However, in grades 6 and 7, students tend to write their thoughts about a proof plan even before checking its appropriateness.

In problem P2, the level of difficulty related to Setting Proof Plans (SPP) varied from 21.88% to 5% across grades 6 to 9, as shown in Table 20 and Chart 2. The third level of difficulty was in grade 6 (9.09%). The second level of difficulty was in grade 7 (10.58%). The least level of difficulty was in grade 8 (5%). Finally, the highest level of difficulty of setting proof plans was in grade 9 (21.88%). It is clear that the level of difficulty in P2, at each grade level, is less than that in P1
except at grade 9 where it was almost the same. This is due to the simple proof tasks addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1. According to the students’ proficiency in mathematical language, the complex mathematical language used in P2 affected their ability to set proof plans.

As a conclusion, the students’ ability to set proof plans is much more affected by the proof task complexity level than by the language complexity level.

**4.3.3 Conducting Deductive Reasoning (CDR) across grades 6 to 9**

In problem P1, the level of difficulty related to Conducting Deductive Reasoning (CDR) varied from 31.82% to 12.5% across grades 6 to 9, as shown in Table 19 and Chart 1. The highest level of difficulty of conducting deductive reasoning was in grade 6 (31.82%). The second level of difficulty was in grade 7 (22.22%). The least level of difficulty was in grade 8 (12.5%). Finally, the third level of difficulty was in grade 9 (18.75%). This shows that the students’ difficulty in Conducting Deductive Reasoning (CDR), at the problem P1 level of complexity, is affected and reduced as students grow and experience more proving across grade levels. This might be due to two factors, the actual practice
of developing proofs, and the fact that the students’ cognitive abilities and logical reasoning have developed as they grew up and hence, they understand better what mathematical reasoning requires. However, some external factors, such as the understanding of some mathematical concepts, and the increasing level of complexity of geometric figures might hinder the students’ ability to conduct deductive reasoning. This was clear in grade 9.

In problem P2, the level of difficulty related to Conducting Deductive Reasoning (CDR) varied between 7.41% and 28.03% across grades 6 to 9, as shown in Table 20 and Chart 2. The highest level of difficulty of conducting deductive reasoning was in grade 6 (28.03%). The least level of difficulty was in grade 7 (7.41%). The third level of difficulty was in grade 8 (10%). Finally, the second level of difficulty was in grade 9 (12.5%). It is clear that the level of difficulty in P2, at each grade level, is less than that in P1. This is due to the simple proof tasks addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1 except for grade 6; it maintained the highest level of difficulty in conducting deductive reasoning. As for the other grade levels, the complexity of language affected the students’ ability to conduct deductive reasoning.
As a conclusion, irrespective of the proof or language complexity levels, among the Lebanese middle school students, grade 6 students have the highest level of difficulty of Conducting Deductive Reasoning (CDR) due to their little experience in constructing and formulating proofs. Nevertheless, other factors, such as understanding mathematical concepts, affect the students’ ability, at any grade level, to conduct deductive reasoning.

4.3.3.4 Understanding-Applying Mathematical Concepts (UAMC) across grades 6 to 9

In problem P1, the level of difficulty related to Understanding-Applying Mathematical Concepts (UAMC) varied from 37.5% to 18.75% across grades 6 to 9, as shown in Table 19 and Chart 1. The second level of difficulty was in grade 6 (31.82%). The third level of difficulty was in grade 7 (27.78%). The highest level of difficulty of understanding-applying mathematical concepts was in grade 8 (37.5%). Finally, the least level of difficulty was in grade 9 (18.75%). This shows that the students’ difficulty in Understanding-Applying Mathematical Concepts (UAMC), at the problem P1 level of complexity, is affected and varies across grade levels depending on the load of mathematical concepts addressed in each grade level. For instance, according to the Lebanese national curriculum,
students at grade 8 are exposed to the biggest number of new geometrical concepts compared to other classes while in grade 9, students are exposed to the least number of new geometrical concepts compared to the other classes. Thus, students at grade 8 might not be able to grasp or develop appropriate conceptual understanding of all taught mathematical concepts.

In problem P2, the level of difficulty related to Understanding-Applying Mathematical Concepts (UAMC) varied from 28.03% to 7.5% across grades 6 to 9, as shown in Table 20 and Chart 2. The highest level of difficulty of understanding-applying mathematical concepts was in grade 6 (28.03%). The third level of difficulty was in grade 7 (8.99%). The least level of difficulty was in grade 8 (7.5%). Finally, the second level of difficulty was in grade 9 (12.5%). It is clear that the level of difficulty in P2, at each grade level, is less than that in P1. This is due to the simple proof tasks addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1. It is noticed that the complex language addressed in P2, language of premises and used definitions, theorems, and properties, reflected a kind of difficulty that grade 6 students have in understanding mathematical concepts regardless of the proof complexity level.
As a conclusion, complex mathematical language and proof tasks showed the grade 6 students’ inability to understand-apply mathematical concepts. However, grade 8 students’ ability to show understanding of mathematical concepts is affected by the proof complexity level.

4.3.3.5 Comprehending Mathematical Texts (CMT) across grades 6 to 9

In problem P1, the level of difficulty related to Comprehending Mathematical Texts (CMT) varied from 20.83% to 4.55% across grades 6 to 9, as shown in Table 1 and Chart 1. The least level of difficulty was in grade 6 (4.55%). The third level of difficulty was in grade 7 (5.56%). The second level of difficulty was in grade 8 (25.93%). Finally, the highest level of difficulty of comprehending mathematical texts was in grade 9 (20.83%). This shows that the students’ difficulty in Comprehending Mathematical Texts (CMT), at the problem P1 level of complexity, is affected and became more as students grow across grade levels. This might be due to the huge amount of mathematical vocabulary and terms that the students have to consider year after year.

In problem P2, the level of difficulty related to Comprehending Mathematical Texts (CMT) varied from 37.5% to 9.38% across grades 6 to 9, as shown in Table 20 and Chart 2. The highest level of difficulty of
comprehending mathematical texts was in grade 6 (37.5%). The third level of difficulty was in grade 7 (10.58%). The least level of difficulty was in grade 8 (9.38%). Finally, the second level of difficulty was in grade 9 (21.88%). It is clear that the level of difficulty in P2, at each grade level, is higher than that in P1. This is due to the complex mathematical language addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1. In grade 6, though the number of mathematical terms and vocabulary is considered to be minimal as compared to the other grade levels, grade 6 students are still considered to have the least experience of dealing with geometric mathematical texts. This explains the high level of difficulty grade 6 students have in comprehending mathematical texts. As for grade 9, they almost maintained the level of difficulty compared to P1 due to the huge amount of mathematical vocabulary and terms that the students have to consider year after year.

As a conclusion, irrespective of the proof or language complexity levels, among the Lebanese middle school students, grade 6 students have the highest level of difficulty of Comprehending Mathematical Texts (CMT) due to the little experience of dealing with mathematical texts, and grade 9 students have a high level of difficulty of comprehending
mathematical texts due to the huge amount of mathematical terms and vocabulary they have to consider.

4.3.3.6 Writing Mathematical Texts (WMT) across grades 6 to 9

In problem P1, the level of difficulty related to Writing Mathematical Texts (WMT) varied from 12.5% to 6.25% across grades 6 to 9, as shown in Table 1 and Chart 1. The second level of difficulty was in grade 6 (11.36%). The third level of difficulty was in grade 7 (9.26%). The highest level of difficulty of writing mathematical texts was in grade 8 (12.5%). Finally, the least level of difficulty was in grade 9 (6.25%). This shows that the students’ difficulty in Writing Mathematical Texts (WMT), at the problem P1 level of complexity, is not affected by the proof complexity level. Rather, it is considered to be minimal.

In problem P2, the level of difficulty related to Writing Mathematical Texts (WMT) varied from 8.99% to 0% across grades 6 to 9, as shown in Table 2 and Chart 2. The second level of difficulty was in grade 6 (7.58%). The highest level of difficulty of writing mathematical texts was in grade 7 (8.99%). The third level of difficulty was in grade 8 (7.5%). Finally, the least level of difficulty was in grade 9 (0%). It is clear that the level of difficulty in P2, at each grade level, is less than that
in P1. This is due to the simple proof tasks addressed in P2. However, the ranking of the grade levels according to the level of difficulty is different from that in P1 except for grade 9; it maintained the lowest level of difficulty in writing mathematical texts. This could be due the huge number of proofs that grade 9 students have to write as they have to sit for official exams.

As a conclusion, irrespective of the proof or language complexity levels, among the Lebanese middle school students, grade 6 students have the highest level of difficulty of Writing Mathematical Texts (WMT) and grade 9 students have lowest level. This might be considered to be a normal difference between grade levels who are considered to be just developing proofs (grade 6) and those who should have mastered writing proofs (grade 9).

4.4 Clinical Interviews’ Analysis

As mentioned before, during each session within which a class was completing a test, the researcher was holding a clinical interview with one of the students of the class. The selected student from each grade level was clinically interviewed when completing both Tests 1 and 2. The clinical interview was videotaped and transcribed for later data analysis. Similar to the test analysis, a detailed analysis of the clinical interviews of
grades 6 and 7 is presented. The grade 6 student is referred to as S6. The clinical interviews for Test 1 and Test 2 are referred to as 6C1 and 6C2 respectively. The grade 7 student is referred to as S7. The clinical interviews for Test 1 and Test 2 are referred to as 7C1 and 7C2 respectively.

The analysis of the clinical interviews is carried out according to the following two steps: First, the errors committed by S6 and S7 in each of the problems P1, P2, P3, & P4 are categorized according to the adopted framework. These errors are included in the results of grades 6 and 7 tests analysis. Second, the transcript of the video-tape is used to present a brief description of the S6’s and S7’s steps while constructing and formulating the required proofs.

4.4.1 Grade 6 clinical interviews

4.4.1.1 6C1

In Test 1, S6 is given the two problems P1 and P4.

4.4.1.1.1 Difficulties faced by S6 while solving problem P1

In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE].

\[ \hat{DBE} = 30^\circ, \ AB = 3\text{cm.} \]
What is the length of [BC]? Justify your answer.

S6 started reading the questions carefully. On the given figure, she coded the properties provided in the premises of the problem P1, as shown in the adjacent figure. She coded all the given properties except for the point D being the midpoint of [BC]. This shows that she was able to comprehend the mathematical text of the problem.

She read the question in P1 and answered by writing the following: “The length of [BC] is 3 cm because.” She did not write any justification. The researcher asked S6 to explain how she got the length of [BC]. S6 replied that it was because triangle ABC was an equilateral triangle. However, she wasn’t able to explain why. This shows that S6 had difficulty in Understanding the Notion of Proof (UNP) and in Setting Proof Plans (SPP).
Then the researcher started gradually providing S6 with hints (such as asking her to justify every step, reading the premises again and deriving appropriate inferences, and reviewing some geometrical concepts related to the proof at hand) in order to explore the source of difficulty that S6 had regarding constructing and formulating the geometric proof. According to the responses of S6 and the errors committed after being provided with hints, the difficulties that S6 had pertaining to her ability to construct and formulate proofs were identified according to the adopted framework.

1. Difficulty in Understanding the Notion of Proof (UNP):

There were several instances that showed that S6 had difficulty in understanding the notion of proof. The first example was that S6 did not organize her ideas logically (NL). S6 wrote the following, “AB= 3cm because”. She was not able to complete her solution. When the researcher tried to enhance her ability to justify her answer, she said, “AB= 3cm
because triangle ABC is an equilateral triangle.’’

Furthermore, when S6 was asked the following question, “Why is triangle ABC equilateral?” she replied, “It is equilateral because triangle ABD is semi-equilateral”. However, if triangle ABD is semi-equilateral, this does not necessarily imply that triangle ABC is an equilateral triangle. The second example was when she wrote that the measure of \(\overline{ABE}\) was given to be 90\(^{\circ}\). In fact, it was given that (AB) is perpendicular to \([BE]\). This showed the inability of S6 to differentiate between the premises of the proof and their inferences (NDPI). The third example was the fact that S6 neither justified her answer nor did she present the way she came up with her solution. This showed the existence of a gap in the proof developed by S6 (NJCS) and her inability to justify statements (NJEV).

2. Difficulty in Setting Proof Plans (SPP):

S6 was not able to set a complete proof plan to find the solution for the posed question in P1 even when
provided with hints. This was clear because she directly wrote the answer without any previous justification. When the researcher started providing her with hints such as “How did you know that triangle ABC is equilateral?”, S6 answered, “I knew because triangle ABD is a semi-equilateral triangle.” However, she was not able to present, even orally, any proof plan that she had followed to come up with her conclusion. Hence, we considered that S6 had an inability to set complete proof plans. She ended up her solution after 25 minutes of discussion without being able to write or set a proof plan.

3. Difficulty in Conducting Deductive Reasoning (CDR):

As for the difficulty of S6 in conducting deductive reasoning, it was clear from the previous examples (finding false inferences, not differentiating between the premises and their inferences, and showing an inability to derive conclusions) that S6 was not able to connect premises to come up with conclusions related to the proof at hand. For instance, when the researcher
noticed that S6 was not able to develop the proof, she offered the following hint: “Let us read the given premises again and try to see whether we may benefit from some of them. D is the midpoint of [BC], and (AD) is perpendicular to [BC] at D. Does that mean anything to you?” S6 said, “This means that $\angle ADB$ is equal to 90°.” She did not state that (AD) was the perpendicular bisector of [BC]. However, when the researcher asked her to define the perpendicular bisector, S6 defined it appropriately. However, she could not conclude that (AD) was the perpendicular bisector of [BC]. This shows that though S6 knew the definition of a perpendicular bisector, she was not able to conclude that a given line was a perpendicular bisector or not.

4. Difficulty in Understanding-Applying Mathematical Concepts (UAMC):

The example of the perpendicular bisector mentioned before showed that she had problems
related to the conceptual understanding of the geometric concept of perpendicular bisector.

The researcher then asked S6 to calculate the measure of $\overline{ABD}$. S6 said that it was equal to $90^\circ$ divided by two. Thus, S6 wrote a false statement in the proof because the measure of $\overline{ABD}$ was $60^\circ$.

5. Difficulty in Writing Mathematical Texts (WMT):

Though S6 did not write a mathematical text due to her inability to develop a mathematical proof, the researcher tried to let her justify some of the statements orally while giving her some hints. The researcher asked S6, “Are segments AC and AB equal?” S6 said, “Yes they are because A is equidistant since it is perpendicular bisector and it is on the perpendicular bisector, and these are the endpoints”. It was clear that S6 had a difficulty in writing mathematical properties correctly. For example, the word “equidistant” has no meaning unless it is completed with “from … and …”. Also, she used the pronoun “it” twice, standing for a different geometric
object every time, once for the line (AD) and the other time for the point A. Furthermore, she said during her explanations that “the measure of the triangle is 180°”, while the appropriate statement of the property was “the sum of angles in a triangle is 180°”.

As a conclusion, the difficulties of S6 in constructing and formulating geometric proofs were related to Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Understanding-Applying Mathematical Concepts (UAMC), Conducting Deductive Reasoning (CDR), and Writing Mathematical Texts (WMT).

**4.4.1.1.2 Difficulties faced by S6 while solving problem P4**

Triangle MNP is isosceles at M. NM = MP = 4cm. \( \hat{\text{NMP}} = 120^\circ \). Construct triangle MNP.

S6 read the questions carefully, and she drew the figure correctly except for the measure of \( \text{NMP} \). Instead of drawing a 120° angle, she drew a 60° angle. The researcher asked S6 about the nature of the angle \( \text{NMP} \) that she had drawn. S6 said that it
was an acute angle. Based on that, the researcher asked her, “Does an acute angle measure 120°?” S6 replied, “No, obtuse”, and she directly noticed the error that she had committed and then modified her figure.

Several errors were committed by S6 when answering question 2 in problem P4 (calculate the measures of $\hat{MNP}$ and $\hat{MPN}$). Show your work. Justify your answer.). The first error was when she wrote, “$\hat{MNP} = 180° − 120° = 160°$.” It was clear that she had a calculation mistake (FSC). Moreover, she did not explain what 180° stood for (NJEV). However, when the researcher asked her, “What does 180° represent?” She replied, “The measure of a triangle is 180°.” It was an inappropriate statement of the property (DPTI). The researcher then asked S6, “If the measure of $\hat{NMP}$ is 120° and that of $\hat{MNP}$ is 160°, what is left for $\hat{MPN}$?” S6 directly noticed the error made and said that the measure of $\hat{MPN}$ should be half of 60°. S6 was still not able to explain why the angles $\hat{MNP}$ and $\hat{MPN}$ were equal. The researcher tried to help her by asking the following question, “Can you define an isosceles triangle?” S6 answered, “It is a triangle having two equal sides,” but she did not mention anything about
its equal base angles. The researcher offered her more guidance to extract more properties of the isosceles triangle, but S6 did not mention anything other than the equal sides. Moreover, when the researcher asked, “Are the base angles of the isosceles triangle equal?”, S6 agreed. It is worth mentioning that S6 did not write any justification for the statements written in part 2 of problem P4.

In part 3 of problem P4 (Through M, draw \((x)\) perpendicular to \([NP]\). \((x)\) cuts \([NP]\) at \(R\). Through \(P\), draw \((d)\) parallel to \((x)\). \((d)\) cuts \([NM]\) at \(Q\).), S6 read the text of the problem aloud without mentioning the nature of the geometric objects reflected by the symbols used. For instance, she read, “Through \(M\), draw \(x\) perpendicular to \(NP\). \(x\) cuts \(NP\) at \(R\). Through \(P\), draw \(d\) parallel to \(x\). \(d\) cuts \(NM\) at \(Q\).”, without taking into consideration the nature of the geometric objects, as in reading: “Through \(point\) \(M\), draw \(straight\) \(line\) \(x\) perpendicular to \(segment\) \(NP\). \(Straight\) \(line\) \(x\) cuts \(straight\) \(line\) \(NP\) at \(point\) \(R\). Through \(point\) \(P\), draw \(straight\) \(line\) \(d\) parallel to \(line\) \(x\). \(Line\) \(d\) cuts \(line\) \(NM\) at \(Q\).
S6 drew the figure correctly but when locating point Q, she stopped because she did not find any intersection between (d) and [MN] as shown in the figure below. Nevertheless, when the researcher asked her why she had stopped drawing, she replied, “Because NM and d do not meet, we cannot extend NM”. She stopped for a while and said, “Ah, it’s a line; we can extend it” and completed drawing the figure correctly.

In part 3(a) of problem P4, when S6 was asked to determine the relative position of (d) and (NP), S6 first wrote that (d) and (NP) were intersecting lines. She stopped to justify her answer and noticed that the lines were perpendicular. Consequently, she erased and wrote the correct answer, but she did not use an appropriate statement of the property to justify her answer. She wrote, “If two lines parallel one of them is perpendicular to a segment so the other line is perpendicular too”. This reflects knowledge of the theorem to be used and understanding of its meaning, but inability to express it in proper
English or mathematical language. For example, “so the other line is perpendicular too”, without mentioning *perpendicular to what*.

In part 3(b) of problem P4 (Name a height of triangle NPQ. Justify your answer.), S6 wrote a wrong answer and used a wrong statement to justify her answer as shown in the figure below.

![Figure showing S6's response]

“RM is a height because a height is a segment who issued from vertex to the midpoint vertically opposite to”

When the researcher asked her, “Is M a vertex of triangle NPQ?” S6 replied, “No”. The researcher then asked, “How can you consider that RM is a height of triangle NPQ?” S6 was not able to explain or change her answer though she knew that MR was a wrong response.

As a summary of the difficulties faced in problem P4, which presented simple proof tasks and a simple mathematical language, the errors committed by S6 showed that she had difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing
Mathematical Texts (WMT). Nevertheless, she did not have any difficulties in Setting Proof Plans (SPP) nor Comprehending Mathematical Texts (CMT).

4.4.1.1.3 Comparison between difficulties faced by S6 in problems P1 and P4

After comparing the difficulties that S6 had in problems P1 and P4, we noticed that some difficulties were not affected by the nature or the complexity of the proof tasks such as the difficulty in: Understanding the notion of proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT). However, the ability of S6 to Set Proof Plans (SPP) was better when the proof tasks were simplified as in P4, whereby the premises and conclusions to reach were closer to each other with less or no intermediary steps. In both problems P1 and P4, S6 did not face any difficulties in comprehending mathematical texts. This might be due to the fact that the mathematical text used in both problems was simple.
4.4.1.2 6C2

In Test 2, S6 is given the two problems P3 and P2.

4.4.1.2.1 Difficulties faced by S6 while solving problem P3

In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE].

\[ \hat{D}BE = 30^\circ, \ AB = 3\text{cm}. \]

S6 started reading the questions carefully. On the given figure, she coded the properties provided in the premises of the problem P3, as shown in the adjacent figure. She coded all the given properties except for the point D being the midpoint of [BC]. This shows that she was able to comprehend the mathematical text of the problem.
She read the question in P1 (a) “Show that (AD) is the perpendicular bisector of [BC]”, stopped, wrote nothing, and started rereading the premises again. She did not write any justification. The researcher asked S6, “What are you thinking of?” S6 replied, “If a line perpendicular to a segment so this line is perpendicular bisector to the segment, cutting the midpoint of the segment. If it cutting vertically opposite to the midpoint and perpendicular then it is perpendicular bisector”. The response of S6 to the researcher’s question showed the inability of S6 to: state definitions correctly (DTPI), use mathematical terms appropriately (VI), and understand the concept of “vertically opposite”. Moreover, the written response of S6 to the question of P3 (a), as shown in the figure below, showed that S6 incorrectly claimed that a statement implied another one (ISIE) because if (AD) cuts [BC] at its midpoint this does not necessarily imply that (AD) is a perpendicular bisector of [BC]. This shows that S6 had difficulty in Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and in Writing Mathematical Texts (WMT).
In P3 (b), which required showing that AB and AC are equal, S6 wrote the following “A is equidistant to [BC] so AB=AC=3cm” and justified her answer by writing “if a line perpendicular to a segment on it perpendicular to point the segment so one other point is perpendicular too”. The responses of S6 showed that S6 had difficulty in writing mathematical statements correctly (SA). For instance, a point is not equidistant from a segment; rather a point is equidistant from the endpoints of the segment. Moreover, she wrote inappropriate statements of the properties (DTPI) and used an inappropriate property to justify a statement (IDTP). The errors committed by S6 show that she had difficulty in Understanding-Applying Mathematical Concepts (UAMC) and in Writing Mathematical Texts (WMT).

In P3 (d) which required showing that triangle ABC is an equilateral triangle, S6 built her proof on an empirical assumption and used a premise that was not given (PNG). She assumed that the angles $\angle ABC$, $\angle ACB$, and $\angle BAC$ were equal without justifying her answer (NJEV) and (NJCS). Thus all what she had done was calculating the measure of each of the angles without any justification and then deducing that triangle ABC was an
equilateral triangle by using the definition of an equilateral triangle. The errors committed by S6 show that S6 had difficulty in Understanding the Notion of Proof (UNP).

As a conclusion, the difficulties of S6 in constructing and formulating geometric proofs were related to Understanding the Notion of Proof (UNP), Understanding-Applying Mathematical Concepts (UAMC), Conducting Deductive Reasoning (CDR), and Writing Mathematical Texts (WMT).

4.4.1.2.2 Difficulties faced by S6 while solving problem P2

Consider an isosceles triangle MNP whose main vertex is M and the length of its equal sides is 4cm. Let $\hat{\text{PMN}} = 120^\circ$. Construct triangle MNP.

S6 read the questions carefully, and drew the figure correctly except for the measure of $\hat{\text{NMP}}$. Instead of drawing a $120^\circ$ angle, she drew a $60^\circ$ angle (DWF). She coded on her figure that the sides MN and MP were equal. The researcher asked her, “How did you know that MN and MP are equal?”. S6 replied, “M is the vertex so MN and MP are the base angles, sorry, equal sides.” This showed that S6 did not have difficulty in
Comprehending Mathematical Texts (CMT), but she had difficulty in Understanding-Applying some mathematical (geometrical) Concepts (UAMC) related to angles which was reflected through the wrong angle drawn.

Several errors were committed by S6 when answering question 2 in problem P4 (calculate the measures of $M\hat{N}P$ and $M\hat{P}N$. Show your work. Justify your answer.). The first error was when she included an extra detail that did not contribute to the proof at hand (ED) which was mentioning that the sides MN and MP were equal without even justifying her statement (NJEV). Moreover, she wrote that the measure of $M\hat{N}P = 60/2$ without justifying why (NJEV). However, when the researcher asked her to justify she said, “Because the base sides are equal so the angles are also equal”. Thus, she incorrectly claimed that a statement implied another one (ISIE). Moreover, she inappropriately used the term “base sides” (VI). It is worth mentioning that S6 did not write any justification for the statements written in part 2 of problem P2. Hence, errors committed by S6 show that she has difficulties in Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-
Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT).

In part 3 of problem P2 (Let R be the foot of the perpendicular drawn from M to [NP] and (d) be the parallel to (MR) drawn through P and cutting (NM) at Q.), S6 read the text of the problem aloud without mentioning the nature of the geometric objects reflected by the symbols used. For instance, she read, “Let R be the foot of the perpendicular drawn from M to NP and d be the parallel to MR drawn through P and cutting NM at Q.”, without taking into consideration the nature of the geometric objects, as in reading: “Let point R be the foot of the perpendicular drawn from point M to segment NP and straight line d be the parallel to straight line MR drawn through point P and cutting straight line NM at point Q.

S6 read the question all together then read each phrase alone in a trial to comprehend the text of the question. At the beginning she was not able to draw (d) correctly. However, when the researcher helped her reading the phrase related to (d) she drew it as shown in the figure below.
Next, S6 did not understand what the word “cutting” stand for, and asked, “Cutting, who is cutting?” The researcher provided her with help in order to identify the appropriate intersecting lines. However, S6 was not able to locate point Q. The researcher drew the figure for S6 to be able to complete answering the rest of the questions. Errors committed by S6 in P2 (3) show that she had difficulty in Comprehending Mathematical Texts (CMT).

In part 3(a) of problem P2, which required determining the relation between lines (d) and (NP), S6 asked, “Related to what?” The researcher replied, “Related to each other.” S6 wrote that (d) and (NP) were perpendicular lines without being able to provide any justification and said that she does not know why. Errors committed by S6 in P2 (3a) reflect difficulties in Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), and Comprehending Mathematical Texts (CMT).

In part 3(b) of problem P2 (Name a height of triangle NPQ. Justify your answer.), S6 wrote a wrong answer and used a wrong statement to justify her answer as shown in the figure below.

“RM because it is issued from the vertex vertically opposite to the midpoint”
As a summary of the difficulties faced in problem P2, which presented simple proof tasks and a complex mathematical language, the errors committed by S6 showed that she had difficulties related to Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT).

4.4.1.2.3 Comparison between difficulties faced by S6 in problems P3 and P2

After comparing the difficulties that S6 had in problems P3 and P2, we noticed that some difficulties were affected by the complexity of the mathematical language such as the difficulty in Comprehending Mathematical Texts (CMT). However, difficulties related to Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT) were not affected by the language complexity of the mathematical texts.
4.4.1.3 *Comparison across problems P1, P2, P3, and P4 during 6C1 and 6C2*

Upon varying the proof and language complexities across problems P1, P2, P3, and P4, the difficulties that S6 faced varied according to the nature of the problem. In P1 (complex proof task; simple mathematical language), S6 had difficulty in Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT). However, reducing the proof difficulty in the isomathic problem P3 eliminated the difficulty in Setting Proof Plans (SPP). In problem P2 (simple proof tasks; complex mathematical language), S6 had difficulty in Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT). However, reducing the language complexity level in the isomathic problem P4 affected the difficulties related to Setting Proof Plans (SPP) and Comprehending Mathematical Texts.
(CMT). The complex text in P2 prevented S6 from drawing correct figures and thus from being able to set proof plans in some cases. Yet, difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT) were not affected.

4.4.2 Grade 7 clinical interviews

4.4.2.1 7C1

In Test 1, S7 is given the two problems P1 and P4.

4.4.2.1.1 Difficulties faced by S7 while solving problem P1

In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

Show that AI= JC.
In problem P1 (a) which requires showing that AI is equal to JC, S7 started reading the questions carefully. On the given figure, she coded the properties provided in the premises of the problem P1, as shown in the adjacent figure. This shows that she was able to comprehend the mathematical text of the problem.

She read the question in P1 (a) and answered by writing the following: “Consider triangles AIB and DJC.” The researcher asked S7 to explain why she chose that pair of triangles particularly. S7 replied that it was because AI and JC are sides of the pairs of the chosen triangles. She added, “After proving the equal triangles, AI and JC will be one of the homologous pairs.” After proving the equality of a pair of sides and angles each belonging to one of the considered triangles, S7 stopped and said that she still needs a pair of either equal sides or angles. S7 wrote the following “(AB) // (DC) (given). AB= DC (by remaining)” and then directly concluded that triangles AIB and DJC were
equal by using the property of “SAS”. The researcher asked S7, “How did you prove that AB and DC are equal?” S7 replied, “By remaining sides”. The researcher said, “Is there a property that says (In two triangles, if two pairs of sides are equal, then the remaining pair of sides should be equal)?” S7 replied, “Yes, like the remaining angles’ property”. It is clear that S7 used a false property (FSC). The error committed by S7 showed that she had difficulty in Understanding-Applying some Mathematical Concepts (UAMC) and in Conducting Deductive Reasoning (CDR). Furthermore, when the researcher asked S7, “What does the property SAS mean?”, S7 replied, “It means that to prove equal triangles we should prove that they have two equal sides and an angle.” The researcher said, “So, can we name the property as SSA or ASS?” S7 did not agree and said that she first proved a side then an angle then another side. It is clear that S7 had a difficulty in understanding what the “SAS” property means. To her, it represents the order of the proved equal parts. However, the “SAS” property requires proving two equal sides and their included angle. Hence, S7 had another difficulty in Understanding-Applying Mathematical Concepts (UAMC).
In problem P1 (b) which requires proving that lines (AD) and (BC) were parallel, S7 said that angles $\angle BAD$ and $\angle ADC$ are corresponding angles and that she has to prove them equal. It is clear that S7 made a false assumption (FA). The researcher asked S7, “Can you define a pair of corresponding angles?” S7 replied, “Angles that are equal and they are facing each other”. This also shows that S7 was not able to define “corresponding angles” correctly and name them correctly thus having a difficulty in Understanding-Applying Mathematical Concepts (UAMC). When the researcher asked her to name another pair of corresponding angles, S7 looked at the figure and noticed that her previous answer was wrong and decided to add a point to the figure to change her proof plan, as shown in the figure below.

![Diagram](image)

After modifying the figure, S7 wrote the following as a proof: “w.r.t. (AB) $\parallel$ (CD) and transversal (DC). $\angle BAD =$
$\angle CDF$ (Corresponding angles). So, $\overline{AD} \parallel \overline{BC}$ (corresponding angles formed are equal)”. The written proof contained several errors. The first error was the name of the transversal. $\overline{DC}$ is not a transversal to the chosen pair of parallel lines (FSC). The second error was incorrectly claiming that equal corresponding angles chosen imply that the lines $\overline{AD}$ and $\overline{BC}$ are parallel (ISIE). Thus, the approach chosen for the proof will not work (ANW).

As a conclusion, the difficulties of S7 in constructing and formulating geometric proofs were related to Setting Proof Plans (SPP), Understanding-Applying Mathematical Concepts (UAMC), and Conducting Deductive Reasoning (CDR).

4.4.2.1.2 Difficulties faced by S7 while solving problem P4

Consider triangle $MNP$ isosceles at $M$. $(y)$ is perpendicular from $M$ to $[NP]$. $(y)$ cuts $[NP]$ at $R$. $S$ is the midpoint of $[MR]$. Through $S$, draw $(x)$ is perpendicular to $(MR)$. $(x)$ cuts $[MN]$ in $J$. $(x)$ cuts $[MP]$ in $F$.

In problem P4 (a) which required drawing the figure, S7 read the questions carefully, and she drew the figure correctly.
In P4 (b) which required showing that (MR) is the angular bisector of \(\overline{MMP}\), S7 wrote a correct proof but that contained an inappropriate statement of a property (IDTP). S7 wrote the following: “in an isosceles triangle, any height is an angular bisector”. It is well known that in an isosceles triangle, the height issued from the main vertex is an angular bisector. The researcher asked S7 to draw another height of triangle MNP and asked S7 to check if the drawn height is an angular bisector. S7 noticed that the drawn height was not an angular bisector. Nevertheless, she did not notice the error that she had committed.

S7 was able to correctly prove parts (c, d, e, & f) of problem P4. However, in P4 (g) S7 stopped writing after reading the question, which required deducing that lines (MN) and (FR) were parallel. The researcher asked S7, “How do you prove that lines are parallel?” S7 replied, “By proving that the two lines are perpendicular to the same line or parallel to the same line and in this case we cannot prove them. So I do not know.” S7 was totally giving up. According to her response, it was clear that S7 lacks knowledge about other methods of proving parallel lines. Thus,
she had inability to understand some mathematical concepts related to proving parallel lines.

As a summary of the difficulties faced in problem P4, which presented simple proof tasks and simple mathematical language, the errors committed by S7 showed that she had difficulties related to Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC).

4.4.2.1.3 *Comparison between difficulties faced by S7 in problems P1 and P4*

After comparing the difficulties that S7 had in problems P1 and P4, we noticed that all the difficulties that S7 had were not affected by the complexity of the proof tasks. In both problems she had difficulties in Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC).
4.4.2.2 7C2

In Test 2, S7 is given the two problems P3 and P2.

4.4.2.2.1 Difficulties faced by S7 while solving problem P3

In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

\[ \begin{array}{c}
A \\
| \quad I \\
| \quad J \\
| \quad B \\
D \\
\end{array} \]

S7 started reading the questions carefully. On the given figure, she coded the properties provided in the premises of the problem P3, as shown in the adjacent figure. She coded all the given properties. This shows that she was able to comprehend the mathematical text of the problem.
She read the question in P1 (a) “Show that BI is equal to DJ”, and correctly found the appropriate proof.

In P3 (b), which required showing that angles $\overline{BTA}$ and $\overline{DTC}$ are equal, S7 decided to prove that triangles ABI and JDC were equal and to use their homologous elements to answer the question. She started finding equal elements of the considered triangles, as shown in the figure below.

S7 stopped and said that she was trying to prove that the sides AB and CD were equal. When S7 figured out that she was not able to prove that AB and CD were equal, she stopped. She decided to move to part (c) of problem P3. It is clear that S7 had a problem related to the understanding of the concept of parallel lines and the inferences derived. To her, parallel segments should be equal.
When S7 started reading the part (c) of problem P3, she noticed that the proof plan that she thought of in problem P3 (b) was wrong. The researcher asked her, “How did you know that your proof was wrong?” S7 replied, “Because in part c we have to prove equal triangles and not in part b. This means that we have to use what we have proved in part b and not repeat it.” S7 directly corrected her proof in part (b) and then proved the required task in P3 (c) correctly. The errors committed by S7 in P3 (b) showed that S7 had difficulty in Setting Proof Plans (SPP) and in Understanding-Applying Mathematical Concepts (UAMC) that are related to properties of parallel lines.

S7 was able to correctly prove the required tasks in parts (d, e, & f) without any difficulties.

In P3 (g) which required showing that lines (AD) and (BC) were parallel S7 stopped because she was not able to find the appropriate approach. She recalled aloud the methods that she knows to prove parallel lines. It took her few minutes of thinking to come up with the appropriate proof.
As a conclusion, the difficulties of S7 in constructing and formulating geometric proofs were related to Setting Proof Plans (SPP) and Understanding-Applying Mathematical Concepts (UAMC).

4.4.2.2 Difficulties faced by S7 while solving problem P2

Consider an isosceles triangle MNP whose main vertex is M and foot of the perpendicular drawn from M to [NP] is R. Through S, the midpoint of [MR], (x) is the perpendicular drawn to (MR) cutting [MN] and [MP] in J and F respectively.

S6 read the questions carefully. She separated the long sentences into phrases before drawing a correct figure. She coded on her figure all properties provided from the premises of the problem as shown in the figure below. This shows that S7 was able to comprehend the complex text of problem P2.
In problem P2 (b), S7 developed a proof that contained lots of errors as shown in the figure below.

The first error committed by S7 was developing a proof approach that will not work (ANW). Showing that [MR] is an angular bisector does not require mentioning that the sides MN and MP were equal. S7 aimed from using the equality of MN and MP to verify that point M belongs to the angular bisector of angle \( \angle NMP \). However, the vertex of any angle belongs to its angular bisector. The second error was that she did not state the property used appropriately (IDTP). She wrote “Any point is equidistant from the endpoints of the segment then it belongs to the angular bisector”. The terms of the written property are not related and it is clear that she mixed between properties related to points belonging to the perpendicular bisector of a segment and points belonging to the angular bisector of an angle.

In part (d) of problem P2, S7 was not able to prove that line (x) was the perpendicular bisector of [MR]. However, when
the researcher asked her to recall the definition of a perpendicular bisector she stopped for a while and was able to find the appropriate proof. This showed that she had difficulty in Setting Proof Plans (SPP) related to particular geometric concepts without any help.

In parts (c, e, f, & g), S7 did not face any difficulty. The researcher was expecting that S7 would face difficulty in part (g) that required proving parallel lines, compared to previous experiences in P1, P3, and P4, but surprisingly S7 was able to explore the solution.

As a summary of the difficulties faced in problem P2, which presented simple proof tasks and a complex mathematical language, the errors committed by S7 showed that she had difficulties related to Setting Proof Plans (SPP), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT).
4.4.2.2.3 Comparison between difficulties faced by S7 in problems P3 and P2

After comparing the difficulties that S6 had in problems P3 and P2, we noticed that the difficulties faced by S7 were not affected by the complexity of the mathematical language. In problems P3 and P2, S7 had difficulties in Setting Proof Plans (SPP) and Understanding-Applying Mathematical Concepts (UAMC). However, in P2 she faced a difficulty in writing some mathematical texts. However, this difficulty can be ignored because S7 was able to write the same property correctly in part (e) of problem P2.

4.4.2.3 Comparison across problems P1, P2, P3, and P4 during 7C1 and 7C2

Upon varying the proof and language complexities across problems P1, P2, P3, and P4, the difficulties that S7 faced varied according to the nature of the problem. In P1 (complex proof task; simple mathematical language), S7 had difficulty in Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts
(WMT). However, reducing the proof difficulty in the isomathic problem P3 did not affect her difficulty in Setting Proof Plans (SPP). In problem P2 (simple proof tasks; complex mathematical language), S7 had difficulty in Setting Proof Plans (SPP) and Understanding-Applying Mathematical Concepts (UAMC). However, reducing the language complexity level in the isomathic problem P4 did not affect the difficulties that S7 had.

The current chapter presented a detailed analysis and interpretation of the data collected. The next chapter provides conclusions related to the research questions built on the data analysis results.
CHAPTER FIVE

CONCLUSIONS

5.1 Introduction

The current study aimed at exploring difficulties that Lebanese middle school students face when constructing and formulating geometric proofs using their non-native language (English). There were many facts that promoted carrying out this research. The first fact, and as previously mentioned, was that middle school students struggle when constructing geometric proofs. According to Senk (1985), “only 30% of students in full-year geometry courses that teach proof reach a 75% mastery level in proof writing” (p. 168). The second fact was that communicating mathematics using a non-native language is considered to be cognitively demanding for students. Adegoke and Ibode (2011) state that the language that is used to transmit knowledge plays a major role in that transmission. According to Freeman and Crawford (2008), when students learn mathematics using a non-native language, they will face language barriers. Thus, Lebanese middle school students who learn, construct, and formulate geometric proofs using their non-native language were expected to have difficulties when constructing and formulating geometric proofs since they were exposed to two major proof writing barriers, proof and language difficulties.
In the current chapter, discussion of results and elements of answers for the three research questions will be provided.

5.2 Discussion of Results Based on Research Questions

5.2.1 Research question 1: Do Lebanese middle school students face difficulties when constructing geometric proofs? What are these difficulties?

In order to answer this research question the researcher have to consider the levels of difficulty that the students had in each of the problems P1, P2, P3 and P4, according to each of the identified six difficulties in the framework (Understanding the Notion of Proof “UNP”, Setting Proof Plans “SPP”, Conducting Deductive Reasoning “CDR”, Understanding-Applying Mathematical concepts “UAMC”, Comprehending Mathematical Texts “CMT”, and Writing Mathematical Texts “WMT”) at each of the grades 6 and 7.

In grade 6 and according to the simple language and varying proof-complexity tests’ analysis results, students had difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC) irrespective of the proof complexity level in each of the isomathic problems P1 and P3 which addressed proof tasks in a simple language. The percentages of the students having these difficulties almost maintained high values in problems P1
and P3, though varying from P1 to P3. Yet, difficulties related to Setting Proof Plans (SPP), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT) were affected by varying the proof complexity level between problems P1 and P3 but maintained values that were considered to be minimal.

According to the results of the clinical interviews 6C1 and 6C2 and irrespective of varying the proof complexity level across the isomathic problems P1 and P3, S6 had difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT). However, difficulty related to Setting Proof Plans (SPP) was affected by the proof complexity level. As for Comprehending Mathematical Texts (CMT), S6 did not face any difficulty in Comprehending simple Mathematical Texts (CMT).

Thus, irrespective of the proof complexity level, grade 6 students do face difficulties that are related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC). Nevertheless, when the proof tasks are complex, grade 6 students have additional difficulties such as Setting Proof Plans (SPP) and Writing Mathematical Texts (WMT).
In grade 6 and according to the simple proof task and varying language-complexity tests’ analysis results, students had difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Comprehending Mathematical Texts (CMT), irrespective of the language complexity level in each of the isomathic problems P2 and P4 which addressed simple proof tasks. The percentages of the students having these difficulties almost maintained high values in problems P2 and P4, though varying from P2 to P4. Yet, difficulties related to Setting Proof Plans (SPP) and Writing Mathematical Texts (WMT) were not affected by varying the language complexity level between problems P2 and P4 and maintained values that were considered to be minimal.

According to the results of the clinical interviews 6C1 and 6C2 and irrespective of varying the language complexity level across the isomathic problems P2 and P4, S6 had difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Understanding-Applying Mathematical Concepts (UAMC), and Writing Mathematical Texts (WMT). However, difficulties related to Setting Proof Plans (SPP) and Comprehending Mathematical Texts (CMT) were affected by the language complexity level.
Thus, irrespective of the language complexity level, grade 6 students do face difficulties that are related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC). Nevertheless, when the mathematical language is complex, grade 6 students have additional difficulties such as Setting Proof Plans (SPP), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT).

As a conclusion, irrespective of proof and language complexity levels, grade 6 students face difficulties related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC). However, proof and language complexities generate difficulty related to Setting Proof Plans (SPP) while language complexity blurs the students’ ability to Comprehend Mathematical Texts (CMT). Yet, difficulty related to Writing Mathematical Texts (WMT) varies according to individual differences between students.

In grade 7 and according to the simple language and varying proof-complexity tests’ analysis results, when the proof tasks were complex in problem P1, students had difficulties related to Understanding the Notion of Proof (UNP), Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC). However, these
difficulties declined when the proof tasks were simplified in the isomathic
problem P3 of P1. The percentages of the students having these difficulties
dropped from their high values in problem P1 to have minimal values in problem
P3. Yet, difficulties related to Comprehending Mathematical Texts (CMT) and
Writing Mathematical Texts (WMT) were considered to have minimal values in
problems P1 and P3.

According to the results of the clinical interviews 7C1 and 7C2 and
irrespective of varying the proof complexity level across the isomathic problems
P1 and P3, S7 had difficulties related to Setting Proof Plans (SPP) and
Understanding-Applying Mathematical Concepts (UAMC). Yet, when the proof
task was complex in problem P1, S7 had difficulty in Conducting Deductive
Reasoning (CDR). On the other hand, and irrespective of the proof complexity
levels in problems P1 and P3, S7 did not face difficulties related to
Understanding the Notion of Proof (UNP), Comprehending Mathematical Texts
(CMT), or Writing Mathematical Texts (WMT).

Thus, irrespective of the proof complexity level, grade 7 students do face
difficulties that are related to Setting Proof Plans (SPP). Nevertheless, when the
proof tasks are complex, grade 7 students face more difficulties related to
Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning
(CDR), and Understanding-Applying Mathematical Concepts (UAMC).
In grade 7 and according to the simple proof task and varying language-complexity tests’ analysis results, students did not have difficulties in constructing and formulating geometric proofs. Rather, percentages of the students reflecting each difficulty were considered to be minimal irrespective of the language complexity level addressed in the isomathic problems P2 and P4.

According to the results of the clinical interviews 7C1 and 7C2 and irrespective of varying the language complexity level across the isomathic problems P2 and P4, S7 had difficulties related to Setting Proof Plans (SPP) and Understanding-Applying Mathematical Concepts (UAMC). However, difficulty related to difficulty in Writing Mathematical Texts (WMT) was affected by the language complexity level.

Thus, irrespective of the language complexity level, grade 7 students do face difficulties that are related to Setting Proof Plans (SPP).

As a conclusion, when the proof tasks are complex, and irrespective of the language complexity level, grade 7 students face difficulties related to Setting Proof Plans (SPP), Conducting Deductive Reasoning (CDR), and Understanding-Applying Mathematical Concepts (UAMC). However, language complexity did not affect the students’ ability to Comprehend Mathematical Texts (CMT). Yet,
difficulty related to Writing Mathematical Texts (WMT) varied according to individual differences between students.

According to the results of the analysis of the interview with the math coordinator, the difficulties that students face when constructing and formulating geometric proofs are related to Understanding the Notion of Proof (UNP), Conducting Deductive Reasoning (CDR), Comprehending Mathematical Texts (CMT), and Writing Mathematical Texts (WMT).

These results coincide with results in the literature. For instance, Weber (2001) and Moore (1994) assert that students’ conception of the nature of proof, which represents Understanding the Notion of Proof (UNP), blurs their ability to construct and formulate proofs. Furthermore, Weber (2001) and Moore (1994) state that students’ Understanding-Applying of the Mathematical Concepts (UAMC) affects their ability to construct proofs. Andrew (2009), Heinze, Cheng, Ufer, Lin, & Reiss (2008), and Moore (1994) claim that students face difficulty in Setting Proof Plans (SPP) when the proof task requires writing a big number of arguments, when the proof task is complex; they then won’t know how to start a proof. According to Mariotti (2006), Herbst and Brach (2006), Hoyles and Healy (2007), and Harel and Sowder (2007) difficulties faced by students when constructing proofs are due to the fact that students do not recognize that proofs require certifying that something is true by giving reasons from theorems and
properties. Moreover, they have difficulty in connecting statements and conclusions. Thus, they have difficulty in Conducting Deductive Reasoning (CDR). As for Comprehending and Writing Mathematical Texts (CMT) and (WMT), Moore (1994), Slavit and Ernst-Slavit (2007), and Morgan (1996) state that students have difficulty in comprehending and using mathematical notations, terminology, and language and thus won’t be able to use them correctly. Moreover, Balacheff (2000), Selden and Selden (2003), Gueudet (2008), and Yang and Lin (2008) claim that the language used to formulate and produce the proof blurs the students’ ability to express their thoughts and ideas about the proof tasks.

5.2.2 Research question 2: Is it possible to classify, differentiate or distinguish the difficulties that are due to language and those that are due to proofs’ cognitive complexity?

The six major difficulties identified in the framework were reflected through errors committed by the students. As mentioned before, some errors reflect more than one difficulty. In order to determine whether it is possible to classify, differentiate or distinguish the difficulties that are due to language and those that are due to proofs’ cognitive complexity or not, we need to refer to the nature of the errors that reflected each difficulty, to the students’ tests results, and
to the clinical interviews’ results to provide explanation of each difficulty and its sources.

5.2.2.1 Understanding the Notion of Proof (UNP)

According to the tests’ analysis results of grade 6, it was found that students’ difficulty related to Understanding the Notion of Proof (UNP) maintained high values across problems P1, P2, P3, and P4. However, its highest values were attained in solving problems P1 and P2. Thus, we may consider that Understanding the Notion of Proof (UNP) is affected by the proof and language complexity levels. Yet, it persists as we vary both proof and language complexity levels. Moreover, and according to the grade 6 clinical interviews’ results, it was found that irrespective of proof and language complexity levels, the student faced difficulty in Understanding the Notion of Proof (UNP). Hence, as a conclusion, students’ difficulty related to Understanding the Notion of Proof (UNP) is not related to either proof or language complexity levels.

According to the tests’ analysis results of grade 7, it was found that students’ difficulty related to Understanding the Notion of Proof (UNP) maintained minimal values across problems P1, P2, P3, and P4. However, its highest value was in solving problem P1. Thus, we may
consider that Understanding the Notion of Proof (UNP) is affected by the proof complexity level. Yet, it persists as we vary both proof and language complexity levels. Moreover, and according to the grade 7 clinical interviews’ results, it was found that irrespective of proof and language complexity levels, the student did not have difficulty in Understanding the Notion of Proof (UNP). Hence, as a conclusion, students’ difficulty related to Understanding the Notion of Proof (UNP) is not related to either proof or language complexity levels. However, and according to the results of interview with the math coordinator, it might be due to the minimal time allocated by teachers to present proving as a process irrespective of any mathematical context. Moreover, the analysis of the Lebanese national curriculum and textbooks at the middle school level highlighted the lack of any introductory section for proving and reasoning. Thus, not helping in developing the students’ ability to understand what a proof means and requires.

The current research results are aligned with research in literature. For instance, Weber (2001) categorized students’ difficulties in constructing proofs into two categories. The first category arises from the students’ conception of the nature of mathematical proof. The second difficulty category by Weber (2001) arises from the students’
misunderstanding of either the concept or the theorem. The two categories identified by Weber (2001) show that students’ difficulty in Understanding the Notion of Proof (UNP) is a difficulty by itself and not affected by other factors.

5.2.2.2 Setting Proof Plans (SPP)

According to the results of tests’ analysis of grade 6, it was found that students’ ability to set proof plans was affected by proof complexity level. The highest level of difficulty was in solving problem P1. However, the difficulty level dropped at problems P2, P3, and P4. As for the clinical interviews’ results, the student had difficulty in Setting Proof Plans (SPP) in solving problems P1 and P2. However, she did not have difficulty in Setting Proof Plans (SPP) in solving problems P3 and P4. This showed that students’ ability to set proof plans was affected by both proof and language complexity levels.

According to the results of the tests’ analysis of grade 7, the highest level of proof difficulty related to Setting Proof Plans (SPP) was in solving problem P1. However, it dropped to its minimal values in problems P2, P3, and P4. As for the results of the clinical interviews of grade 7, the student had difficulty in Setting Proof Plans (SPP) across
problems P1, P2, P3, and P4 irrespective of both proof and language complexity levels. Discussions with the student showed that her inability to set proof plans was due to lack of understanding of some mathematical concepts.

As a conclusion, difficulty in Setting Proof Plans (SPP) has several sources. It might be due to proof complexity as in grades 6 and 7 tests, language complexity as in grade 6 clinical interviews, understanding of mathematical concepts as in grade 7 tests. This result might not be surprising since the Lebanese national textbooks do not provide students with guidelines that help them overcome the obstacle of setting proof plans at various levels of proof complexity.

The current research results are aligned with the research in the literature. For instance, Senk (1985), Morgan (2005), Knapp (2005), and Harel and Sowder (2007) claim that when students don’t understand definitions, theorems, and properties derived appropriately, they will not be able to identify which one the proof involves. In addition, Heinze, Cheng, Ufer, Lin and Reiss (2008) and McCrone and Martin (2004), and Andrew (2009), state that one of the important predictors of a proof’s difficulty is the number of arguments that a student has to combine in order to attain validity of the required statement. Moreover, Cirillo,
Bruna, and Herbel-Eisenmann (2010) state that lexile difficulties blur students’ understanding of mathematical texts and block their ability to write and communicate mathematically.

5.2.2.3 Conducting Deductive Reasoning (CDR)

According to the results of grade 6 tests’ analysis, it was found that the level of difficulty related to Conducting Deductive Reasoning (CDR) maintained high values across problems P1, P2, P3, and P4, though it varied from one to another. Thus, varying proof and language complexities do affect the students’ ability to conduct deductive reasoning but if somebody had this difficulty, it still appears at different degrees, regardless of the proof and language complexity levels. As for the results of the grade 6 clinical interviews, the student had difficulty in conducting deductive reasoning throughout problems P1, P2, P3, and P4. Her inability to conduct deductive reasoning was sometimes due to lack of understanding of some mathematical concepts. These results are aligned with the results of the interview with the math coordinator.

According to the math coordinator, students have difficulty in differentiating between premises and conclusions.
Results of research in literature such as Yang and Lin (2008), Andrew (2009), and Powers, Craviotto, and Grassel (2010), show that students don’t realize the existence of a relation between statements and conclusions of a proof or statements and conclusions. Moreover, Yang and Lin, 2008; Hanna and De Villiers, 2008; and Andrew, 2009, state that mathematical proof is a challenge because it requires certifying that something is true and explaining why it is true. Moreover, it requires giving reasons from definitions, theorems, and properties.

5.2.2.4 Understanding-Applying Mathematical Concepts (UAMC)

Results of the tests’ analysis of grade 6 showed that students’ difficulty in Understanding-Applying Mathematical Concepts (UAMC) maintained high values across problems P1, P2, P3, and P4 though varying from one problem to another. However, results of tests’ analysis of grade 7 showed that students had difficulty in Understanding-Applying Mathematical Concepts (UAMC) at problem P1. Yet, at problems P2, P3, and P4 difficulty levels were of minimal values. As for the grade 6 clinical interviews’ results, the student showed difficulty in Understanding-Applying Mathematical Concepts (UAMC) throughout solving problems P1, P2, P3, and P4. However, the analysis of the grade 7 clinical interviews showed that the student had difficulty in
Understanding-Applying Mathematical Concepts (UAMC) across problems P1, P2, P3, and P4 irrespective of the proof and language complexity levels.

As a conclusion, students’ ability to construct and formulate geometric proofs was affected by students’ understanding-Applying of Mathematical Concepts (UAMC) addressed in the proof tasks. Proof and language complexity levels were sources of this difficulty in some cases. Yet, in other cases, students had this difficulty irrespective of complexity levels of proof and language.

The current research results are aligned with research in literature. According to Senk (1985), Morgan (2005), Knapp (2005), and Harel and Sowder (2007), when students don’t understand definitions, theorems, and properties derived appropriately, they will not be able to identify which one the proof involves. Thus, students will misapply these concepts and theorems.

5.2.2.5 Comprehending Mathematical Texts (CMT)

Results of the tests’ analysis of grade 6 showed that students had difficulty in Comprehending mathematical Texts (CMT) in solving problems P2 and P4. Though problem P4 addresses simple mathematical
language, students had difficulty in comprehending some of its mathematical texts that required using mathematical symbols. However, according to the results of the tests’ analysis of grade 7, students did not face difficulties related to Comprehending Mathematical Texts (CMT) regardless of proof and language complexity levels. As for the results of the clinical interviews at grade 6, the student had difficulty in Comprehending Mathematical Texts (CMT) that were considered to be complex as in P2, and those that required using mathematical concepts as in P4. However, the results of the clinical interviews of grade 7 showed that the student did not face difficulty in Comprehending Mathematical Texts (CMT).

As a conclusion, students’ difficulty in constructing geometric proofs might be due to their inability to comprehend mathematical texts. Moreover, and according to the math coordinator’s interview, students’ inability to comprehend the text of the definitions, theorems, or properties blurs their ability to utilize these definitions, theorems, and properties appropriately in proofs. This result is aligned with research results. For instance, Morgan (1996), Zack (1999), and Cirillo, Bruna, and Herbel-Eisenmann (2010) present several lexile difficulties that might blur
students’ understanding of mathematical texts and block their ability to write and communicate mathematically.

5.2.2.6 Writing Mathematical Texts (WMT)

Results of the tests’ analysis of grades 6 and 7 showed that the level of difficulty related to Writing Mathematical Texts (WMT) had minimal values across problems P1, P2, P3, and P4. This shows that irrespective of proof and language complexity levels, students’ ability to write mathematical texts is an individual issue. This might be explained according to the grade 6 clinical interviews’ results. The student had difficulty in writing mathematical texts across problems P1, P2, P3, and P4. This could be due to her weakness regarding the non-native language used (English) and not to the mathematical language itself.

This result is explained in research results. For instance, Harel and Sowder (1998) and Szendrei-Radnai and Török (2007) claim that even when students produce right proofs, it is hard to understand the written proof due to problems in wording and inability to express themselves. They add that some students use words that don’t match with the proof’s context because of their multiple meanings. Hence, it is vital to recognize the nested relation between mathematical language and proofs in terms of
content, structure, and coherence in order to enhance students’ abilities to construct and formulate well-developed geometric proofs, and communicate mathematically.

5.2.3 Research question 3: Are there any developments or changes in the nature and extent of difficulties through the four grade levels of middle school (grades 6 to 9)?

Results of the comparison between the tests’ analysis across grades 6 to 9 showed that students’ difficulty in Understanding the Notion of Proof (UNP) had its highest values at grade 6 and that the students’ ability to Understand the Notion of Proof (UNP) developed gradually as students’ experience with proving as a process developed across grade levels. Difficulty related to setting proof plans (SPP) varied across grade levels and was affected by the proof complexity level of the proof tasks. This difficulty increased at grades 8 and 9 levels since despite of the experience that students gain from previous levels, the proof problems become more and more complex. Difficulty related to conducting deductive reasoning (CDR) had its highest values at grade 6. However, it varied across grades 7, 8, and 9 according to the nature of the proof tasks at each grade level. For instance, when the proof tasks were simple, students’ difficulty in Conducting Deductive Reasoning (CDR) had its minimal values at grades 7, 8, and 9. However, when the proof tasks were complex, the difficulty level of
Conducting Deductive Reasoning (CDR) was high in grades 6, 7, and 9. As for the difficulty related to Understanding-Applying Mathematical Concepts (UAMC) it had its highest values at grade 8 and varied across grade levels. Thus, it is not related to the proving process development across grades levels. Yet, it might be due to the load of the curriculum across grade levels. As mentioned before, in grade 8, students have to learn many more new topics, properties, and relationships in a short time. Difficulty related to comprehending mathematical texts (CMT) had its highest values at grade 6 when the language of the text was complex and dropped in other grade levels but at different rates. Finally, difficulty related to writing mathematical texts (WMT) also had its highest values at grade 6 and dropped in other grade levels but at different rates.

Research in literature showed that students’ ability to construct and formulate proofs develops as age and experience with proving as a process develop. For instance, Balacheff presents the four levels of proof cognitive development and which were named later as “Balacheff’s taxonomy of proof”. Moreover, Aydin and Halat (2009) present the Van Hieles’ theory related to the levels of development of reasoning in geometry. They assert that the levels presented by the Van Hieles are hierarchal and continuous. Similarly, in their study, Lin and Yang (2007) tested the ability of grade 9 and 10 to comprehend written geometric proofs and developed a model that consists of levels of
“reading comprehension of geometry proofs (RCGP)”. This model considers student’s ability to comprehend and analyze written geometric proofs.

5.3 Recommendations

The current research showed that students face difficulties when constructing and formulating geometric proofs. According to the explored difficulties, several recommendations that might help in reducing these difficulties are presented.

- Since students face difficulty related to Understanding the Notion of Proof (UNP), it is recommended that proving (mainly reasoning, argumentation and justification) as a process be introduced regardless of any mathematical context before introducing geometrical proofs. Informal, real-life exercises on using logic and inferences would be useful to set the stage for later more formal deductive work. Moreover, reasoning and proving might be taught at early grade levels in different content areas and contexts.

- Since the Lebanese national textbooks and teachers’ guides do not provide guidelines for introducing proofs or proof samples or methods, it is recommended that ECRD develop a new edition of the Lebanese national textbooks that presents and allocates a section for strategies of introducing proofs, proof guidelines, and proof samples.
• Since students face difficulties related to Understanding-Applying Mathematical Concepts (UAMC), it is recommended that teachers allocate enough time needed for students to comprehend learnt mathematical concepts prior to starting constructing and formulating geometric proofs related to the taught concepts. This will help in decreasing the effect of difficulties related to Understanding-Applying Mathematical Concepts (UAMC) on students’ abilities to construct and formulate proofs.

• Since students face difficulties related to Setting Proof Plans (SPP), it is recommended to have students evaluate and discuss in classroom various proof plans for a certain situation and identify those that are appropriate for the proof task at hand and those that are faulty or too long or indirect. This would help students recognize appropriate proof plans and negotiate with their teacher and their classmates issues related to development of proper proof plans.

• Have teachers analyze difficulties that students have through identification of the errors committed and design accordingly appropriate remedial plans. Moreover, teachers should be aware of the stages of proof and reasoning development and the characteristics of each stage and try
to remediate the gaps that the students have in reference to these characteristics.

5.4 Limitations of the Study

Though the researcher tried to investigate the validity for all instruments used, there are still some limitations for the study. First, the researcher is currently a teacher of two classes of participants and taught the other two classes in previous academic years. In this case, the researcher is somehow familiar with the difficulties faced by her students, thus might not think of other difficulties at times.

Second, the time that separated the administration of the two isomathic tests might be another limitation. The tests for each grade level tested the same objectives but in different situations. Since there is time separating the administration of the tests, students might have mastered some of the concepts tested, thus affecting the results of the tests.

Third, the participants selected belong to one school. In this case, results cannot be generalized to all Lebanese middle school students. Fourth, all participants selected were girls. This might affect the validity of the results as being gender bias.
5.5 Recommended Future Research

The current study presented difficulties faced by Lebanese middle school students who learn mathematics using their non-native language (English) and their possible source of difficulties. According to the results of the current research, several recommendations would be presented for future research.

First, it is recommended to conduct this research in other schools that vary in terms of their curriculum and socio-economic status in order to test and compare other factors that might affect students’ ability to construct and formulate geometric proofs.

Second, it is recommended to conduct this research on each grade levels of the middle school at different schools that teach math using the native and non-native language and that adopts different curricula and textbooks. This will help in providing a detailed exploration of the sources of difficulties that the students face at each grade level in different circumstances.

Third, it is recommended that after providing students with a unit that teaches proving irrespective of any mathematical context and developing math books that present strategies for geometric proof development, to conduct an experimental research that explores the development of proof difficulties at various grade levels.
Fourth, it is recommended to conduct a comparative study between different teacher’s approaches to teaching of proof and their impact on the students’ ability to construct and formulate geometric proofs.

Fifth, it is proposed to conduct a research that explores the students’ difficulties in constructing and formulating geometric proofs related to specific mathematical concepts across the grade levels of the middle school. This will help providing a better insightful exploration of the effect of the cognitive level of certain mathematical concepts on students’ ability to construct and formulate geometric proofs.
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Appendix A

Math Coordinator’s Interview

The math coordinator of the middle school was asked the following questions:

1. At which grade level do you start introducing proofs?
2. What forms of proofs does the school adopt?
3. Do you accept using other forms? Why/Why not?
4. What strategies do you follow to develop proving skills?
5. What are the difficulties you face as a teacher when teaching students to construct geometric proofs?
6. In your opinion or from your experience, what difficulties do students face, particularly difficulties of language source, when constructing geometric proofs?
7. What do you do, as a teacher; to emphasize appropriate use of language when reading or listening to students’ constructed geometric proofs?
8. What kind of help do you provide for students who have proof difficulties?
9. Do you think that students don’t understand the notion of proof?
10. Do students construct proofs in algebra? In what forms? At which grade levels?
11. Do you accept introducing the notion of proof regardless of the mathematical domain?
Appendices B to E
Tests 1 and 2
Grades 6 to 9
Appendix B

Grade 6 Tests 1 and 2

Grade: 6                      Name: _______________
Section: A & B                  Test 1                      Date: ________________
Duration: 50 minutes

Read the following instructions carefully:

- Read the given problems carefully.
- Make sure you answer all the questions.
- Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.
- Write your opinion and comments about each of the given problems.

Problem 1:

In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE]. \( \hat{DBE} = 30^\circ \). AB = 3cm.
What is the length of [BC]? Justify your answer.
Problem 2:

Triangle MNP is isosceles at M. NM = MP = 4cm. NMP = 120°.

1. Draw triangle MNP.

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2. Calculate the measures of MNP and MPN. Show your work. Justify your answer.

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3. Through M, draw (x) perpendicular to [NP]. (x) cuts [NP] at R. Through P, draw (d) parallel to (x). (d) cuts (NM) at Q.

   a) What is the relative position of (NP) and (d)? Justify your answer.
b) Name a height of triangle NPQ. Justify your answer.
Grade: 6                                      Name: _______________
Section: A & B                                Test 2                     Date: ________________
Duration: 50 minutes

Read the following instructions carefully:

- Read the given problems carefully.
- Make sure you answer all the questions.
- Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.
- Write your opinion and comments about each of the given problems.

**Problem 1:**

In the given figure, D is the midpoint of [BC]. (AD) is perpendicular to [BC]. (AB) is perpendicular to [BE]. \( \angle DBE = 30^\circ \). AB= 3cm.

a) Show that (AD) is the perpendicular bisector of [BC].

b) Show that AB= AC. Justify your answer.
c) Calculate the measure of $\hat{ABC}$. Show your work. Justify your answer.

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d) Show that triangle ABC is equilateral. Justify your answer.

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Problem 2:

Consider an isosceles triangle MNP whose main vertex is M and the length of its equal sides is 4cm. Let $\angle NMP = 120$.

1. Draw triangle MNP.

2. Calculate the measures of $\angle MNP$ and $\angle MPN$. Show your work and justify your answer.
3. Let R be the foot of the perpendicular drawn from M to [NP] and (d) be the parallel to (MR) drawn through P and cutting (NM) at Q.
   a) How are (NP) and (d) related? Justify your answer.

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   b) Name a height of triangle NPQ. Justify your answer.

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Appendix C

Grade 7 Tests 1 and 2

Grade: 7                        Name: _______________
Section: A & B                  Test 1                      Date: ________________
Duration: 50 minutes

Read the following instructions carefully:

- Read the given problems carefully.
- Make sure you answer all the questions.
- Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.
- Write your opinion and comments about each of the given problems.

**Problem 1:**

In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

![Diagram](image)

a) Show that AI = JC.
b) Show that (AD) is parallel to (BC).
Problem 2:

Consider triangle MNP isosceles at M. (y) is perpendicular from M to [NP]. (y) cuts [NP] at R. S is the midpoint of [MR]. Through S, draw (x) is perpendicular to (MR). (x) cuts [MN] in J. (x) cuts [MP] in F.

a) Draw the figure.

b) Show that (MR) is the angular bisector of $\hat{P}M\hat{N}$.

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c) Show that (x) is parallel to (NP).

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d) Show that (x) is the perpendicular bisector of [MR].

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e) Compare MF and FR. Justify your answer.

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f) Show that triangles MSF and SFR are congruent. Write their homologous parts.

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Grade: 7                                Name: ______________
Section: A & B                              Test 2                      Date: ______________

Duration: 50 minutes

Read the following instructions carefully:

- Read the given problems carefully.
- Make sure you answer all the questions.
- Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.
- Write your opinion and comments about each of the given problems.

**Problem 1:**

In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

![Diagram of the figure]

**a)** Show that BI = DJ.
b) Show that $BÎA = DÎC$.

c) Show that triangles CDJ and ABI are congruent.

d) Show that AI = JC.

e) Show that $AÎD = CÎB$. 
f) Prove that triangles ADI and BCJ are congruent. Write their homologous parts.

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g) Deduce that (AD) is parallel to (BC).
Problem 2:

Consider an isosceles triangle MNP whose main vertex is M and foot of the perpendicular drawn from M to [NP] is R. Through S, the midpoint of [MR], (x) is the perpendicular drawn to (MR) cutting [MN] and [MP] in J and F respectively.

a) Draw the figure.

b) Show that [MR) is the angular bisector of $\angle PMN$.

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e) Compare MF and FR. Justify your answer.

f) Show that triangles MSF and SFR are congruent. Write their homologous parts.

g) Deduce that (MN) and (FR) are parallel.
Appendix D
Grade 8 Tests 1 and 2

Grade: 8                                                                                     Name: _______________
Section: A                                                  Test 1                      D
Date: ________________
Duration: 50 minutes

Read the following instructions carefully:

- Read the given problems carefully.
- Make sure you answer all the questions.
- Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.
- Write your opinion and comments about each of the given problems.

Problem 1:

Given: ABC is any triangle. [AM] is a median. [AH] is a height. E is the symmetric of A with respect to M. D is the symmetric of A with respect to H.

Show that BE = DC.
Problem 2:

ABC is a right triangle at C such that AC = 6cm, BC = 8cm, and AB = 10cm. [AM] is the median relative to [BC]. Through B, draw a perpendicular to (AM) at E. Through C, draw a perpendicular to (AM) at F.

1) Draw the figure.

2) Prove that triangles FCM and BEM are congruent. Write their homologous parts.

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3) Show that quadrilateral CFBE is a parallelogram.
4) Let I be the midpoint of [AB]. R the symmetric of E with respect to I.
   a) Show that quadrilateral BRAE is a rectangle.

   b) Calculate EI and CI.
c) Deduce the nature of triangle EIC.
Problem 1:

Given: ABC is any triangle. [AM] is a median. [AH] is a height. E is the symmetric of A with respect to M. D is the symmetric of A with respect to H.

a) Show that (DE) is parallel to (BC).

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b) Show that BA = BD.

c) Show that ABEC is a parallelogram.

d) Show that BCED is an isosceles trapezoid.
e) Deduce that BE = CD.
Problem 2:
Consider a triangle ABC right angled at C such that AC= 6cm, BC= 8cm, and AB= 10cm, in which [AM] is a median relative to [BC]. E and F are the respective feet of the perpendicularly drawn from B and C to (AM).

1) Draw the figure.

2) Prove that triangles FCM and BEM are congruent. Write their homologous parts.

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3) Show that quadrilateral CFBE is a parallelogram.

4) Let I be the midpoint of [AB] and R the symmetric of E with respect to I.
   a) Show that quadrilateral BRAE is a rectangle.
b) Calculate EI and CI.

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Appendix E

Grade 9 Tests 1 and 2

Grade: 9                                    Name: _______________

Section: A, B & C                           Test 1           Date: _______________

Duration: 50 minutes

Read the following instructions carefully:

• Read the given problems carefully.
• Make sure you answer all the questions.
• Don’t use a draft paper; write all your thoughts on the answer sheet. Cross, using a pencil, answers that you want to cancel.

Problem 1:

Given: Consider a semi-circle (C) of center O, radius R, and diameter [AB]. C is point on (C) such that (CO) is perpendicular to [AB]. M is any point on arc \( \overline{BC} \). (CM) and (AB) intersect at point P.
1) Show that $\overline{CMA} = \overline{BMP}$.

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2) Let I be the point of intersection of (AM) and (OC).
   Show that $AI \times AM = 2R^2$. 

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3) Suppose that I is the midpoint of [OC]. Calculate the length of [AI] and that of [MB] in terms of R.
Problem 2:

In triangle ABC, P is a point on [BC]. [AD] is a median. Through P, draw (x) parallel to [AD]. (x) cuts (AB) in M and (AC) in N.

1) Draw the figure.

2) Prove that \( \frac{AM}{AB} = \frac{AN}{AC} \).

3) Show that PM x BD = DA x DP.
Problem 1:

Given: Consider a semi-circle (C) of center O, radius R, and diameter [AB]. C is a point on (C) such that (CO) is perpendicular to [AB]. M is any point on arcBC. (CM) and (AB) intersect at point P.

1) Show that \(\widehat{CM}A = \widehat{BMP} = 45^\circ\).
2) Let I be the point of intersection of (AM) and (OC).
   a) Show that triangles AMB and AOI are similar. Write their ratio of similarity.
   b) Deduce that $AI \times AM = 2R^2$. 
3) Suppose that I is the midpoint of [OC].
   a) Calculate the length of [AI] in terms of $R$.
   b) Deduce the length of [AM] then that of [MB] in terms of $R$. 
**Problem 2:**

In triangle ABC, a line through P, a point on [BC], is drawn parallel to the median [AD] cutting (AB) and (AC) in M and N respectively.

1) Draw the figure.

2) Prove that the ratio of AM to AB is equal to that of AN to AC.

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3) Show that PM and BD have the same product as DA and DP.

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Appendices F and G

Proof and Language Complexity Rubrics
## Appendix F

### Proof’s Complexity Rubric

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The given</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
<td>The given is provided using mathematical statements but no figure is provided.</td>
</tr>
<tr>
<td><strong>The figure</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
<td>The figure is complex (too many imbricated parts)</td>
</tr>
<tr>
<td><strong>The required to prove</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
<td>The proof task is not stated in a direct way and requires evaluation level according to Bloom’s Taxonomy</td>
</tr>
<tr>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
<td>The proof requires considering imbricated parts of a figure</td>
</tr>
<tr>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
<td>The proof task involves too many geometric objects</td>
</tr>
<tr>
<td><strong>Number of statements required in the proof</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
<td>Multi-step proof that requires partial proofs for the statements and reasons for the proof task</td>
</tr>
<tr>
<td><strong>Usage of definitions, theorems &amp; postulates</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
<td>Requires usage of definitions, properties, and theorems</td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
<td>Easy 1</td>
<td></td>
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</tbody>
</table>

**Average proof complexity:** 14/21
**Appendix G**

**Text Complexity Rubric**

Grade Level: _________  Problem #: _________

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Easy 1</th>
<th>Average 2</th>
<th>Difficult 3</th>
<th>Score / 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Semantic features of the text</strong></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td>Compound sentence</td>
<td>Complex sentence</td>
<td></td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
<td>Usage of more than two connectors in a statement</td>
<td></td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
<td>Infrequently used tenses</td>
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</tr>
<tr>
<td>prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
<td>Frequent use of prepositions</td>
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<tr>
<td>pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
<td>Frequent use of pronouns</td>
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<tr>
<td><strong>Syntactic features of the text</strong></td>
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<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
<td>The sentence used can be subdivided into two or more sentences</td>
<td></td>
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<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
<td>A sentence has a dependent clause as a modifier</td>
<td></td>
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<tr>
<td><strong>Total Score</strong></td>
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</tbody>
</table>

Average text complexity: 14/21
Appendix H

Comparison of the Isomathic Problems

Grades 7, 8, and 9
Appendix H

Comparison of the Problems

Grade: 7

P1: (Complex proof, simple language)
In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

a) Show that AI = JC.

P3 (Iso P1): (Simplified proof, simple language)
In the given figure, (AB) is parallel to (CD). I is the midpoint of [DJ]. J is the midpoint of [IB]. BÂI = JĈD.

a) Show that BI = DJ.
b) Show that BÎA = DĴC.
c) Show that triangles CDJ and ABI are congruent.
d) Show that AI = JC.

Change 1
b) Show that (AD) is parallel to (BC).

e) Show that \( \triangle ADI = \triangle CJB \).

f) Prove that triangles ADI and BCJ are congruent. Write their homologous parts.

g) Deduce that (AD) is parallel to (BC).
<table>
<thead>
<tr>
<th>P2: (Complex language, simple proof)</th>
<th>P4 (Iso P2): (Simplified language, simple proof)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider an isosceles triangle MNP whose main vertex is M and foot of the perpendicular drawn from M to [NP] is R. Through S, the midpoint of [MR], (x) is the perpendicular drawn to (MR) cutting [MN] and [MP] in J and F respectively.</td>
<td>Consider an isosceles triangle MNP whose main vertex is M and foot of the perpendicular drawn from M to [NP] is R. Through S, the midpoint of [MR], (x) is the perpendicular drawn to (MR) cutting [MN] and [MP] in J and F respectively.</td>
</tr>
</tbody>
</table>

| Change 1 |
|------------------------------------|-----------------------------------------------|
|  |  |
| a) Draw the figure. | a) Draw the figure. |
| b) Show that [MR] is the angular bisector of $\hat{N}\hat{M}\hat{P}$. | b) Show that [MR] is the angular bisector of $\hat{N}\hat{M}\hat{P}$. |
| c) Show that (x) is parallel to (NP). | c) Show that (x) is parallel to (NP). |
| d) Show that (x) is the perpendicular bisector of [MR]. | d) Show that (x) is the perpendicular bisector of [MR]. |
| e) Compare MF and FR. Justify your answer. | e) Compare MF and FR. Justify your answer. |
| f) Show that triangles MSF and SFR are congruent. Write their homologous parts. | f) Show that triangles MSF and SFR are congruent. Write their homologous parts. |
| g) Deduce that (MN) and (FR) are parallel. | g) Deduce that (MN) and (FR) are parallel. |
Grade: 8

**P1: (Complex proof, simple language)**

**Given:** ABC is any triangle. [AM] is a median. [AH] is a height. E is the symmetric of A with respect to M. D is the symmetric of A with respect to H.

*Show that BE = DC.*

**P3 (Iso P1): (Simplified proof, simple language)**

**Given:** ABC is any triangle. [AM] is a median. [AH] is a height. E is the symmetric of A with respect to M. D is the symmetric of A with respect to H.

a) *Show that (DE) is parallel to (BC).*
b) *Show that BA = BD.*
c) *Show that ABEC is a parallelogram.*
d) *Show that BCED is an isosceles trapezoid.*
e) *Deduce that BE = CD.*
<table>
<thead>
<tr>
<th>P2: (Complex language, simple proof)</th>
<th>P4 (Iso P2): (Simplified language, simple proof)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider a triangle ABC right angled at C such that AC= 6cm, BC= 8cm, and AB= 10cm, in which [AM] is</td>
<td>ABC is a right triangle at C such that AC= 6cm, BC= 8cm, and AB= 10cm. [AM] is the median relative to [BC].</td>
</tr>
<tr>
<td>a median relative to [BC]. E and F are the respective feet of the perpendiculares drawn from B and C</td>
<td>Through B, draw a perpendicular to (AM) at E. Through C, draw a perpendicular to (AM) at F.</td>
</tr>
<tr>
<td>to (AM).</td>
<td></td>
</tr>
</tbody>
</table>

1) Draw the figure.
2) Prove that triangles FCM and BEM are congruent. Write their homologous parts.
3) Show that quadrilateral CFBE is a parallelogram.

<table>
<thead>
<tr>
<th>4) Let I be the midpoint of [AB] and R the symmetric of E with respect to I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Show that quadrilateral BRAE is a rectangle.</td>
</tr>
<tr>
<td>b) Calculate EI and CI.</td>
</tr>
<tr>
<td>c) Deduce the nature of triangle EIC.</td>
</tr>
</tbody>
</table>

Change 1

<table>
<thead>
<tr>
<th>Change 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Show that quadrilateral BRAE is a rectangle.</td>
</tr>
<tr>
<td>b) Calculate EI and CI.</td>
</tr>
<tr>
<td>c) Deduce the nature of triangle EIC.</td>
</tr>
</tbody>
</table>
P1: (Complex proof, simple language)

Given: Consider a semi-circle (C) of center O, radius R, and diameter [AB]. C is point on (C) such that (CO) is perpendicular to [AB]. M is any point on arc $\overline{BC}$. (CM) and (AB) intersect at point P.

1) Show that $\overline{CMA} = \overline{BMP}$.

2) Let I be the point of intersection of (AM) and (OC).

Show that $AI \times AM = 2R^2$.

P3 (Iso P1): (Simplified proof, simple language)

Given: Consider a semi-circle (C) of center O, radius R, and diameter [AB]. C is point on (C) such that (CO) is perpendicular to [AB]. M is any point on arc $\overline{BC}$. (CM) and (AB) intersect at point P.

1) Show that $\overline{CMA} = \overline{BMP} = 45^\circ$.

2) Let I be the point of intersection of (AM) and (OC).

a) Show that triangles AMB and AOI are similar. Write their ratio of similarity.

b) Deduce that $AI \times AM = 2R^2$. 
3) Suppose that I is the midpoint of [OC].

   Calculate the length of [AI] and that of [MB] in terms of R.

3) Suppose that I is the midpoint of [OC].

   a) Calculate the length of [AI] in terms of R.
   b) Deduce the length of [AM] then that of [MB] in terms of R.
In triangle ABC, a line through P, a point on [BC], is drawn parallel to the median [AD] cutting (AB) and (AC) in M and N respectively.

1) Draw the figure.

2) Prove that the ratio of AM to AB is equal to that of AN to AC.

3) Show that PM and BD have the same product as DA and DP.
Appendix I

List of Topics Included in Tests 1 and 2

Grade 6:

- Drawing and classifying triangles.
- Calculating angles of a triangle.
- Remarkable segments in a triangle.
- Relative position of two lines.
- Perpendicular Bisector.

Grade 7:

- Perpendicular bisector.
- Congruent triangles.
- Properties of equalities.
- Angles and parallel lines.

Grad 8:

- Quadrilaterals.
- Midpoint theorem.
- Median in a right triangle.

Grade 9:

- Arcs and Angles.
- Pythagorean Theorem.
- Thales Theorem.
- Similar Triangles.
Appendix J

List of Definitions, Properties, and Theorems Used

Definitions, Theorems, and Properties that might be used in the proof:

Grade 6:

- Definition of midpoint.
- Sum of angles in a triangle is $180^\circ$.
- Definition of perpendicular bisector.
- Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.
- Definition of perpendicular lines.
- Definition of isosceles triangle.
- An isosceles triangle having one of its angles $60^\circ$ is an equilateral triangle.
- Base angles.
- If two lines are parallel, then any line perpendicular to one of them is perpendicular to the other.
- A height is a segment issued from the vertex of a triangle perpendicular to the opposite side.

Grade 7:

- Alternate interior angles are equal.
- Addition property.
- Definition of midpoint.
- Substitution.
- Supplements of equals are equal.
- Remaining angles property.
- ASA.
- Homologous parts.
- Definition of perpendicular bisector.
- Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.
- Definition of midpoint.
- If two lines are perpendicular to the same line, then they are parallel.
- By alternate interior angles.
• By S.A.S
• By S.S.S

**Grade 8:**

• By symmetry.
• A quadrilateral whose diagonals bisect each other is a parallelogram.
• Opposite sides in a parallelogram are equal.
• By substitution.
• By midpoint theorem in a triangle.
• Definition of a trapezoid.
• A trapezoid whose non-parallel sides are equal is an isosceles trapezoid.
• Diagonals in an isosceles trapezoid are equal.
• Definition of a median.
• By symmetry.
• If two lines are perpendicular to the same line, then they are parallel.
• By alternate interior angles.
• Vertically opposite angles.
• Remaining angles property.
• By A.S.A.
• A quadrilateral with one pair of opposite sides parallel and equal is a parallelogram.
• A quadrilateral whose diagonals bisect each other is a parallelogram.
• A parallelogram with a right angle is a rectangle.
• Diagonals in a rectangle are equal.
• In a right triangle, the median relative to the hypotenuse is equal to half the hypotenuse.
• Opposite sides in a parallelogram are equal.
• Definition of isosceles triangle.

**Grade 9:**

• Inscribed angle facing diameter.
• Inscribed angle.
• Central angle.
• Pythagorean Theorem.
• Thales Theorem.
• Ratio of sides of similar triangles.
Appendix K

Judging Results

Grades 6 to 8
## Appendix K

### Judging Results

**Text Complexity Rubric**

Test #: __1__  
Grade level: __6__

Text of the question: 6 P1 “Complex proof, simple language”

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<th>Levels of Difficulty</th>
<th>Score</th>
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<td><strong>Semantic features of the text</strong></td>
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<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
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<tr>
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<td>Simple sentence</td>
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<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
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<tr>
<td></td>
<td>Compound sentence</td>
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<td></td>
<td>Difficult 3</td>
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<td>Complex sentence</td>
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<td>Commonly used tenses</td>
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<td>Prepositions</td>
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<td>Sufficient number of words to form a complete sentence</td>
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<tr>
<td>Usage of modifiers</td>
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<td>A sentence has no modifiers</td>
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<td>A sentence has one word or phrase as a modifier</td>
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<td>A sentence has a dependent clause as a modifier</td>
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<td><strong>Total Score</strong></td>
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<td>8/21</td>
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Average text complexity: 14/21
Text of the Definitions, Theorems, and Properties involved:

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<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
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<td>Compound sentence</td>
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<td>Difficult 3</td>
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<td>Logical connectors</td>
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<td>Usage of two connectors in a statement</td>
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<tr>
<td>Prepositions</td>
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<tr>
<td>Pronouns</td>
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<td>1</td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>2</td>
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<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
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<td>Usage of modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
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<td>Usage of modifiers</td>
<td>A sentence has a dependent clause as a modifier</td>
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<td>Total Score</td>
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</table>

Average Definitions, Theorems, and Properties text complexity: 14/21
### Proof's Complexity Rubric

**Test #: __1___  Grade level: __6__**

**Question:** 6 P1 “Complex proof, simple language”

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Easy 1</th>
<th>Average 2</th>
<th>Complex 3</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The given</strong></td>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
<td>The given is provided using mathematical statements but no figure is provided</td>
<td>2</td>
</tr>
<tr>
<td><strong>The figure</strong></td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
<td>The figure is complex (too many imbricated parts)</td>
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<tr>
<td><strong>The required to prove</strong></td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
<td>The proof task is not stated in a direct way and requires evaluation level according to Bloom’s Taxonomy</td>
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<tr>
<td></td>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
<td>The proof requires considering imbricated parts of a figure</td>
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<tr>
<td></td>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
<td>The proof task involves too many geometric objects</td>
<td>3</td>
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<tr>
<td><strong>Number of statements required in the proof</strong></td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
<td>Multi-step proof that requires partial proofs for the statements and reasons for the proof task</td>
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<tr>
<td><strong>Usage of definitions, theorems &amp; postulates</strong></td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
<td>Requires usage of definitions, properties, and theorems</td>
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**Total Score**

17/21

**Average proof complexity:** 14/21
### Text Complexity Rubric

**Test #: ___2__**  
**Grade level: ___6__**

**Text of the question:** 6 P3 “Simplified proof, simple language”

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<td>/ 3</td>
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<td>Usage of two connectors in a</td>
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<td>statement</td>
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<td>Usage of more than two connectors</td>
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<td>in a statement</td>
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<td>Verbs (passive or active)</td>
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<tr>
<td>Commonly used tenses</td>
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<tr>
<td>Less frequently used tenses</td>
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<tr>
<td>Frequent use of tenses</td>
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<tr>
<td>Prepositions</td>
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<td>No use of prepositions</td>
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<td>Frequent use of prepositions</td>
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<tr>
<td>sentences</td>
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<tr>
<td>Usage of modifiers</td>
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<tr>
<td>A sentence has no modifiers</td>
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<td></td>
</tr>
<tr>
<td>A sentence has one word or phrase</td>
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<tr>
<td>as a modifier</td>
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<td>10/21</td>
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**Average text complexity:** 14/21
Text of the Definitions, Theorems, and Properties involved:

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<th>Item Description</th>
<th>Easy 1</th>
<th>Average 2</th>
<th>Difficult 3 / 3</th>
<th>Score</th>
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<tr>
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<td>Simple sentence</td>
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<td>Complex sentence</td>
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</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
<td>Usage of more than two connectors in a statement</td>
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<tr>
<td>Verbs (passive or active)</td>
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<td>Less frequently used tenses</td>
<td>Infrequently used tenses</td>
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</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
<td>Frequent use of prepositions</td>
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<tr>
<td>Pronouns</td>
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<td>Frequent use of pronouns</td>
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<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
<td>The sentence used can be subdivided into two or more sentences</td>
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<td>A sentence has a dependent clause as a modifier</td>
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Average Definitions, Theorems, and Properties text complexity: 14/21
**Proof’s Complexity Rubric**

**Test #: 2**  
**Grade level: 6**

**Question:** 6 P3 “Simplified proof, simple language”

<table>
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<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
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<td><strong>Easy 1</strong></td>
<td><strong>Average 2</strong></td>
<td><strong>Complex 3</strong> / 3</td>
</tr>
<tr>
<td>The given</td>
<td>The given is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
</tr>
<tr>
<td></td>
<td>The figure is simple (no imbricated parts)</td>
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</tr>
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<td>The required to prove</td>
<td>The proof requires considering simple parts of the figure</td>
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<td>The proof task involves a limited number of geometric objects</td>
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<tr>
<td>Number of statements required in the proof</td>
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</tr>
<tr>
<td>Usage of definitions, theorems &amp; postulates</td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
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</tbody>
</table>

**Total Score**  
10/21

**Average proof complexity:** 14/21

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313
Text Complexity Rubric

Test #: __2__  
Grade level: __6__

Text of the question: 6 P2 “Complex language, simple proof”

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<th>Score</th>
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</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td>1</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
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</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
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</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>2</td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td>2</td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td>Easy 1</td>
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</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>3</td>
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<td>Usage of modifiers</td>
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</table>

Total Score: 16/21

Average text complexity: 14/21
Text of the Definitions, Theorems, and Properties involved:

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<td></td>
<td>Compound sentence</td>
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<td>Difficult 3</td>
<td>3</td>
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<td></td>
<td>Complex sentence</td>
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</tr>
<tr>
<td>Logical connectors (conjunctions/</td>
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</tr>
<tr>
<td>transitions)</td>
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<td>Usage of one connector in a statement</td>
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<td>Usage of two connectors in a statement</td>
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<td></td>
<td>Less frequently used tenses</td>
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<td></td>
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<td>2</td>
</tr>
<tr>
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<td>The sentence used can be subdivided into two or more sentences</td>
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</tr>
<tr>
<td>Usage of modifiers</td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has no modifiers</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has one word or phrase as a modifier</td>
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<tr>
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<td>Difficult 3</td>
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<tr>
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<td>A sentence has a dependent clause as a modifier</td>
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Average Definitions, Theorems, and Properties text complexity: 14/21
Proof’s Complexity Rubric

Test #: __2___  Grade level: __6__

Question: 6 P2 “Complex language, simple proof”

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<tbody>
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</tr>
<tr>
<td>The given</td>
<td>The figure is given and coded to provide given relations</td>
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<tr>
<td></td>
<td>Average 2</td>
<td></td>
</tr>
<tr>
<td>The figure</td>
<td>The figure is simple (no imbricated parts)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Complex 3</td>
<td></td>
</tr>
<tr>
<td>The required to prove</td>
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Average proof complexity: 14/21
# Text Complexity Rubric

Test #: ___1__                                                                 Grade level: __6__

Text of the question: 6 P4 “Simplified language, simple proof”

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<tr>
<th>Item Description</th>
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<td><strong>Semantic features of the text</strong></td>
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<td></td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
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</tr>
<tr>
<td></td>
<td>Compound sentence</td>
<td>2</td>
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<tr>
<td></td>
<td>Complex sentence</td>
<td>3</td>
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<tr>
<td>Logical connectors</td>
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</tr>
<tr>
<td>(conjunctions/ transitions)</td>
<td>Usage of two connectors in a statement</td>
<td>2</td>
</tr>
<tr>
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<td>Usage of more than two connectors in a statement</td>
<td>3</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
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<tr>
<td></td>
<td>Less frequently used tenses</td>
<td>2</td>
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<tr>
<td></td>
<td>Infrequently used tenses</td>
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</tr>
<tr>
<td>Prepositions</td>
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<td></td>
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<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
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</tr>
<tr>
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<td>A sentence has one word or phrase as a modifier</td>
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<td></td>
<td>A sentence has a dependent clause as a modifier</td>
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**Total Score** | **11/21**

Average text complexity: 14/21
Text of the Definitions, Theorems, and Properties involved:

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<td>Usage of more than two connectors in a statement</td>
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Average Definitions, Theorems, and Properties text complexity: 14/21
Proof's Complexity Rubric

Test #: __1__                                                                                      Grade level: __6__

**Question:** 6 P4 “Simplified language, simple proof”

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**Total Score**

10/21

**Average proof complexity:** 14/21
Text Complexity Rubric

Test #: ___1__                                               Grade level: ___7___

Text of the question: 7 P1 “Complex proof, simple language”

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<td><strong>Syntactic features of the text</strong></td>
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<tr>
<td>Length of the sentence</td>
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<td></td>
<td>Sufficient number of words to form a complete sentence</td>
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</tr>
<tr>
<td></td>
<td>The sentence used can be subdivided into two or more sentences</td>
<td></td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has one word or phrase as a modifier</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has a dependent clause as a modifier</td>
<td>2</td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
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</tbody>
</table>

Average text complexity: 14/21
**Text of the Definitions, Theorems, and Properties involved:**

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<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
</tr>
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<tbody>
<tr>
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<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td>Compound sentence</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
</tr>
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**Total Score** 9/21

**Average Definitions, Theorems, and Properties text complexity:** 14/21
Proof's Complexity Rubric

Test #: __1__  
Grade level: __7__

Question: 7 P1 “Complex proof, simple language”

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<tr>
<th>Item Description</th>
<th>Easy 1</th>
<th>Average 2</th>
<th>Complex 3</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The given</strong></td>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
<td>The given is provided using mathematical statements but no figure is provided</td>
<td>2</td>
</tr>
<tr>
<td><strong>The figure</strong></td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
<td>The figure is complex (too many imbricated parts)</td>
<td>2</td>
</tr>
<tr>
<td><strong>The required to prove</strong></td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
<td>The proof task is not stated in a direct way and requires evaluation level according to Bloom’s Taxonomy</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
<td>The proof requires considering imbricated parts of a figure</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
<td>The proof task involves too many geometric objects</td>
<td>3</td>
</tr>
<tr>
<td><strong>Number of statements required in the proof</strong></td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
<td>Multi-step proof that requires partial proofs for the statements and reasons for the proof task</td>
<td>3</td>
</tr>
<tr>
<td><strong>Usage of definitions, theorems &amp; postulates</strong></td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
<td>Requires usage of definitions, properties, and theorems</td>
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Average proof complexity: 14/21
Text Complexity Rubric

Test #: ___2__  Grade level: __7__

Text of the question: 7 P3 “Simplified proof, simple language”

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<td></td>
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<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
<td>1</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Average 2</td>
<td>2</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Average 2</td>
<td>2</td>
</tr>
<tr>
<td>Prepositions</td>
<td>Difficult 3</td>
<td>3</td>
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<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
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<tr>
<td>Length of the sentence</td>
<td>Minimal use of prepositions</td>
<td>2</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>Sufficient use of prepositions</td>
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Average text complexity: 14/21
Text of the Definitions, Theorems, and Properties involved:

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<th>Levels of Difficulty</th>
<th>Score</th>
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<tbody>
<tr>
<td><strong>Sentence structure</strong></td>
<td>Easy 1 - Simple sentence</td>
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<tr>
<td></td>
<td>Average 2 - Compound sentence</td>
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<td></td>
<td>Difficult 3 - Complex sentence</td>
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<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
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<td></td>
<td>Usage of two connectors in a statement</td>
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<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
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<tr>
<td></td>
<td>Less frequently used tenses</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Infrequently used tenses</td>
<td></td>
</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Minimal use of prepositions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequent use of prepositions</td>
<td></td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Minimal use of pronouns</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequent use of pronouns</td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
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<td>Sufficient number of words to form a complete sentence</td>
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<tr>
<td></td>
<td>The sentence used can be subdivided into two or more sentences</td>
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<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td></td>
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<tr>
<td></td>
<td>A sentence has one word or phrase as a modifier</td>
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<td></td>
<td>A sentence has a dependent clause as a modifier</td>
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<td><strong>Total Score</strong></td>
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Average Definitions, Theorems, and Properties text complexity: 14/21
Proof's Complexity Rubric

Test #: 2                     Grade level: 7

Question: 7 P3 “Simplified proof, simple language”

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<tr>
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</tr>
<tr>
<td>The given</td>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>The figure</td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>The required to prove</td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
</tr>
<tr>
<td></td>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
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<td></td>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Number of statements required in the proof</td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
</tr>
<tr>
<td></td>
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<tr>
<td>Usage of definitions, theorems &amp; postulates</td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
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<tr>
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Average proof complexity: 14/21
## Text Complexity Rubric

**Test #: 2**  
**Grade level: 7**

**Text of the question:** 7 P2 “Complex language, simple proof”

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<td><strong>Semantic features of the text</strong></td>
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<tr>
<td>Sentence structure</td>
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<td>Average 2</td>
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</tr>
<tr>
<td></td>
<td>Difficult 3</td>
<td>/3</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>2</td>
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<tr>
<td></td>
<td>Usage of two connectors in a statement</td>
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<tr>
<td></td>
<td>Usage of more than two connectors in a statement</td>
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</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
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<tr>
<td></td>
<td>Less frequently used tenses</td>
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<tr>
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<td>Infrequently used tenses</td>
<td></td>
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<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
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</tr>
<tr>
<td></td>
<td>Minimal use of prepositions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequent use of prepositions</td>
<td></td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Minimal use of pronouns</td>
<td></td>
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<tr>
<td></td>
<td>Frequent use of pronouns</td>
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<tr>
<td><strong>Syntactic features of the text</strong></td>
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<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>3</td>
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<td></td>
<td>Sufficient number of words to form a complete sentence</td>
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<tr>
<td></td>
<td>The sentence used can be subdivided into two or more sentences</td>
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<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>A sentence has one word or phrase as a modifier</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has a dependent clause as a modifier</td>
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**Average text complexity:** 14/21
### Text of the Definitions, Theorems, and Properties involved:

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<th>Levels of Difficulty</th>
<th>Score</th>
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<tr>
<td><strong>Semantic features of the text</strong></td>
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<tr>
<td>Sentence structure</td>
<td>Easy: Simple sentence</td>
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<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
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<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
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<td>Pronouns</td>
<td>No use of pronouns</td>
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<tr>
<td><strong>Syntactic features of the text</strong></td>
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**Average Definitions, Theorems, and Properties text complexity:** 14/21
Proof's Complexity Rubric

Test #: __2__  
Grade level: __7__

Question: 7 P2 “Complex language, simple proof”

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</tr>
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<td>The figure is given and coded to provide given relations</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
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<td></td>
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</tr>
<tr>
<td>The given is provided using mathematical statements but no figure is provided</td>
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<td></td>
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<tr>
<td><strong>The figure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The figure is simple (no imbricated parts)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The figure includes imbricated parts</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>The figure is complex (too many imbricated parts)</td>
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<td></td>
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<tr>
<td><strong>The required to prove</strong></td>
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</tr>
<tr>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
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<tr>
<td>The proof task is not stated in a direct way and requires evaluation level according to Bloom’s Taxonomy</td>
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<td>The proof requires identifying parts to be considered in a more complex figure</td>
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<td>The proof requires considering imbricated parts of a figure</td>
<td>3</td>
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<tr>
<td>The proof task involves a limited number of geometric objects</td>
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<td></td>
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</tr>
<tr>
<td>The proof task involves several number of geometric objects</td>
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<tr>
<td>The proof task involves too many geometric objects</td>
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<tr>
<td><strong>Number of statements required in the proof</strong></td>
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<td></td>
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</tr>
<tr>
<td>The proof task involves one single inference</td>
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<td></td>
<td></td>
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<tr>
<td>The proof task involves more than one single inference</td>
<td>2</td>
<td></td>
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<tr>
<td>Multi-step proof that requires partial proofs for the statements and reasons for the proof task</td>
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<tr>
<td><strong>Usage of definitions, theorems &amp; postulates</strong></td>
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<td></td>
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<tr>
<td>Requires usage of definitions only</td>
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<td></td>
<td></td>
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<tr>
<td>Requires usage of definitions and properties</td>
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<td>Requires usage of definitions, properties, and theorems</td>
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Text Complexity Rubric

Test #: 1  Grade level: 7

Text of the question: 7 P4 “Simplified language, simple proof”

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<th>Levels of Difficulty</th>
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<tbody>
<tr>
<td>Semantic features of the text</td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td>Compound sentence</td>
</tr>
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<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
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<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
</tr>
<tr>
<td>Syntactic features of the text</td>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
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Total Score: 11/21

Average text complexity: 14/21
### Text of the Definitions, Theorems, and Properties involved:

<table>
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<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
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<tbody>
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<td><strong>Semantic features of the text</strong></td>
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<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
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<tr>
<td></td>
<td>Average 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Difficult 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>/ 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Simple sentence</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Compound sentence</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Complex sentence</td>
<td></td>
</tr>
<tr>
<td>Logical connectors</td>
<td>Usage of one</td>
<td>3</td>
</tr>
<tr>
<td>(conjunctions/</td>
<td>connector in a</td>
<td></td>
</tr>
<tr>
<td>transitions)</td>
<td>statement</td>
<td></td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Less frequently used tenses</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Infrequently used tenses</td>
<td></td>
</tr>
<tr>
<td>Prepositions</td>
<td>No use of</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>prepositions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Minimal use of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>prepositions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequent use of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>prepositions</td>
<td></td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>pronouns</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Minimal use of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>pronouns</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequent use of</td>
<td></td>
</tr>
<tr>
<td></td>
<td>pronouns</td>
<td></td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>words to form a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>complete sentence</td>
<td></td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>modifiers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has one</td>
<td></td>
</tr>
<tr>
<td></td>
<td>word or phrase as a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>modifier</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has a</td>
<td></td>
</tr>
<tr>
<td></td>
<td>dependent clause as</td>
<td></td>
</tr>
<tr>
<td></td>
<td>a modifier</td>
<td></td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
<td></td>
<td>14/21</td>
</tr>
</tbody>
</table>

**Average Definitions, Theorems, and Properties text complexity:** 14/21
Proof’s Complexity Rubric

Test #: __1__                                         Grade level: __7__

Question: 7 P4 “Simplified language, simple proof”

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>The given</td>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
</tr>
<tr>
<td>The figure</td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
</tr>
<tr>
<td>The required to prove</td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
</tr>
<tr>
<td></td>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
</tr>
<tr>
<td></td>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
</tr>
<tr>
<td>Number of statements required in the proof</td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
</tr>
<tr>
<td>Usage of definitions, theorems &amp; postulates</td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
</tr>
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</table>

Total Score: 12/21

Average proof complexity: 14/21
Text Complexity Rubric

Test #: ___1__                                                       Grade level: ___8__

Text of the question: 8 P1 “Complex proof, simple language”

<table>
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<th>Levels of Difficulty</th>
<th>Score</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>Semantic features of the text</td>
<td>Sentence structure</td>
<td>Simple sentence</td>
</tr>
<tr>
<td></td>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
</tr>
<tr>
<td></td>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
</tr>
<tr>
<td></td>
<td>prepositions</td>
<td>No use of prepositions</td>
</tr>
<tr>
<td></td>
<td>pronouns</td>
<td>No use of pronouns</td>
</tr>
<tr>
<td>Syntactic features of the text</td>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
</tr>
<tr>
<td></td>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
</tr>
<tr>
<td>Total Score</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average text complexity: 14/21
### Text of the Definitions, Theorems, and Properties involved:

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
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</thead>
<tbody>
<tr>
<td><strong>Semantic features of the text</strong></td>
<td>Easy 1</td>
<td></td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td></td>
</tr>
<tr>
<td>Logical connectors</td>
<td>Usage of one connector in a statement</td>
<td></td>
</tr>
<tr>
<td>prepositions</td>
<td>No use of prepositions</td>
<td></td>
</tr>
<tr>
<td>pronouns</td>
<td>No use of pronouns</td>
<td></td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td></td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td></td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
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<td>10/21</td>
</tr>
</tbody>
</table>

Average Definitions, Theorems, and Properties text complexity: 14/21
Proof's Complexity Rubric

Test #: 1

Grade level: 8

Question: 8 P1 “Complex proof, simple language”

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Easy 1</strong></td>
<td><strong>Average 2</strong></td>
<td><strong>Complex 3</strong></td>
</tr>
<tr>
<td>The given</td>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
</tr>
<tr>
<td>The figure</td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
</tr>
<tr>
<td>The required to prove</td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
</tr>
<tr>
<td></td>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
</tr>
<tr>
<td></td>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
</tr>
<tr>
<td>Number of statements required in the proof</td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
</tr>
<tr>
<td>Usage of definitions, theorems &amp; postulates</td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
</tr>
<tr>
<td>Total Score</td>
<td>18/21</td>
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</tbody>
</table>

Average proof complexity: 14/21
### Text Complexity Rubric

**Test #: ___2__**  
**Grade level: ___8___**

**Text of the question:** 8 P3 “Simplified proof, simple language”

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<tr>
<th>Item Description</th>
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</thead>
<tbody>
<tr>
<td><strong>Semantic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td></td>
<td>Simple sentence</td>
<td>Compound sentence</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
</tr>
<tr>
<td>prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
</tr>
<tr>
<td>pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
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<td></td>
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</tbody>
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**Average text complexity:** 14/21
### Text of the Definitions, Theorems, and Properties involved:

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<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
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<tbody>
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<td><strong>Semantic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Simple sentence</td>
<td>Compound sentence</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
</tr>
<tr>
<td>Prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
</tr>
<tr>
<td>Pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
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</table>

**Total Score** | **11/21**

**Average Definitions, Theorems, and Properties text complexity:** 14/21
# Proof's Complexity Rubric

**Test #: __2__  Grade level: __8__**

**Question:** 8 P3 “Simplified proof, simple language”

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Easy</strong></td>
<td><strong>Average</strong></td>
<td><strong>Complex</strong></td>
</tr>
<tr>
<td>The given</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>The figure is given and coded to provide given relations</td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
<td>The given is provided using mathematical statements but no figure is provided</td>
</tr>
<tr>
<td>The figure</td>
<td>The figure is simple (no imbricated parts)</td>
<td>The figure includes imbricated parts</td>
</tr>
<tr>
<td>The required to prove</td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
</tr>
<tr>
<td>The proof requires considering simple parts of the figure</td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
<td>The proof requires considering imbricated parts of a figure</td>
</tr>
<tr>
<td>The proof task involves a limited number of geometric objects</td>
<td>The proof task involves several number of geometric objects</td>
<td>The proof task involves too many geometric objects</td>
</tr>
<tr>
<td>Number of statements required in the proof</td>
<td>The proof task involves one single inference</td>
<td>The proof task involves more than one single inference</td>
</tr>
<tr>
<td>Usage of definitions, theorems &amp; postulates</td>
<td>Requires usage of definitions only</td>
<td>Requires usage of definitions and properties</td>
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</tbody>
</table>

**Total Score**

**Average proof complexity:** 14/21
### Text Complexity Rubric

**Test #: 2**  
**Grade level: 8**  

**Text of the question:** 8 P2 “Complex language, simple proof”

<table>
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<th>Item Description</th>
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<th>Score</th>
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</thead>
<tbody>
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<td></td>
<td></td>
</tr>
<tr>
<td>Sentence structure</td>
<td>Easy 1</td>
<td>Average 2</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
</tr>
<tr>
<td>prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
</tr>
<tr>
<td>pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
</tr>
<tr>
<td><strong>Syntactic features of the text</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
<td></td>
<td></td>
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</tbody>
</table>

**Average text complexity:** 14/21
Text of the Definitions, Theorems, and Properties involved:

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Easy 1</th>
<th>Average 2</th>
<th>Difficult 3</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sentence structure</td>
<td>Simple sentence</td>
<td>Compound sentence</td>
<td>Complex sentence</td>
<td>1</td>
</tr>
<tr>
<td>Logical connectors (conjunctions/ transitions)</td>
<td>Usage of one connector in a statement</td>
<td>Usage of two connectors in a statement</td>
<td>Usage of more than two connectors in a statement</td>
<td>1</td>
</tr>
<tr>
<td>Verbs (passive or active)</td>
<td>Commonly used tenses</td>
<td>Less frequently used tenses</td>
<td>Infrequently used tenses</td>
<td>1</td>
</tr>
<tr>
<td>prepositions</td>
<td>No use of prepositions</td>
<td>Minimal use of prepositions</td>
<td>Frequent use of prepositions</td>
<td>2</td>
</tr>
<tr>
<td>pronouns</td>
<td>No use of pronouns</td>
<td>Minimal use of pronouns</td>
<td>Frequent use of pronouns</td>
<td>2</td>
</tr>
<tr>
<td>Length of the sentence</td>
<td>Minimal number of words to form a complete sentence</td>
<td>Sufficient number of words to form a complete sentence</td>
<td>The sentence used can be subdivided into two or more sentences</td>
<td>1</td>
</tr>
<tr>
<td>Usage of modifiers</td>
<td>A sentence has no modifiers</td>
<td>A sentence has one word or phrase as a modifier</td>
<td>A sentence has a dependent clause as a modifier</td>
<td>2</td>
</tr>
<tr>
<td>Total Score</td>
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<td></td>
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Average Definitions, Theorems, and Properties text complexity: 14/21
Proof’s Complexity Rubric

Test #: __2__  
Grade level: __8__

Question: 8 P2 “Complex language, simple proof”

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<th>Item Description</th>
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<tbody>
<tr>
<td><strong>Easy</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Complex</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Score</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>The given</strong></td>
<td>Easy 1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Average 2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Complex 3 / 3</td>
<td>3</td>
</tr>
<tr>
<td><strong>The figure</strong></td>
<td>The figure is given and coded to provide given relations</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The given is provided through mathematical statements and the figure is also provided with no coding.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The given is provided using mathematical statements but no figure is provided</td>
<td></td>
</tr>
<tr>
<td><strong>The required to prove</strong></td>
<td>The proof task is stated in a direct way and requires analysis level according to Bloom’s Taxonomy</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>The proof task is stated in a direct way and requires synthesis level according to Bloom’s Taxonomy</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The proof task is not stated in a direct way and requires evaluation level according to Bloom’s Taxonomy</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The proof requires identifying parts to be considered in a more complex figure</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The proof requires considering imbricated parts of a figure</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>The proof task involves several number of geometric objects</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The proof task involves too many geometric objects</td>
<td></td>
</tr>
<tr>
<td><strong>Number of statements required in the proof</strong></td>
<td>The proof task involves one single inference</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>The proof task involves more than one single inference</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Multi-step proof that requires partial proofs for the statements and reasons for the proof task</td>
<td></td>
</tr>
<tr>
<td><strong>Usage of definitions, theorems &amp; postulates</strong></td>
<td>Requires usage of definitions only</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Requires usage of definitions and properties</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Requires usage of definitions, properties, and theorems</td>
<td>3</td>
</tr>
<tr>
<td><strong>Total Score</strong></td>
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<td>13/21</td>
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Average proof complexity: 14/21
## Text Complexity Rubric

**Test #: ___1__**  
**Grade level: ___8__**

**Text of the question:** 8 P4 “Simplified language, simple proof”

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Levels of Difficulty</th>
<th>Score</th>
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<tbody>
<tr>
<td><strong>Semantic features of the text</strong></td>
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<td></td>
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</tbody>
</table>
| Sentence structure                        | Easy  
Simple sentence                    | Average  
Compound sentence                | Difficult  
Complex sentence                | 1 |
| Logical connectors (conjunctions/ transitions) | Usage of one connector in a statement | Usage of two connectors in a statement | Usage of more than two connectors in a statement | 1 |
| Verbs (passive or active)                  | Commonly used tenses               | Less frequently used tenses           | Infrequently used tenses           | 1 |
| prepositions                               | No use of prepositions            | Minimal use of prepositions          | Frequent use of prepositions       | 2 |
| pronouns                                   | No use of pronouns                | Minimal use of pronouns              | Frequent use of pronouns           | 1 |
| **Syntactic features of the text**         |                                     |       |
| Length of the sentence                     | Minimal number of words to form a complete sentence | Sufficient number of words to form a complete sentence | The sentence used can be subdivided into two or more sentences | 2 |
| Usage of modifiers                         | A sentence has no modifiers        | A sentence has one word or phrase as a modifier | A sentence has a dependent clause as a modifier | 2 |
| **Total Score**                            |                                     |       |
|                                           |                                     | 10/21 |

**Average text complexity:** 14/21
### Text of the Definitions, Theorems, and Properties involved:

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<td><strong>Semantic features of the text</strong></td>
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<td>Difficult 3: Complex sentence</td>
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<td>A sentence has one word or phrase as a modifier</td>
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**Average Definitions, Theorems, and Properties text complexity:** 14/21
# Proof’s Complexity Rubric

**Test #: ___1___  Grade level: ___8___**

**Question:** 8 P4 “Simplified language, simple proof”

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<td>relations</td>
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<td>properties, and</td>
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<td>theorems.</td>
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</table>

**Average proof complexity:** 14/21
Appendix L

Transcript of the interview with the Middle School Math Coordinator

The following section presents the transcript of the responses of the middle school math coordinator during the interview conducted with her.

1) At grade 6.

2) They vary according to the teacher. The preferable is the paragraph form and not the statement reason.
   
   → You don’t care about the reasons?
   
   Reasons like properties and theorems and not definitions.

3) Yes, I accept as long as they are used correctly.

4) A) We start with the notion of conclusion and the difference between given and conclusion. We introduce it through real-life examples and situations. For example, “somebody carrying a wet umbrella does that imply that it is raining? No. Maybe somebody is throwing water”. We try to build logical reasoning through presenting different situations and possible implications. From these examples, we try to build the If…, then… form.

   B) We start next presenting geometric definitions. Identify the given and derive conclusions from the definition. We next try to vary cases. Example, given
triangle ABC isosceles at A, what do you conclude? Given an isosceles triangle
ABC, “without specifying at what” what do you conclude?

C) Next, we introduce properties of equalities.

D) Finally, we start with geometric proofs.

5) A) They don’t use the appropriate property with conclusion.

B) They are misled by some keywords that lead them to misuse them. For
example, if students are given a pair of complementary angles, students directly
think of using the “property of complements of equals are equal”, which might
not be applicable in that case.

C) Students derive implications from the figure.

D) They add statements that have no evidence.

E) The path of the proof is incomplete, gap.

→Do you think that that reflects misconceptions about using some statements
without proving them?

No, not necessary. They know that they need it and when asked about it they are
able to justify it. This means that it is not a matter of misconception or
misunderstanding; rather it is carelessness because they are not interested in
proving as a process.

6) A) Difficulty in differentiating between definitions, properties, and theorems. i.e.
they use the properties of particular quadrilaterals and it is not given. They use
the theorem without referring to its conditions. To avoid this difficulty, students should write a complete statement of the definition, property, or theorem and not refer to it. This will help much in understanding and recognizing the uses of each. Example: instead of referring to the midpoint theorem, have students write it as “the segment joining the …”

B) Reversibility of the “if… then” statements. They use converse of it while it is not reversible.

→ What about mathematical symbols? Does misusing them reflect misconceptions?

No, if not appropriately used, students might be in a hurry, but not misconception. We are demanding much. These generations are not mathematicians. No need for all such details. Geometric proofs by themselves are hard to be introduced at the middle school. If postponed to the secondary level, students would perform better.

7) & 8) A) While correcting proofs and problems, one of the students presents the solution orally, others correct it, and another writes it on the board to highlight mistakes done.

B) Showing a model of mistakes done and ask students to detect errors and suggest corrections for these errors.
C) We emphasize on the appropriateness of the statement of the theorems and properties.

D) When giving and introducing theorems, we focus on the appropriateness of the use of pronouns and the mathematical terms used.

E) If a student insisted on the truth of a statement, we present it as a hypothesis and we assume it true then we try to prove that we have contradictory results.

F) Help students derive the theorems and properties by setting conjectures and testing particular cases.

Don’t you think that might affect the students’ notion of proof? They might think that they can generalize from particular cases?

No, because we prove the property later after conjecturing without particular cases. And if there are particular cases that make the conjecture not true, I try to present cases in the introductory activity to show that it is not always true. Especially in the converse of the “if…then” properties and theorems.

9) A) No, but they have problem why we learn it since we are not using it. This is why they miss steps.

B) To accept it better and value it, it should start from KG and relate it to algebra.

C) They should start proofs and reasoning by models and picture. I.e. follow the development of Piaget.
10) We use it in grade 7 when solving equations; they solve equations as proofs by giving reason for each step. Example: $3x + 5 = 8$

$3x + 5 - 5 = 8 - 5$ (subtraction property)

$3x = 3$ (substitution property)

$3x/3 = 3/3$ (division property)

$x = 1$ (substitution property)

11) Sure, through logical reasoning without any mathematical operation. We start in grade 3 by giving logical reasoning problems because they help in developing proof abilities. Finally, it’s the role of the teacher to help students develop the notion of proof through connecting them to problem solving (logical reasoning). Usually, good problem solvers are clever in geometric proofs.
### Appendix M

#### Research Adopted Framework

*Adopted Classification of Students’ Errors for Data Analysis*

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<tr>
<th>Code</th>
<th>Error Description</th>
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<tr>
<td>DWF</td>
<td>Drew a Wrong Figure.</td>
</tr>
<tr>
<td>SOR</td>
<td>Proved Statements Other than the Required ones.</td>
</tr>
<tr>
<td>NSP</td>
<td>Did Not Start a Proof.</td>
</tr>
<tr>
<td>SPNC</td>
<td>Started what could be a Proof plan but did Not Complete the proof.</td>
</tr>
<tr>
<td>ANW</td>
<td>“The Approach taken in proving a statement will Not Work”.</td>
</tr>
<tr>
<td>NL</td>
<td>“Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.</td>
</tr>
<tr>
<td>ED</td>
<td>“The proof contained Extraneous Details or steps that didn’t really contribute to the proof”.</td>
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<tr>
<td>UL</td>
<td>“The length of the proof was Unnecessarily Long and thus extremely difficult to follow”.</td>
</tr>
<tr>
<td>ISIE</td>
<td>“Incorrectly claimed that one Statement Implied or Equaled another statement”.</td>
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<tr>
<td>PNG</td>
<td>Used a Premise that is Not Given.</td>
</tr>
<tr>
<td>NDPI</td>
<td>Did Not Differentiate between the Premises of a proof and their Inferences and considered an inference as if it were given.</td>
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<tr>
<td>IDTP</td>
<td>Used Irrelevant Definition, Theorem, or Property to Justify a Statement.</td>
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<tr>
<td>FA</td>
<td>“Made a False Assumption somewhere in the proof”.</td>
</tr>
<tr>
<td>FSC</td>
<td>“Made a False Statement or incorrect Computation in the proof”.</td>
</tr>
<tr>
<td>NJEV</td>
<td>“Wrote a statement that wasn’t Justified, Explained or Verified”.</td>
</tr>
<tr>
<td>NJCS</td>
<td>“Did not sufficiently Justify a Crucial Step in a proof”.</td>
</tr>
<tr>
<td>SA</td>
<td>“Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex”.</td>
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<tr>
<td>MSI</td>
<td>Used Mathematical Symbols Incorrectly.</td>
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<tr>
<td>VI</td>
<td>Used Mathematical terms, words, or Vocabulary Incorrectly.</td>
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<tr>
<td>DPTI</td>
<td>Wrote statement of Definitions, Properties, or Theorems Incorrectly.</td>
</tr>
<tr>
<td>LCI</td>
<td>Used Logical Connectors Incorrectly.</td>
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</tbody>
</table>
Students’ Errors and Proof Difficulties

Understanding the Notion of Proof (UNP)

Writing Mathematical Texts (WMT)

Comprehending Mathematical Texts (CMT)

Setting Proof Plans (SPP)

Conducting Deductive Reasoning (CDR)

Understanding-Appling Mathematical Concepts (UAMC)
Appendices N and O

Analysis of Tests 1 and 2 (Grade 8 and 9)
Appendix N

Analysis of Tests 1 and 2 (Grade 8)

*Grade 8 Errors Related to Understanding the Notion of Proof (UNP)*

<table>
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<th>P3</th>
<th>P2</th>
<th>P4</th>
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</table>

NL: “Didn’t Proceed through the proof in a Linear fashion, and ideas were not in logical order”.
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PNG: Used a Premise that is Not Given.
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NJCS: “Did not sufficiently Justify a Crucial Step in a proof”.
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LD: Level of the difficulty.
### Grade 8 Errors Related to Setting Proof Plans (SPP)

<table>
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NSP: Did Not Start a Proof.
SPNC: Started what could be a Proof plan but did Not Complete the proof.
ANW: “The Approach taken in proving a statement will Not Work”.
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**Grade 8 Errors Related to Conducting Deductive Reasoning (CDR)**

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<thead>
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Grade 8 Errors Related to Understanding-Applying Mathematical Concepts (UAMC)

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SOR: Proved Statements Other than the Required ones.
ISIE: “Incorrectly claimed that one Statement Implied or Equaled another statement”.
IDTP: Used Irrelevant Definition, Theorem, or Property to Justify a Statement.
FA: “Made a False Assumption somewhere in the proof”.
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LD: Level of the difficulty.
Grade 8 Errors Related to Comprehending Mathematical Texts (CMT)

<table>
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</tbody>
</table>

DWF: Drew a Wrong Figure.
SOR: Proved Statements Other than the Required ones.
NSP: Did Not Start a Proof.
N/A: Not applicable. The problem doesn’t require testing this error.
PS: Percentage of Students.
LD: Level of the difficulty.
Grade 8 Errors Related to Writing Mathematical Text (WMT)

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<td><strong>Error</strong></td>
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<td>VI</td>
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<tr>
<td>DPTI</td>
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<td>LCI</td>
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SA: “Wrote a Statement or paragraph that was Ambiguous, confusing, and/or unnecessarily complex”.  
MSI: Used Mathematical Symbols Incorrectly.  
VI: Used Mathematical terms, words, or Vocabulary Incorrectly.  
DPTI: Wrote statement of Definitions, Properties, or Theorems Incorrectly.  
LCI: Used Logical Connectors Incorrectly.  
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Appendix O

Analysis of Tests 1 and 2 (Grade 9)

*Grade 9 Errors Related to Understanding the Notion of Proof (UNP)*

<table>
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<th>Understanding the Notion of Proof (UNP)</th>
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<td>NJCS</td>
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</tr>
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<tr>
<td>FSC</td>
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<td>12.5%</td>
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</table>

LD (CDR)  18.75%  18.75%  12.5%  6.25%

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