On the Convergence of the Collatz Conjecture Sequence

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***Abstract*—The Collatz conjecture, also known as the $3n+1$ problem, is a famous unsolved problem in mathematics. This work converts the conjecture dynamics into its corresponding difference equation. To determine the boundedness of the sequence, we commence with a boundedness analysis. This process allows us to identify a necessary and sufficient condition for the sequence to be bounded. Following this, we employ standard mathematical techniques to demonstrate conclusively that the 4-2-1 cycle is the only one. Additionally, we show that it is impossible for the sequence to diverge given any positive starting point. Ultimately, we demonstrate that the sequence invariably converges to 1.**

***Index Terms*—Collatz conjecture, 3n+1 problem, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, Syracuse problem**

I. Introduction

The Collatz Conjecture, named after the mathematician Lothar Collatz who introduced the concept in 1937 [1], is a well-known problem in mathematics. This problem is also referred to by several other names, such as Ulam's Conjecture, the Hailstone Problem, the Syracuse Problem, Kakutani's Problem, Hasse's Algorithm, $3n+1$ problem, and the Collatz Conjecture [2]. Despite the variety of names, they all describe the same sequence generation process and the question of whether or not, for any positive integer, the sequence eventually reaches the number 1. The different names come from various researchers and contexts in which the problem has been discussed or studied, but they all relate to the same underlying problem.

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The conjecture applies to sequences generated by repeatedly applying specific rules:

1) Start with any positive integer n.
2) If n is even, divide it by 2.
3) If n is odd, multiply it by 3 and add 1.
4) Repeat steps 2) and 3) until you reach the number 1.

The Collatz Conjecture posits that a specific sequence invariably converges to the number 1. This conjecture remains one of the most intriguing unsolved mysteries in mathematics [3]. Despite its simplicity, it claims that a particular iterative process, when applied to any positive integer, will ultimately lead to the number one. Though straightforward in its premise, the conjecture has resisted a definitive mathematical proof, even though empirical evidence supports its validity for an extraordinary range of cases.

In the quest to resolve the Collatz Conjecture, researchers face several formidable challenges [4]. One of the key difficulties lies in analyzing an infinite sequence. The conjecture generates a never-ending series of numbers, which poses significant challenges for both analysis and proof, as traditional methods of mathematical proof often rely on a finite structure for verification. Adding to the complexity is the exhaustive search for a counterexample. Due to the conjecture's infinitely expansive nature, finding a counterexample is a daunting task that stretches the limits of computational resources and search methodologies [5]. Further complicating matters are the pattern irregularities observed in the generated sequence. While certain special cases of the sequence exhibit discernible patterns [6], these patterns are not universally applicable across all instances of the sequence. This lack of consistency renders many traditional mathematical approaches ineffective, leaving researchers to grapple with the unpredictable and elusive nature of the conjecture. The Collatz Conjecture shows how math isn't always neat and perfect. There's no shortcut to a solution, and mathematicians have to explore different approaches and ideas to get closer to finding an answer [7].

Terras [8] demonstrated that for almost all positive integers, the Collatz sequence either settles into a cycle or concludes at 1. He achieved this by delineating limits on the "stopping time"—the requisite number of steps to either reach 1 or commence a cycle. Although the findings do not provide a definitive solution to the Collatz Conjecture, it introduces a robust framework for examining stopping times associated with this captivating problem in number theory.

Tao's work [6] represents a significant foray into probabilistic methods applied to the Collatz Conjecture, suggesting that while individual sequences can behave unpredictably, they exhibit a form of statistical regularity when considered in aggregate. This approach opens new avenues for exploring the conjecture and demonstrates the potential of leveraging probabilistic and statistical methods to tackle long-standing problems in number theory.

Our literature review provides essential background in a succinct format. While it does not encompass the wide range of perspectives found in the extensive existing literature, our study carves out a new pathway for addressing the problem.

The remainder of this paper is organized as follow. In Section II, we transform the conjecture's dynamics into an nonlinear equivalent difference equation, laying the groundwork for our analysis. From a control theory viewpoint, the aim is to demonstrate the sequence's stability and its eventual convergence towards 1.

Specifically, we derive a closed-form solution for the difference equation, enabling a more effective handling of the sequence. Our initial step in this transformation involves a detailed examination of the boundedness of the resulting sequence. Through a comprehensive boundedness analysis, we successfully establish the necessary and sufficient conditions under which the sequence remains bounded. This crucial determination forms the basis for our subsequent investigations.

Proceeding to Section III, we leverage established mathematical methodologies to provide irrefutable evidence supporting the uniqueness of the 4-2-1 cycle within the conjecture's framework. This section rigorously proves that the 4-2-1 cycle stands alone, unchallenged by any potential rivals. In addition to elucidating the singularity of this cycle, we further elucidate the impossibility of sequence divergence when initiated from any positive starting point. This finding negates the possibility of unbounded behavior, thus reinforcing the conjecture's stability.

Moreover, our analysis extends to demonstrate the inevitable convergence of the sequence towards the unity. This convergence, irrespective of the initial value provided it is finite, underscores a fundamental characteristic of the conjecture's behavior. The sequence, through a series of transformations and transitions, invariably seeks the simplicity and finality of the number one. Furthermore, it is demonstrated that the maximum number of steps required for the sequence to converge to 1 is bounded by the highest value attainable by the sequence, corresponding to its initial starting point.

II. Preliminary Results

We begin by converting the conjecture into its corresponding difference equation, as follows:

$$x_{k+1} = \begin{cases} \frac{1+\cos(\pi x_k)}{2}\frac{x_k}{2} + \frac{1-\cos(\pi x_k)}{2}(3x_k+1), & \text{if } x_k > 1 \\ 1, & \text{if } x_k = 1. \end{cases} \tag{1}$$

In this scenario, x_k denotes a positive integer at the discrete-time index k. When x_k is odd, then $\frac{1+\cos(\pi x_k)}{2} = 0$ and $\frac{1-\cos(\pi x_k)}{2} = 1$. Similarly, when x_k is even, then $\frac{1+\cos(\pi x_k)}{2} = 1$ and $\frac{1-\cos(\pi x_k)}{2} = 0$. Hence, when the initial state x_0 represents any starting positive integer, Equation (1) effectively models the conjecture across all corresponding steps with each step denoted by k. The inclusion of the second case in Equation (1), the total stopping time [9], serves to prevent the sequence from entering the well-known loop of 4-2-1.

If the iteration process terminates or if N does not approach infinity, then according to (1), it implies that the sequence has reached 1 in a finite number of steps.

Combining terms in (1) for $x_k > 1$, we have:

$$x_{k+1} = (1.75 - 1.25\cos(\pi x_k))x_k + \frac{1-\cos(\pi x_k)}{2}. \quad (2)$$

Iterating (2), we obtain

$$x_N = \left(\prod_{i=0}^{N-1} a_i\right) x_0 + \sum_{k=0}^{N-1}\left(\prod_{j=k+1}^{N-1} a_j\right) b_k, \quad (3)$$

where $a_i = 1.75 - 1.25\cos(\pi x_i) \in \left\{\frac{1}{2}, 3\right\}$ and $b_i = \frac{1-\cos(\pi x_i)}{2} \in \{0,1\}$, with $a_i = \frac{1}{2}$ and $b_i = 0$ for even x_i, and $a_i = 3$ and $b_i = 1$ for odd x_i. We denote $\prod_{j=N}^{N-1} a_j = 1$.

We define O_N as the total count of steps x_N assumes odd integer values, and E_N as the number of instances x_N assumes even integer values. Thus, $N = E_N + O_N$,

$$P_N \triangleq \prod_{i=0}^{N-1} a_i = \left(\frac{1}{2}\right)^{E_N} 3^{O_N}, \text{and } S_N \triangleq \sum_{k=0}^{N-1}\left(\prod_{j=k+1}^{N-1} a_j\right) b_k.$$

In the following, we establish a relationship between S_k and P_k. We adjust the indices in Equation (3) to enhance flexibility, as our work utilizes either x_N or x_k, depending on the derivation context.

Proposition 1: For any given $x_0 \in \mathbb{N}$, we have

$$S_k = P_k \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j + b_{k-1}, \quad (4)$$

where $\bar{E}_j \triangleq E_k - E_{j-1}$ and $\bar{O}_j \triangleq O_k - O_{j-1}$.

Proof. By expanding the terms in S_k, we obtain:

$$\begin{aligned} S_k &= b_0 \prod_{j=1}^{k-1} a_j + b_1 \prod_{j=2}^{k-1} a_j + b_2 \prod_{j=3}^{k-1} a_j + \cdots \\ &\quad + b_{k-2}\cdot a_{k-1} + b_{k-1} \\ &= b_0 \frac{3^{O_1}}{2^{E_1}} + b_1 \frac{3^{O_2}}{2^{E_2}} + \cdots + b_{k-2}\frac{3^{O_{k-1}}}{2^{E_{k-1}}} + b_{k-1} \\ &= \frac{3^{O_k}}{2^{E_k}}\left(b_0 \frac{2^{E_k - E_1}}{3^{O_k - O_1}} + b_1 \frac{2^{E_k - E_2}}{3^{O_k - O_2}} + \cdots + b_{k-2}\frac{2^{E_k - E_{k-1}}}{3^{O_k - O_{k-1}}}\right) \\ &\quad + b_{k-1} \end{aligned}$$

In the final step, we multiplied by $\frac{3^{O_k}}{2^{E_k}} \cdot \frac{2^{E_k}}{3^{O_k}} = 1$ and utilized the identity $\frac{3^{O_i}}{2^{E_i}} = \frac{2^{-E_i}}{3^{-O_i}}$. Recognizing that $P_k = \frac{3^{O_k}}{2^{E_k}}$ concludes the proof. □

Example 1. To illustrate our derivation, we begin with an example assuming both x_0 and x_2 are odd. From this, we derive:

$x_1 = 3x_0 + 1.$

$x_2 = \frac{3}{2}x_0 + \frac{1}{2}.$

$x_3 = \frac{9}{2}x_0 + \frac{5}{2}.$

$x_4 = \frac{9}{4}x_0 + \frac{5}{4} = P_4 x_0 + S_4.$

We analyze x_4, noting that $b_0 = b_2 = 1$, $b_1 = b_3 = 0$, $a_0 = a_2 = 3$, and $a_1 = a_3 = \frac{1}{2}$. Employing Equation (3), we find $P_4 = a_0 a_1 a_2 a_3 = \frac{9}{4}$, leading to $O_4 = E_4 = 2$. Consequently, $S_4 = b_0 a_1 a_2 a_3 + b_2 a_3 = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$. Given that $a_1 a_2 a_3 = \frac{3}{4}$, we deduce $O_1 = 1$ and $E_1 = 2$, while for the term $a_3 = \frac{1}{2}$, it follows that $O_3 = 0$ and $E_3 = 1$. Applying Equation (4), we derive $S_4 = P_4\left(b_0 \frac{2^{E_4 - E_1}}{3^{O_4 - O_1}} + b_2 \frac{2^{E_4 - E_3}}{3^{O_4 - O_3}}\right) = \frac{9}{4}\left(\frac{1}{3} + \frac{2}{9}\right) = \frac{5}{4}$. □

Lemma 1: For any given initial value $x_0 \in \mathbb{N}$, the following inequalities hold:

1) Upper bound for x_N: $x_N \le 3\left(\frac{1}{2}\right)^{E_N} 3^{O_N} x_0 + 1 = 3P_N x_0 + 1$, where $P_N = \left(\frac{1}{2}\right)^{E_N} 3^{O_N}$.
2) Upper bound for S_N: $S_N \le 2P_N x_0 + 1$.

Proof. The inequality stated in Lemma 1 is equivalent to:

$$x_N \le \left(\frac{1}{2}\right)^{E_N} 3^{N+1-E_N} x_0 + 1.$$

We first observe that $N + 1 - E_N = O_N + 1$. The proof of the first part proceeds via induction.

This is valid for $N = 1$, as $x_1 \le 3x_0 + 1$. if x_0 is odd, $x_1 = 3x_0 + 1 < 9x_0 + 1$; conversely, $x_1 = \frac{x_0}{2} < \frac{3}{2}x_0 + 1$ if x_0 is even.

In this case, we have $E_1 = 0$, $O_1 = 1$ and $P_1 = 3$ if x_0 is odd, where $x_1 = 3x_0 + 1 < 9x_0 + 1$; conversely, $x_1 = \frac{x_0}{2} < \frac{3}{2}x_0 + 1$ if x_0 is even, resulting in $E_1 = 1$, $O_1 = 0$ and $P_1 = \frac{1}{2}$.

We assume its validity for $N-1$. Therefore, we have $x_{N-1} \le \left(\frac{1}{2}\right)^{E_N} 3^{N-E_N} x_0 + 1$. By following a similar argument for cases where x_{N-1} is odd or even, it results in

$$\begin{aligned} x_N \le 3x_{N-1} + 1 &\le \left(\frac{1}{2}\right)^{E_N} 3^{N+1-E_N} x_0 + 1 \\ &= 3\left(\frac{1}{2}\right)^{E_N} 3^{O_N} x_0 + 1. \end{aligned}$$

For the second part of this lemma, we have $x_N = P_N x_0 + S_N \le 3P_N x_0 + 1$, then $S_N \le 2P_N x_0 + 1$. This concludes the proof. □

Theorem 1. Given any $x_0 \in \mathbb{N}$, the boundness of Equation (3) is guaranteed *if only if* $\lim_{N\to\infty} \prod_{i=0}^{N-1} a_i$ is bounded.

Proof.

Necessary condition: The boundedness of x_N necessitates that $\lim_{N\to\infty} \prod_{i=0}^{N-1} a_i$ be bounded; otherwise, $\lim_{N\to\infty} x_N = \lim_{N\to\infty}(P_N x_0 + S_N)$ becomes unbounded.

Sufficient condition: This follows directly from Lemma 1, as $x_N \leq 3\left(\frac{1}{2}\right)^{E_N} 3^{O_N} x_0 + 1$ is bounded if and only if $\prod_{i=0}^{N-1} a_i = \left(\frac{1}{2}\right)^{E_N} 3^{O_N}$ is bounded. □

Remark 1. The convergence of this nonlinear system, as governed by Equation (1), towards a specific equilibrium point is influenced by its initial conditions. Setting $x_{k+1} = x_k = x \in \mathbb{R}$ in Equation (2) to deduce the equilibrium points, we identify potential points at $x = 0$ and $x = -0.5$. However, within the framework of Equation (1), the scenario for $x = -0.5$ or $x = 0$ is not applicable since the system is confined to states where x_k can only assume positive values, making the non-positive equilibrium points irrelevant. However, the system in (1) restricts that as $x_k = 1$, $x_{k+m} = 1$ for all $m \in \mathbb{N}$. Therefore, when considering the system's behavior with respect to positive integers, the legitimate equilibrium point identified is at $x = 1$. □

III. Main Results

We initially define what constitutes a cycle and prove that the 4-2-1 cycle is unique, as stated in Theorem 2. Subsequently, in Theorem 3, we demonstrate the sequence's boundedness. Lemma 2 establishes that the convergence of the sequence $\{x_k\}$ necessarily implies convergence to 1. Finally, we establish that the sequence converges to 1 from any finite starting point.

Cycle: A cycle occurs when the sequence of numbers generated by repeatedly applying the Collatz function eventually returns to a previously encountered number, and from that point on, the sequence keeps repeating the same set of numbers in a specific order.

In a cycle, with number of steps equals to N, we have:

$$x_0 \to x_1 \to x_2 \to x_3 \to \cdots \to x_{N-1} \to x_N = x_0.$$

For example, in the 4-2-1 cycle, if $x_0 = 2$, then $x_3 = 2$. Similarly, if $x_0 = 4$, we obtain $x_3 = 4$.

Clearly, any starting number x_i in a cycle $x_i \in \{x_0, x_1, \ldots x_{N-1}\}$ returns to itself taking the same number of steps.

Theorem 2. For any initial value $x_0 \in \mathbb{N}$ with subsequent terms $x_k \in \mathbb{N}$, the 4-2-1 cycle is the only cycle.

Proof. We begin by demonstrating that if $x_0 = x_N$ for some $N \in \mathbb{N}$, then $P_N < 1$. Employing an inductive approach, we examine possible classes of loops where $x_k > x_N$ for $1 \leq k \leq N-1$. Ultimately, we identify a contradiction by showing that $P_N > 1$ for any loop other than the 4-2-1 loop, thereby establishing the uniqueness of the 4-2-1 loop under these conditions.

Assume $\exists N \in \mathbb{N}$ and $x_0 \in \mathbb{N}$ such that $x_N = x_0$. Then, Equation (3) implies that

$$x_0 = P_N x_0 + S_N, \tag{5}$$

where $P_N = \prod_{i=0}^{N-1} a_i$ and $S_N = \sum_{k=0}^{N-1}\left(\prod_{j=k+1}^{N-1} a_j\right) b_k$. Therefore,

$$x_0 = \frac{S_N}{1 - P_N}.$$

Given that $S_N > 0$, $P_N > 0$, and $x_0 \in \mathbb{N}$, it follows that $0 < P_N < 1$. Note that $P_N \neq 0$, otherwise, it would imply $S_N = 0$ and $x_0 = 0$, which contradicts the given condition. Additionally, $P_N = \left(\frac{1}{2}\right)^{E_N} 3^{O_N}$, where O_N is the total number of steps x_k assumes odd integer vales, and E_N is the number of instances x_k assumes even integer values. If $P_N = 1$, then $\frac{O_N}{E_N} = \frac{\log(2)}{\log(3)} \notin \mathbb{Q}$. Given that $N = E_N + O_N$, a finite N cannot yield $P_N = 1$. Additionally, if $P_N = 1$, then from (3), we deduce that $S_N = 0$, which implies that x_k is even $\forall k > 0$. The latter implies that $x_k \to 1$ in $N = \frac{\log(x_0)}{\log(2)}$ steps.

We use the established bound for P_N, $0 < P_N < 1$, which is related to the sufficient condition for boundedness provided in Theorem 1.

In what follows, we use an induction argument to demonstrate that x_0 cannot be part of any cycle for all $x_0 > 4$, we acknowledge the segment of the conjecture as valid for any starting point greater than 4 and less or equal to X_0, where

$X_0 \gg 1$. For simplification, we set $x_0 = X_0 + 1$, aiming to prove that x_0 does not become entrapped in a cycle of any period.

If x_0 is even then $x_1 = \frac{x_0}{2} < X_0$. Therefore, x_0 must be odd. We examine two scenarios: the first where $x_{N-1} < x_0$ and the second one where $x_{N-1} > x_0$.

Scenario 1: If $x_{N-1} < x_N = x_0$ while noting that $x_i \in \mathbb{N}$, then by the induction argument, this configuration cannot be part of any cycle. Another perspective can be explained as follows: for x_{N-1} to progress to its next step $x_N = x_0$, it necessitates that x_{N-1} is odd, while x_0 is even. Hence, we deduce $x_{N+1} = \frac{x_0}{2}$. Based on the induction context, $\frac{x_0}{2} < X_0$ cannot be a part of any cycle.

Scenario 2: When $x_{N-1} > x_0 = x_N$, it implies x_{N-1} is even. Thus, $x_{N-1} = 2x_0$. We explore the case where x_{N-2} can be either odd or even where we provide a general examination. If x_{N-2} is odd, then $x_{N-1} = 3x_{N-2}+1$, or $x_{N-2} = \frac{2x_0}{3} - \frac{1}{3}$. The latter implies that $x_{N-2} < x_0$, which, according to the induction argument, cannot enter into a cycle.

In light of the recent findings, the following is established:

$x_{N-2} = 4x_0$.

$x_{N-1} = 2x_0$.

$x_N = x_0$.

$x_1 = 3x_0 + 1$.

$x_2 = \frac{3}{2}x_0 + \frac{1}{2}$.

$x_3 = \frac{9}{2}x_0 + \frac{5}{2}$. Note that if x_2 were even, then $x_3 < x_0$.

$x_4 = \frac{9}{4}x_0 + \frac{5}{4}$.

It is important to highlight the following:

if $x_1 = x_{N-2}$, then $x_0 = 1$ as expected from the 4-2-1 loop.

if $x_2 = x_{N-2}$, then $x_0 = \frac{1}{5} < 1$.

if $x_3 = x_{N-2}$, then $x_0 = -5 < 0$.

if $x_4 = x_{N-2}$, then $x_0 = \frac{5}{7} < 1$.

Before the sequence begins its non-stop descent to its initial value, an odd value of x_k must be encountered. Therefore, $\exists k > 4$ such that $x_{k+1} = 3x_k + 1 = 2^n x_0$ where $n \geq 2$. We can express x_{k+1} as follows:

$$x_{k+1} = P_{k+1}x_0 + S_{k+1} = 2^n x_0.$$

Assume that, for $N > 4$, $\exists N$ such that $x_0 = x_N = P_N x_0 + S_N$. It follows that:

$$P_{k+1}x_0 + S_{k+1} = 2^n P_N x_0 + 2^n S_N.$$

Thus, $P_N = \frac{P_{k+1}}{2^n}$ and $S_N = \frac{S_{k+1}}{2^n}$. Subsequently, we can express x_0 as follows:

$$\begin{aligned} x_0 &= \frac{S_N}{1-P_N} = \frac{\frac{S_{k+1}}{2^n}}{1-\frac{P_{k+1}}{2^n}} \\ &= \frac{S_{k+1}}{2^n - P_{k+1}} = \frac{3S_k + 1}{2^n - 3P_k} \\ &= \frac{S_k + \frac{1}{3}}{\frac{2^n}{3} - P_k}. \end{aligned}$$

In this derivation, we leverage the relationships $P_{k+1} = 3P_k$ and $S_{k+1} = 3S_k + 1$ because we are considering the case where x_k is odd. Additionally,

$$\begin{aligned} x_k &= P_k x_0 + S_k \\ &= \frac{x_{k+1} - 1}{3} \\ &= \frac{P_{k+1}}{3}x_0 + \frac{S_{k+1} - 1}{3} \\ &= \frac{2^n P_N}{3}x_0 + \frac{2^n S_N - 1}{3} \end{aligned}$$

Consequently, $P_k = 2^n P_N$ and $S_k = \frac{2^n S_N - 1}{3}$ or $S_N = \frac{3}{2^n}S_k + \frac{1}{2^n}$. Since $n \geq 2$, then $S_N < S_k + \frac{1}{3}$. The latter inequality is based on the fact that $\left(1 - \frac{3}{2^n}\right) S_k > 0 > \frac{1}{2^n} - \frac{1}{3}$.

Next, consider

$$x_k = \frac{2^n x_0 - 1}{3} = P_k x_0 + S_k.$$

Therefore,

$$x_0 = \frac{S_k + \frac{1}{3}}{\frac{2^n}{3} - P_k} = \frac{S_N}{1 - P_N}.$$

Since $x_0 > 0$ and $S_k + \frac{1}{3} > 0$, then $\frac{2^n}{3} - P_k > 0$. We have $S_N < S_k + \frac{1}{3}$, which implies that $1 - P_N < \frac{2^n}{3} - P_k$ or $P_N > 1 + \left(\frac{2^n}{3} - P_k\right) > 1$ since $\frac{2^n}{3} - P_k > 0$.

Given that $P_N > 1$, we encounter a contradiction, which bolsters our induction argument. This completes the proof. □

Remark 2. There exists a class of enormously vast starting points that converge to 1. For example, based on the literature (see, e.g., [10] and [5]), we assume that sequences with starting points up to $y < x_0$ converge to 1. If $x_0 = 2^m y, y < x_0, \forall m \in \mathbb{N}$, then the sequence, with starting point equals $x_0 = 2^m y$, would converge to 1.

This is due to the fact that if $x_k = 2^m y$, where $y < x_0$ for all

$m \in \mathbb{N}$, x_0 is even, hence $x_1 = 2^{m-1}y$. Subsequently, after an additional $m-1$ divisions by 2, we arrive at $x_{k+n} = y$. Based on the induction argument, it is inferred that y converges to 1, implying that the sequence starting with x_0 also converges to 1. □

Motivation. We provide context for Theorem 3, focusing on the boundedness of the sequence $\{x_k\}$. Given any *positive* $x_0 \in \mathbb{N}$, if the sequence $\{x_k\}$ diverges, then as $k \to \infty$, we have $E_k \to \infty$ and $O_k \to \infty$.

It follows that there will be infinite odd numbers and infinite divisions by 2, as whenever x_k is odd, then $x_{k+2} = \frac{x_{k+1}}{2}$. Additionally, as stipulated by Theorem 2, the sequence is required to produce an infinite number of distinct values, prohibiting the recurrence of any identical numbers. Consequently, one can infer that the only viable initial point is zero.

Theorem 3. Consider the system governed by Equation (1). For any initial finite value $x_0 \in \mathbb{N}$ with subsequent terms $x_k \in \mathbb{N}$, the sequence $\{x_k\}$ is bounded.

Proof. The fundamental premise of the proof restricts that $\{x_k\} \subset \mathbb{N}$, and if the sequence $\{x_k\}$ diverges, then as $k \to \infty$, we have $E_k \to \infty$ and $O_k \to \infty$.

We show that if x_k diverges, then the starting point x_0 cannot be positive.

As in the proof of Proposition 1, we consider the expansion of $x_k - b_{k-1} \in \mathbb{N}$ since $b_{k-1} \in \{0, 1\}$. Thus, $x_k - b_{k-1} = P_k x_0 + S_k - b_{k-1}$. Making use of (4), we obtain

$$\begin{aligned} x_k - b_{k-1} &= P_k x_0 + P_k \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j \\ &= \frac{3^{O_k}}{2^{E_k}} x_0 + \frac{3^{O_k}}{2^{E_k}} \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j \\ &= \frac{3^{O_k} x_0 + 3^{O_k} \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j}{2^{E_k}}. \end{aligned}$$

Note that since $O_k \geq \bar{O}_{j+1}, 0 \leq j \leq k-2$, then $\sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j$ is a positive integer. For $x_k - b_{k-1} \in \mathbb{N}$, then the numerator must by a multiple of 2^{E_k}, that is,

$$3^{O_k} x_0 + 3^{O_k} \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j = 2^{E_k} m, \quad m \in \mathbb{N}.$$

Since the two terms in the left-hand side are positive integers, then $m \geq 1$. Therefore,

$$x_0 + \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j = \frac{2^{E_k}}{3^{O_k}} m = \frac{1}{P_k} m.$$

Theorem 1 implies that if $x_k \to \infty$ diverges, then $P_k \to \infty$. Consequently,

$$\lim_{P_k \to \infty} \left(x_0 + \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j \right) = 0. \quad (6)$$

We next show that $\frac{S_k - b_{k-1}}{P_k} = \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j$ is bounded. Using Lemma 1, we obtain

$$\frac{S_k - b_{k-1}}{P_k} \leq \frac{2P_k x_0 + 1 - b_{k-1}}{P_k} = 2x_0 + \frac{1 - b_{k-1}}{P_k}.$$

As $P_k \to \infty$, then

$$\sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j \to 2x_0. \quad (7)$$

Consequently, utilizing the limits introduced in (6) and (7), we infer that as $P_k \to \infty$, it follows that x_0 cannot be a positive integer. Therefore, P_k must be bounded. we can directly invoke Theorem 1. This theorem ensures that the sequence x_k is also bounded. Therefore, we have proven that the sequence x_k is bounded. □

Lemma 2. Consider the system governed by Equation (1). For any $x_0 \in \mathbb{N}$, if $\{x_k\}$ is convergent, then $\lim_{k\to\infty} x_k = 1$.

Proof. If iteration ceases, (1) indicates the sequence reaches 1 in finite steps; otherwise, we assume k approaches infinity. Given that $x_0 \in \mathbb{N}$, Theorem 3 guarantees that for all $k > 0$, $x_k = P_k x_0 + S_k$ is bounded, and in the context of this lemma, the sequence $\{x_k\}$ is assumed to be convergent. It is worthwhile noting that x_0, P_k and S_k are all positive and bounded. If, in the limit, either P_k or S_k does not converge, then the sequence x_k will undergo either division by two or multiplication by three plus one. Consequently, x_k cannot converge if either S_k or P_k fails to converge. Consequently, $\lim_{k\to\infty} S_k$ is also convergent. Utilizing Equation (4), we deduce

$$\lim_{k\to\infty} S_k - b_{k-1} = \lim_{k\to\infty} P_k \sum_{j=0}^{k-2} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j$$

must likewise be convergent.

Given that the terms in the summation are bounded, it follows that $\exists C > 0$ such that $\forall j$, $0 < \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j \leq C < \infty$ or $\frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j P_k \leq C P_k$. Therefore, for some integers n and m with $0 < n < m < \infty$, we can express

$$\sum_{j=n}^{m} \lim_{k\to\infty} \frac{2^{\bar{E}_{j+1}}}{3^{\bar{O}_{j+1}}} b_j P_k \leq C(m-n+1) \lim_{k\to\infty} P_k. \quad (8)$$

Using the Cauchy criterion for series, specifically, $\sum_{k=0}^{\infty} p_k$ converges if and only if, for every $\epsilon > 0$, there exists a positive integer N such that for all $m > n \geq N$, it holds that $\sum_{k=n}^{m} p_k < \epsilon$.

We use the value $\frac{C(n-m+1)\epsilon}{3x_0}$ to further bound the sum in (8). Consequently, $\lim_{k\to\infty} P_k < \frac{\epsilon}{3x_0}$. From Lemma1, we have $\lim_{k\to\infty} x_k \leq \lim_{k\to\infty} 3P_k x_0 + 1 < 1 + \epsilon$. Since $\lim_{k\to\infty} x_k$ must be an integer, then $\lim_{k\to\infty} x_k = 1$. □

Theorem 4. Given the system described by Equation (1), for any initial condition $x_0 \in \mathbb{N}$, the sequence $\{x_k\}$ converges to 1 in a finite number of steps. Furthermore, the number of steps required for x_k to converge to 1 is bounded above by the maximum value of x_k corresponding to the initial value x_0.

Proof. Theorem 3 guarantees that for all $k > 0$, x_k is bounded. Therefore, for each $x_0 \in \mathbb{N}$, $\exists M \in \mathbb{N}$ such that $x_k \leq M < \infty$. In addition, Lemma 2 implies that if $\lim_{k\to\infty} x_k = \mu \leq M$, then the sequence $\{x_k\} \to 1$. Therefore, if $\{x_k\}$ does not converge, then it must span infinite integers.

However, as implied by Theorem 2, if the sequence generates an infinite number of distinct values, thereby preventing any repetition of values, it follows that there cannot be more than M distinct integer values. This is because Theorem 3 dictates that $x_k \leq M < \infty$. Consequently, the sequence $\{x_k\}$ is constrained to span at most M distinct values. Furthermore, as stipulated by Lemma 2 and the system described by Equation (1), the iteration halts upon reaching $x_k = 1$. Thus, the sequence $\{x_k\}$ inevitably converges to 1 within a maximum of M steps. □

IV. Conclusion

Our investigation commenced with the formulation of a discrete-time system model to encapsulate the dynamics embedded in the conjecture. Central to our analysis was the establishment of a necessary and sufficient condition for the sequence's boundedness. Through this framework, we identified a condition for the existence of potential cycles and substantiated the 4-2-1 cycle as the only viable cycle. Moreover, we established the impossibility of sequence divergence from any positive starting point. Ultimately, our findings affirm that the sequence invariably converges to 1.

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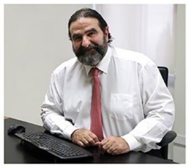

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