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Mathematical and Numerical Simulations of the Lotka-Volterra Equations

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Mathematical and Numerical Simulations of the Lotka-Volterra Equations

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Contents

Abstract

In this study, we perform several Mathematical and numerical simulations of the classic Lotka-Volterra equations and their extensions. We first study the classic Lotka Volterra equations in the context of a two species Predator-Prey model. We then extend the classic two species model into a three-species Lotka Volterra Food chain with a top predator species, a mid-level Predator/Prey species and a lower-level prey. In our two species classic model, we find the critical points of our system, model the phase plot of the two species to understand their behavior around their critical point and then plot the variation of the two species with respect to time. As for our three species Lotka Volterra food chain, we take extreme cases of our system by assuming each of the species to be absent discretely and model the behavior of the two remaining species. We also study the behavior of our three species system around its critical points and vary the constants to understand the fate of each of our species in different contexts. Lastly, we plot, the variation of our three species through time, given varying assumptions. Our two species Lotka-Volterra System shows oscillatory behavior through time, where the predator and prey both persist, and their population oscillates over time. As for our Three species Lotka-Volterra food chain, the destiny of each of our species is dictated by the values of 4 constants in our system.

1. Introduction

In 1926, Vito Volterra, a physicist and mathematician, had taken interest in mathematical biology, and published a system of differential equations to model the interaction of two species population, where one is the prey and the other is the predator [1]. In 1910, Alfred Lotka independently suggested the same equations throughout his work on autocatalytic chemical reactions, where chemical concentrations exhibit oscillatory behavior [2]. Lotka Later extended the usage of these equations in 1920, to include organic system such as a plant species and an herbivorous animal species [3].In 1925, Lotka then published a book on biomathematics where he clearly used this model to analyze predator-prey phenomena [4].This model was extended by Chauvet, Paullet, Previte and Walls (2002) , who proposed an advanced three species Lotka-Volterra food chain, where a prey species x is predated upon by a predator species y, which in turn is predated upon by a predator species z [5].

2. Methods

2.1. Two Species Lotka-Volterra equations

In the study of Population dynamics for two species, we study the growth and decline of a population considering its interaction with the other. The Lotka-Volterra equations, propose a simplistic method of interactions between two species where one is the predator and the other is the prey. The equations are two first order non-linear differential equations defined as follows:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + pxy \end{cases} \tag{1}$$

Where x represents the population of prey, y the population of predators and a, c, b, and p are positive constants. a represents the natural growth rate of the prey when the predator is absent. b represents the effect of the predation on the prey. c represents the natural death of the predator when the prey is absent, and no predation exists. p represents the reproduction rate of the predator when the prey exists and is being consumed by the predator.

The above equations also presuppose the following assumptions:

-The prey is provided with an unlimited food source.

-The only threat to the growth of the prey is the specific predator species.

-The predator has the prey as its only source of food and growth.

-Any interaction (xy) between the predator and the prey will result in the predator eating the prey.

2.1.1. Studying equilibrium and stability

When studying our system (1), It is useful to study its behavior and its stability around its critical points. If a solution that starts close enough to one of our critical points, tends to it, it is said to be a stable critical point. Thus, we take the equations as follows:

$$\begin{cases} \frac{dx}{dt} = 0 = ax - bxy \\ \frac{dy}{dt} = 0 = -cy + pxy \end{cases} \tag{2}$$

It can be found from the above equation that the critical points are (0,0) and $(\frac{c}{p}, \frac{a}{b})$. To test the stability around the following critical points, we first Linearize system (1) according to the following equations:

$$f(x,y) = f(x_{critical}, y_{critical}) + (y - y_{critical})\left(\frac{\partial f}{\partial y}\right) + (x - x_{critical})\left(\frac{\partial f}{\partial x}\right) \tag{3}$$

$$g(x,y) = g(x_{critical}, y_{critical}) + (y - y_{critical})\left(\frac{\partial g}{\partial y}\right) + (x - x_{critical})\left(\frac{\partial g}{\partial x}\right) \tag{4}$$

We Therefore get:

$$\begin{cases} f(x,y) = \frac{dx}{dt} = ax_{critical} - bx_{critical}y_{critical} - (y - y_{critical})bx + (x - x_{critical})(a - by) \\ g(x,y) = \frac{dy}{dt} = -cy_{critical} + px_{critical}y_{critical} + (y - y_{critical})(-\mathrm{c} + \mathrm{p}x) + (x - x_{critical})py \end{cases}$$

(5)

If we set the first derivatives in a matrix called the Jacobian Matrix, and multiply it by a 2x1 matrix containing $(y - y_{critical})$ as element 1x1 and $(x - x_{critical})$ as element 2x1we obtain the following:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = J\begin{bmatrix} y - y_{critical} \\ x - x_{critical} \end{bmatrix} \tag{6}$$

Given that the Jacobian J is:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ py & -c + px \end{bmatrix} \tag{7}$$

The Jacobian at point (0,0) is:

$$J = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \tag{8}$$

The eigenvalues of the following matrix are $\lambda_1 = a$ and $\lambda_2 = -c$ which are of opposite signs, meaning that the point (0,0) is a saddle point and thus unstable.

As for point $(\frac{c}{p}, \frac{a}{b})$, its associated Jacobian matrix is:

$$J = \begin{bmatrix} 0 & \frac{-bc}{p} \\ \frac{p}{ab} & 0 \end{bmatrix} \tag{9}$$

The eigenvalues of this matrix are $\lambda_{1,2} = (\pm\sqrt{ca}\ i)$, which are complex and do not have a real part, meaning that the point $(\frac{c}{p}, \frac{a}{b})$ is a center point of neutral stability.

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2.2. *Three Species Lotka-Volterra food chain equations*

If we now consider a three species food chain model, where a prey x is predated upon by a predator y, which in turn is predated upon by a predator z. Examples of the following model in nature are abundant such as mouse-snake-owl, vegetation-hare-lynx, and worm-robin-falcon. This model consists of three first order non-linear differential equations defined as follows:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + pxy - eyz \\ \frac{dz}{dt} = -fz + gyz \end{cases} \tag{10}$$

Where a, b, c, p, e, f and g are positive constants and:
• e is the effect of predation on y by species z.
• f is the natural death rate of the predator z when the prey y is absent.
• g is the reproduction rate of the predator z when prey y exists and is being consumed by the predator z.

2.2.1. Studying Extreme cases

We study extreme cases of our systems through taking simplifications of two coordinate system rather than three. This is performed through taking three discrete cases where z=0, y=0 and x=0.First, if we take z=0, our model reduces to our previously established two species predator-prey model. Secondly, if we take y=0, we obtain the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = ax \\ \frac{dy}{dt} = 0 \\ \frac{dz}{dt} = -fz \end{cases} \tag{11}$$

The solution of the above system (11) can be determined through dividing its first equation by its third which renders the following:

$$\frac{dx}{dz} = -\frac{ax}{fz} \tag{12}$$

When (12) is reorganized and integrated on both sides, it produces an equation:

$$\ln(x) + K = -\frac{a}{f}\ln(z) + L \tag{13}$$

We reorganize (13) and set K-L=S to obtain:

$$-\frac{f}{a}\ln(x) + S = \ln(z) \tag{14}$$

Solving for z, the following equation is found:

$$z = Q x^{-\frac{f}{a}} \tag{15}$$

for Q=e^S.

For solution in the yz-plane we take x=0 and thus our system will have the following form:

$$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = -\mathrm{c}y - \mathrm{e}yz \\ \frac{dz}{dt} = -\mathrm{f}z + \mathrm{g}yz \end{cases} \tag{16}$$

The solution of (16) can be found through dividing the second equation of the aforementioned system by its third. The solution is therefore of the form:

$$-f ln(\mathrm{y}) + \mathrm{gy} = -c ln(\mathrm{z}) - \mathrm{ez} + K \tag{17}$$

2.2.2. Studying equilibrium and stability

As done with our two species predator prey model, we are to study the behavior of our function around its critical point, to better understand its variations. Setting $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0\ and\ \frac{dz}{dt}$=0, we find that our function admits two critical points $(0,0,0)$ and $\left(\frac{c}{d}, \frac{a}{b}, 0\right)$.Analogous to our previous model, we first linearize (10) according to the following equations:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial z}z \\ \frac{dy}{dt} = \frac{\partial g}{\partial x}x + \frac{\partial g}{\partial y}y + \frac{\partial g}{\partial z}z \\ \frac{dz}{dt} = \frac{\partial h}{\partial x}x + \frac{\partial h}{\partial y}y + \frac{\partial h}{\partial z}z \end{cases} \tag{18}$$

Which gives us:

$$\begin{cases} \frac{dx}{dt} = (a - by)x + (-bx)y + 0 \\ \frac{dy}{dt} = dyx + (-\mathrm{c} + dx - e\mathrm{z})y + (-ey)\mathrm{z} \\ \frac{dz}{dt} = 0 + g\mathrm{z}y + (-f + gy)\mathrm{z} \end{cases} \tag{19}$$

The above system can also be expressed as follows:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \mathrm{z} \end{bmatrix} = J \begin{bmatrix} y \\ x \\ \mathrm{z} \end{bmatrix} \tag{20}$$

Where J is the Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \mathrm{z}} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial \mathrm{z}} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial y}{\partial \mathrm{z}} \end{bmatrix} \tag{21}$$

When we substitute the Jacobian matrix with its proper values, we obtain the following:

$$J = \begin{bmatrix} \mathrm{a} - \mathrm{by} & -\mathrm{xb} & 0 \\ \mathrm{yd} & -\mathrm{c} + \mathrm{dx} - \mathrm{ez} & -\mathrm{ye} \\ 0 & \mathrm{zg} & -\mathrm{f} + \mathrm{gy} \end{bmatrix} \tag{22}$$

The eigenvalues of the above Jacobian matrix at the critical points provide us with information regarding the stability of the system near these equilibrium points. To understand the meaning of the Jacobian matrix eigenvalues, we will utilize the center manifold theorem, which states that each equilibrium point possibly has a unique stable manifold, a unique unstable manifold and a center manifold which is not necessarily unique. The dimensions of the manifolds are determined by the number of eigenvalues with negative, positive and zero real parts. For instance, if a system has two positive eigenvalues, it admits a two-dimensional unstable manifold. Moreover, each of the aforementioned eigenvalues is associated with an eigenvector which is tangents to the corresponding manifold. For an extensive explanation of the center manifold theorem, check Chapter 3 of Guckenheimer and Holmes [6].

Given the following theorem, lets us examine the Jacobian matrix of our critical point (0,0,0):

$$J = \begin{bmatrix} \mathrm{a} & 0 & 0 \\ 0 & -\mathrm{c} & 0 \\ 0 & 0 & -\mathrm{f} \end{bmatrix} \tag{23}$$

The eigenvalues of the following matrix are a, - c, and -f. Based on the aforementioned theorem, the two negative eigenvalues admit a two-dimensional stable manifold. This two-dimensional manifold is tangent to eigenvectors (0,1,0) and (0,0,1) corresponding to eigenvalues λ_2 =- c and λ_3=-f respectively. Therefore, this two-dimensional stable manifold is the yz-plane. Moreover, corresponding to eigenvalue λ_1=a is a one-dimensional unstable manifold, which is tangent to eigenvector (1,0,0). It is apparent that this one-dimensional unstable manifold is the x-axis. Given that the eigenvalues of Jacobian (23) are of different signs, we say that the critical point (0,0,0) is unstable.

As for the point $\left(\frac{c}{d}, \frac{a}{b}, 0\right)$,The Jacobian matrix of this critical point is:

$$J = \begin{bmatrix} 0 & -\frac{cb}{d} & 0 \\ \frac{ad}{b} & 0 & -\frac{ae}{b} \\ 0 & 0 & -\mathrm{f} + \frac{ga}{b} \end{bmatrix} \tag{24}$$

This critical point admits an eigenvalue $\lambda_1 = \frac{ga-fb}{b}$, along with two complex eigenvalues $\lambda_{2,3} = (\pm\sqrt{ca}\ i)$ having their real part equals to zero.

Assuming ga≠fb, means that essentially this critical point admits a one-dimensional manifold. This one-dimensional manifold is stable if fb>ga, and unstable if ga>fb. Moreover, there exists two complex eigenvalues with zero real parts, which indicates that this equilibrium point also admits a two-dimensional center manifold.

If we take ga=fb, The obtained Jacobian matrix will be the following

$$J = \begin{bmatrix} 0 & -\frac{cb}{d} & 0 \\ \frac{ad}{b} & 0 & -\frac{ae}{b} \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

Then the equilibrium point will admit three eigenvalues with their real part equals to zero. This indicates the existence of a three-dimensional center manifold around our critical point.

3. Results

We solve system (1) using the MATLAB ode45 function and plot the variation of predator with respect to prey using to obtain following phase diagram:

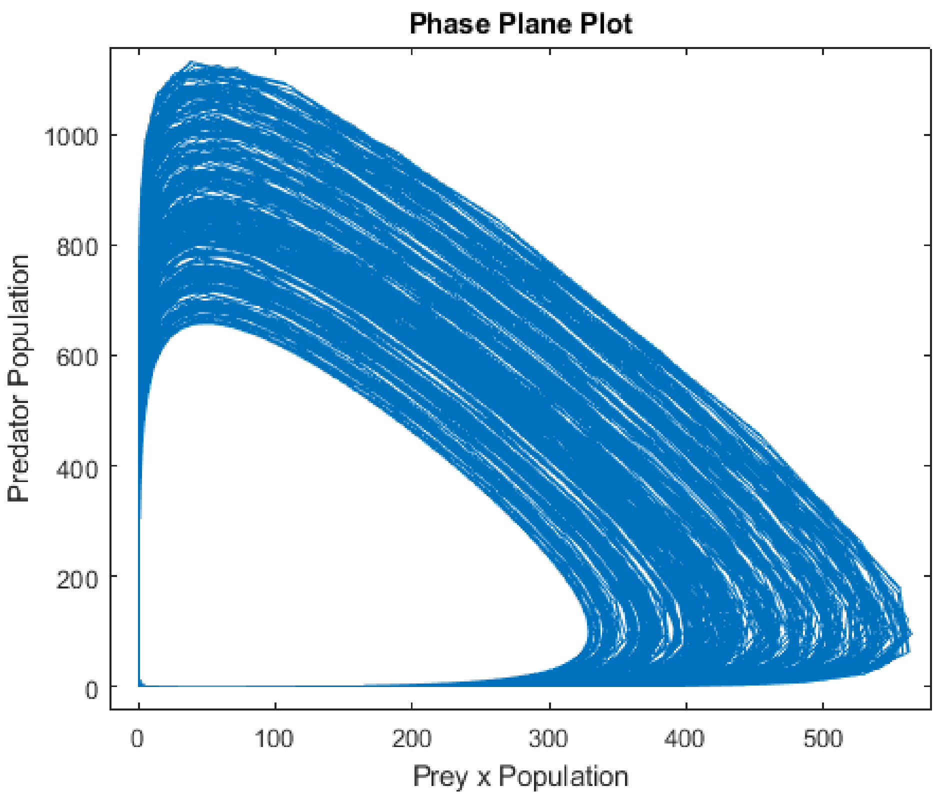

Figure (1) shows the variation of Predator Population with respect to Prey Population for b=0.01, p=0.02, a=1, c=1 and $(x_0,y_0)=(\frac{1}{2},\frac{1}{2})$.

If we then plot the variation of Predator y Population as well as Prey *x* Population with respect to time, we obtain the following figure:

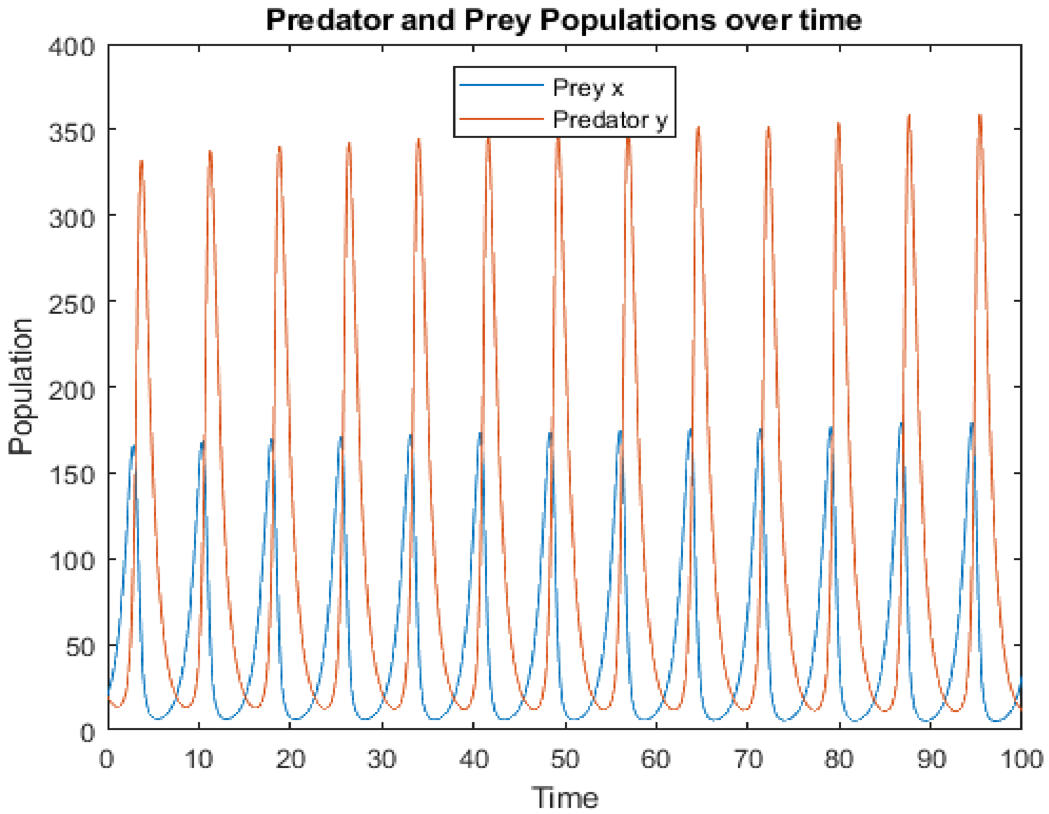

Figure (2) shows the variation of predator population and prey populations with respect to time for b=0.01, p=0.02, a=1, c=1 and (x_0,y_0)=(20,20).

Plotting the solution of (15) in the xz-plane the following figure is obtained:

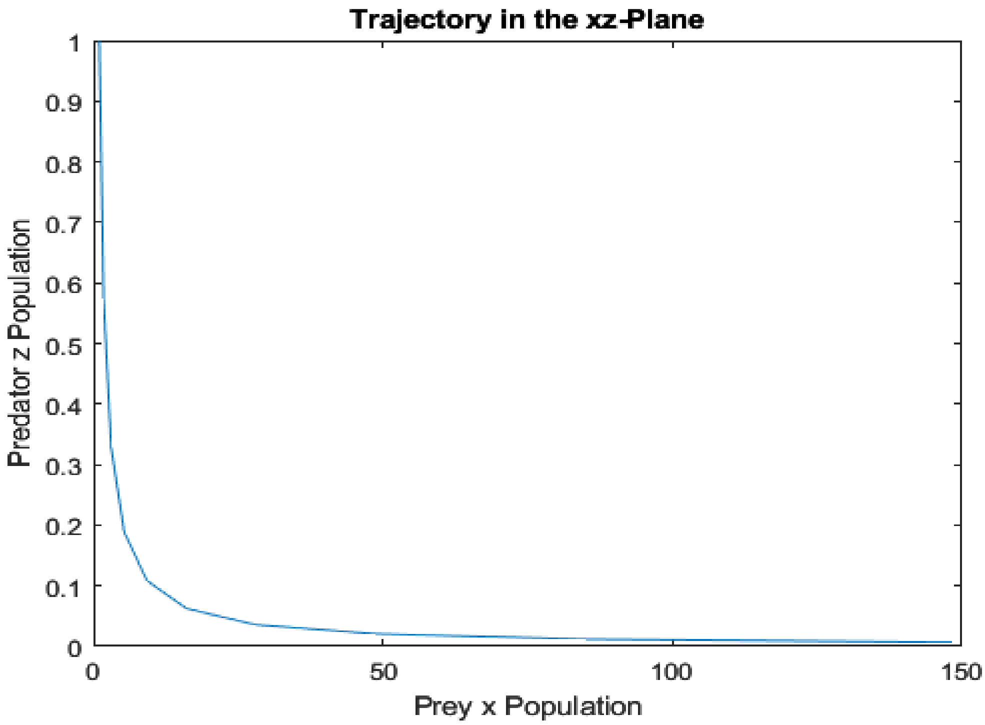

Figure (3) represents the trajectory of our differential equations in the xz-plane, given that a = b = c = p = e = f = g = 1, y=0 and (x_0,y_0,z_0)=(1,0,1).

If we now take x=0,solve system (16), and plot the solutions in the yz-plane we obtain Figure(4).

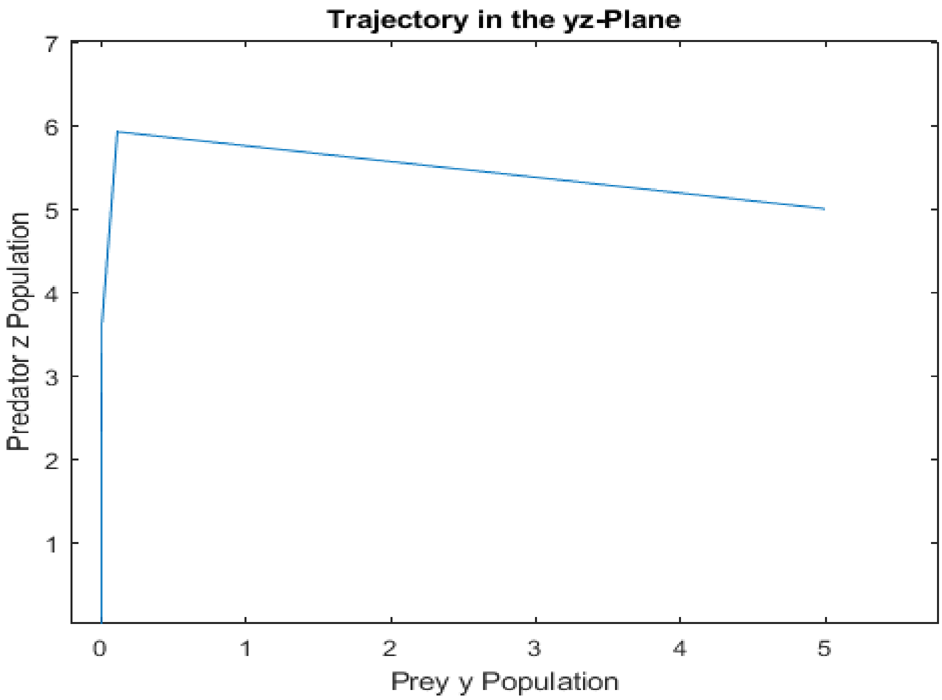

Figure (4) shows the variation of Predator z with respect to Prey y for a = b = c = p= e = f = g = 1, x=0 and (x_0,y_0,z_0)=(0,5,5).

As for (10), if we take ga=fb, solve our system using the MATLAB Ode45 function and then plot the Trajectory in the xyz Space, the following three-dimensional plot is obtained:

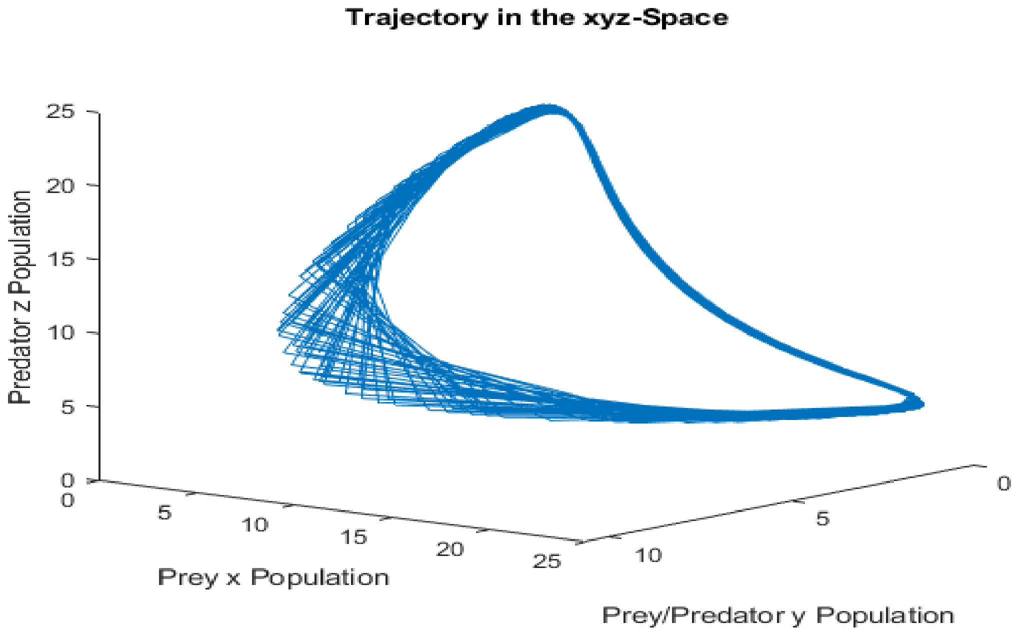

Figure (5) represents the Trajectory of the solution of (10) in the xyz-Space Given that a = b = c = p = e = f = g = 1 and (x_0,y_0,z_0)=(10,10,10).

On the other hand, if we consider ga<fb, find the solution of our system and plot it in the space, we obtain Figure (6).

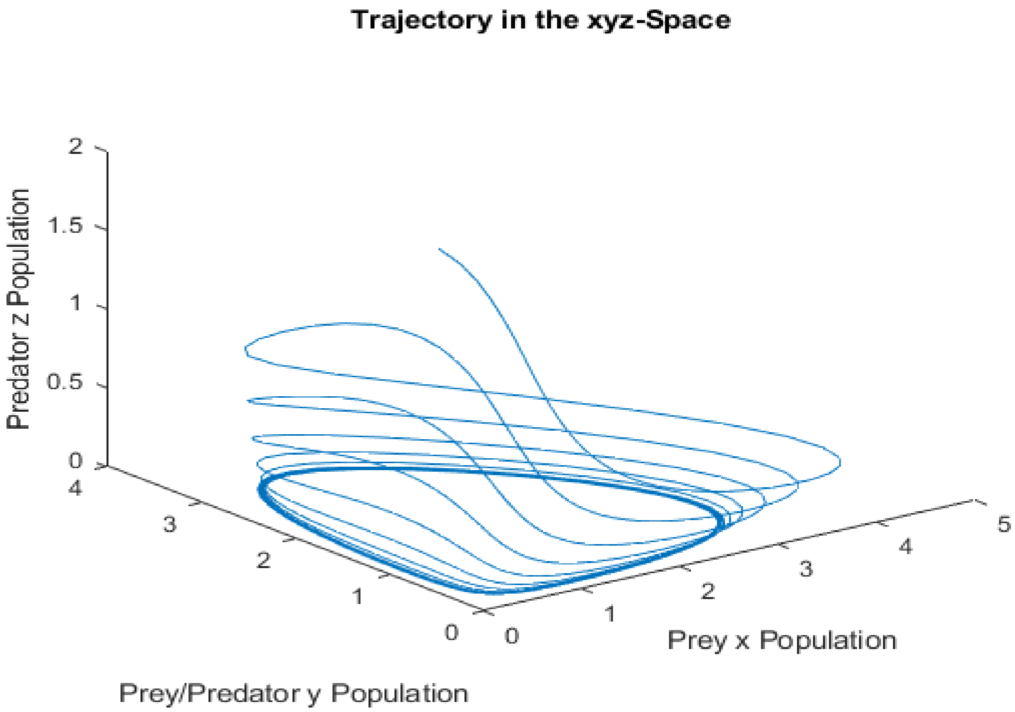

Figure (6) represents the Trajectory of the solution of (10) in the xyz-Space Given that a = b = c = p= e = f, g=0.88 and $(x_0,y_0,z_0)=(\frac{1}{2},1,2)$.

Taking ga>fb and plotting the solution will allow us to obtain Figure (7).

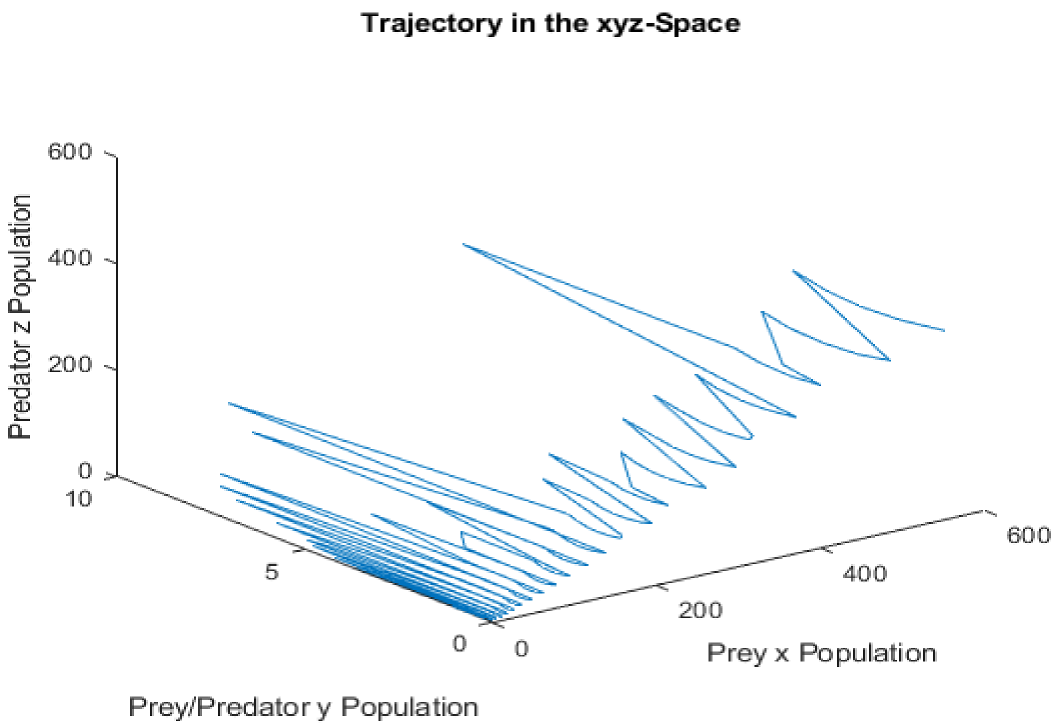

Figure (7) represents the Trajectory of the solution of (10) in the xyz-Space Given that a = b = c = p = e = f, g=1.6 and $(x_0,y_0,z_0)=(\frac{1}{2},1,2)$.

Taking ga<fb, we plot the variation of our three species with respect to time, for us to obtain the following graph:

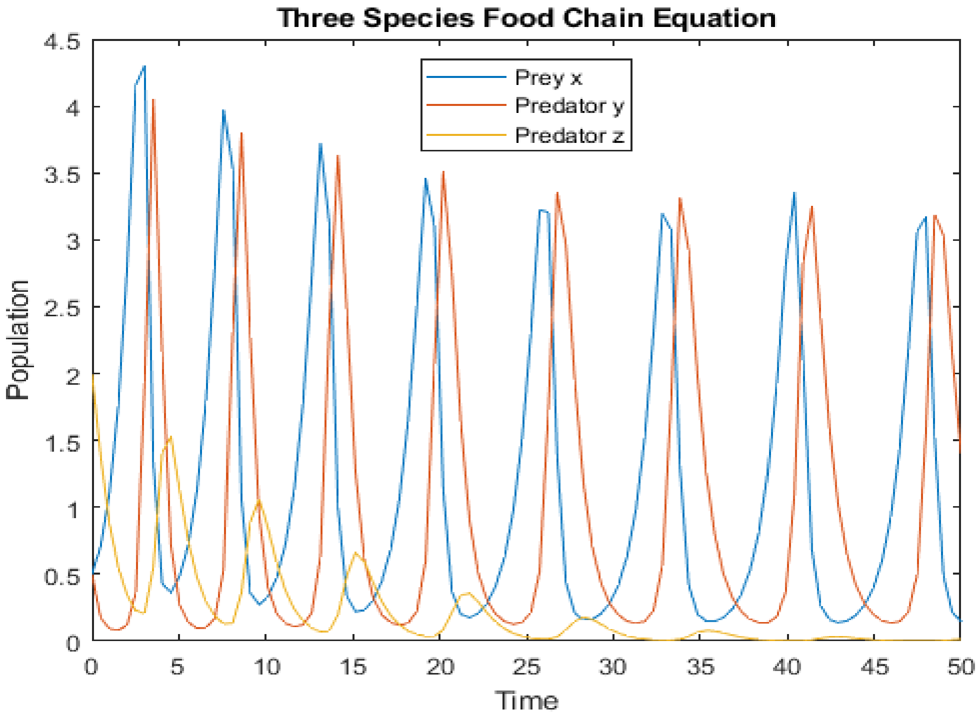

Figure (8) shows the variation of predator population z, predator/prey population y and prey populations x with respect to time, given that $(x_0,y_0,z_0)=(\frac{1}{2}, \frac{1}{2}, 2)$ with a = b = c = p = e = f = 1 and g = 0.88.

Taking ga>fb and plotting the variation of our three species with respect to time allows us to obtain Figure (9).

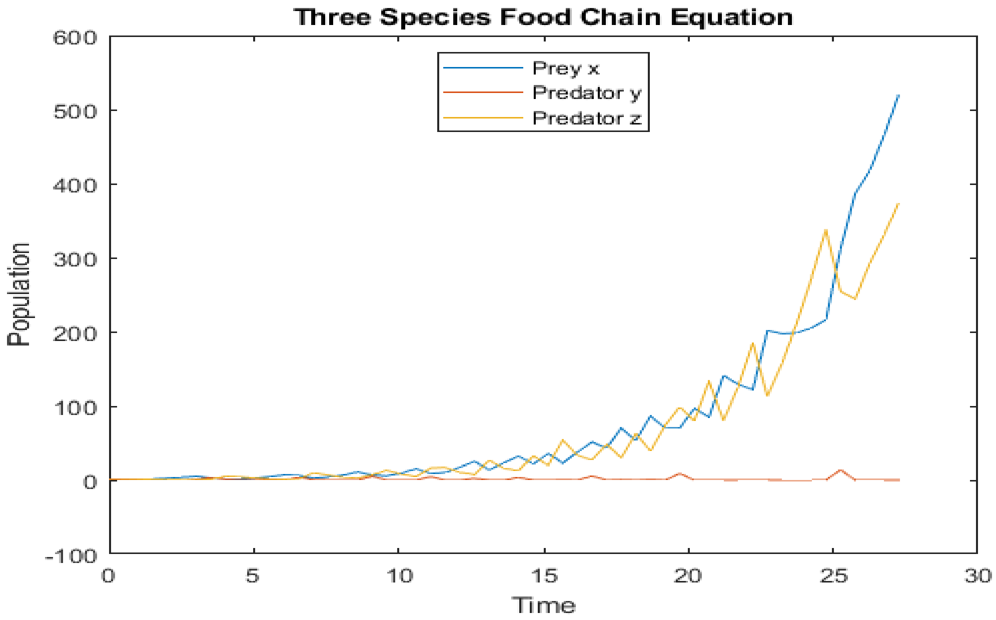

Figure (9) shows the variation of predator population z, predator/prey population y and prey populations x with respect to time, given that $(x_0, y_0, z_0) = (\frac{1}{2}, 1, 2)$ with a = b = c = p = e = f = 1 and g = 1.6.

Lastly, In the case of ga=fb, the variation of our three species with respect to time is shown in Figure (10).

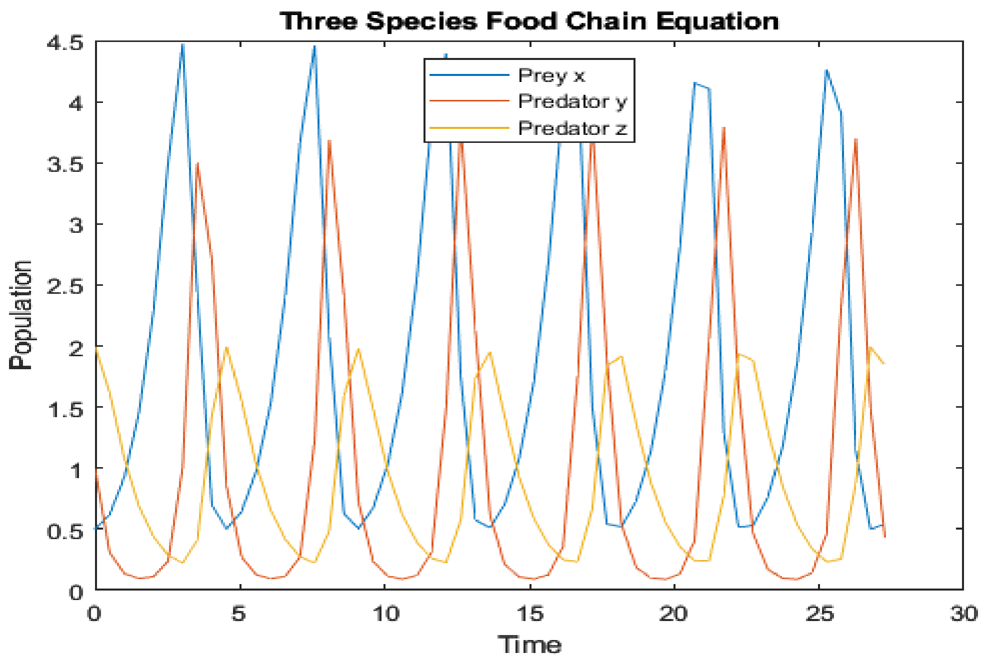

Figure (10) shows the variation of predator population z, predator/prey population y and prey populations x with respect to time, given that (x_0,y_0,z_0)=($\frac{1}{2}$, 1, 2) with a = b = c = p = e = f = g=1.

4. Discussion

Figure (1) shows that around critical point (0,0) the population of Predator experiences natural declines while the Prey experiences natural growth. This makes sense as there are no Predators to consume the prey, which allows the population of prey to grow without limitations. There are also no Preys to be consumed by the Predators, which render the population of predator to decline at the natural rate.

Moreover, if we start with some extra prey population beyond the critical point (x= $\frac{c}{p}, y = \frac{a}{b}$), the prey population will decrease as they are being eaten while the predator population will increase as they are eating the prey. This will cause a decrease in the prey population and an increase in the predator population. This will persist until we have a low number of Prey Population and thus not all Predators will have Prey Population to eat so they number of predators will start to decrease. At the some point the number of predators will be low which will lead to an increase in the number of prey population and the cycle will repeat and the phase plot will have its circular form.

The solution of system (1) with respect to time is plotted in Figure (2) and the latter exhibits oscillatory behavior which is in line with Figure (1) phase plot. Figure (2) shows that the populations of predators and preys exhibit a phase-shifted periodic behavior, having the same period.

As for our Three species Lotka-Volterra System, We first study extreme cases of (10) by taking z =0, where our System reduces to the previously plotted and studied two species Lotka-Volterra System in (1).

If we then take y=0, we obtain system (11) which when plotted renders Figure (3). This plot shows that as time tends to infinity, z tends to zero while x tends to grow in an exponential manner. Therefore, In the absence of species y, species z will have no prey population y to consume and will thus reduce to zero, while there will be no predator population y to consume prey population x which will lead the latter to increase in an unbounded manner.

Taking x=0 and plotting system (16) in the yz-plane, we obtain Figure (4). In this figure we can clearly see that population of prey/predator y is decreasing as it is being consumed by predator z and has no prey x to consume and increase in numbers. As for predator z, it temporarily grows while it consumes prey population y (until the population of Prey y becomes very low), but then tends to extinction rapidly.

As with the two species Lotka-Volterra System, it is useful to study the behavior of our system around its critical point. Using the center manifold theorem to study the behavior of our system (10) around its critical points, we find out that around point (0,0,0) there firstly exists a one-dimensional unstable manifold, which turns out to be the x-axis. Secondly, point (0,0,0) admits a two-dimensional stable manifold which turns out to be the yz-plane. Thus, we conclude that the critical point (0,0,0) is unstable.

*As for our se*cond critical point ($x=\frac{c}{p}, y=\frac{a}{b}, z=\frac{ga - fb}{b}$), assuming ga=fb, our critical admits a three-dimensional center manifold as shown in Figure (5). This means that if ga=fb, all species persist and very periodically over time. This is further reinforced by the solution plot with respect to time as shown in Figure (10) where all the three populations oscillate over time without any species going towards extinction.

If we take ga<fb, then associated with the second critical point is a one-dimensional stable manifold as shown in Figure (6). This Figure indicates than when ga<fb, predator z goes extinct. This makes sense since if we take a=b for instance, then f is greater than g and thus the natural death rate of z is stronger than its propagation rate in the presence of prey y. It would then make sense that z goes extinct. Moreover, when assuming ga<fb, we have another two-dimensional center manifold which turns out to be the xy-plane as shown in Figure (6). This means that in the absence of predator z, x and y exhibit classic Lotka-Volterra two species behavior. If we plot the variation of Populations z, x, and y with respect to time, given ga<fb, we observe in Figure (8) similar behavior to the previously discussed Figure (6), where z tends to extinction and upon extinction the plot shows that x and y exhibit classic two species Lotka-Volterra behavior.

Lastly, if we consider ga>fb, our critical point admits a one-dimensional unstable manifold, which is plotted in Figure (7). This figure shows that as time tends to infinity predator z and prey x tend to infinity, while predator-prey y experiences increasing fluctuations. Moreover, when ga>fb, the solution plot with respect to time (Figure (9)) is in line with the associated plot of Figure (7).

5. Conclusion

We conclude that both the two species Lotka-Volterra system and the three species Lotka-Volterra food chain can be summarized by system (10). If z=0, Our model reduces to a two-species Lotka Volterra System as seen in (1), where both the predator and prey persist and vary periodically. As for our three species model where we do not take any of z=0, we observe that the fate of our top species z is dictated by the values a, b, f and g. If ga<fb, z dies out, and when it becomes extinct prey x and Predator/Prey y exhibit classical Lotka-Volterra two species behavior. When ga>fb, x and z grow unbounded, while y experiences increasing fluctuations. Finally, when ga=fb, all three populations persist and vary periodically.

5.1. Limitations

The two models suggested in the above project contain several unrealistic assumptions which were supposed for the sake of simplicity. To begin with, the first unconsidered premise is that Prey Population is limited by its food source, not only by the predation upon it. Moreover, No predator can consume unlimited quantities of prey. Lastly, it is quite rare to find real life scenarios where prey population can reach very low numbers and yet bounce back.

5.2. Future Work

Regarding future research, we intend to utilize the model proposed by Addison, Bhatt and Owen (2016) [7] to analyze stock behavior between predator and prey companies. This model suggests the following Lotka-Volterra modified equations:

$$\begin{cases} \frac{dX_1}{dt} = s_1 X_1 (1 - \frac{X_1}{K_1}) - m_{12} X_1 X_2 - v_1 X_1 Y\, r_1(X_1, X_2, Y) \\ \frac{dX_2}{dt} = s_2 X_2 (1 - \frac{X_2}{K_2}) - m_{21} X_1 X_2 - v_2 X_2 Y\, r_2(X_1, X_2, Y) \\ \frac{dY}{dt} = -\mu Y + c_1 v_1 X_1 Y\, r_1(X1, X2, Y) + c_2 v_2 X_2 Y\, r_2(X_1, X_2, Y) \end{cases} \tag{26}$$

where $r_1(X_1, X_2, Y) = \frac{1}{1+(\frac{a_2 X_2 + b_2 Y}{X_1})^n}$,$r_2(X_1, X_2, Y) = \frac{1}{1+(\frac{a_1 X_1 + b_1 Y}{X_2})^n}$ and:

- si is the growth rate of the price of prey shares.
- Ki is the carrying capacity of prey shares.
- mij is the competition between prey companies $X_1$and X_2 where i, j = 1, 2.
- Vi is the probability that predator Y invests in prey Xi.
- ai is the harvesting rate of prey shares.
- bi represents anti-predator behavior of prey shares.
- ci: Rate of conversion of prey shares to predator shares.
- μ: Rate of decline of predator share price.
- $r_i(X_1, X_2, Y)$ is the predation term where i, j = 1, 2.

The covid-19 pandemic has demonstrated to be a catalyst which further deepened the monopoly of Big Tech, as the shares of the Big four Tech companies along with Netflix, Tesla and Microsoft increased by $291.66 billion during July 13,2020 only [8]. Our future work will focus on applying these equations in the context of understanding the increase in monopoly during the pandemic.

Acknowledgements

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APPENDIX

Codes:

Figure (1):

-
```
function dxdt = f2(t,x)
dxdt = [0;0];
b=0.01;
p=0.02;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2);

end
```

-
```
y0 = [0.5;0.5];
t = linspace(0,50,100);
[t,x] = ode45(@f2,t,y0);
plot(x(:,1),x(:,2))
title('Phase Plane Plot')
xlabel('Prey x')
ylabel('Predator y')
set(gcf,'color','w');
```

Figure (2):

-
```
y0 = [20;20];
t = linspace(0,50,100);
[t,x] = ode45(@f2,t,y0);
plot(t,x(:,1),t,x(:,2))
title('Predator and Prey Population over Time')
xlabel('Time')
ylabel('Population')
set(gcf,'color','w');
legend('Prey x','Predator y','Location','North')
```

Figure (3):

-
```
function dxdt = f3(t,x)
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1;
dxdt(1) = x(1) - alpha*x(1)*x(2);
```

```
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);

end
```

- y0 = [1;0;1];

```
t = linspace(0,50,100);
x(2)=0;
[t,x] = ode45(@f3,t,y0);
plot(x(:,1),x(:,3))

title('Trajectory in the xz-Plane')
xlabel('Prey x Population')
ylabel('Predator y Population')
```

Figure (4):

- function dxdt = f3(t,x)

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);

end
```

- y0 = [0;5;5];

```
t = linspace(0,5,5);
x(1)=0;
[t,x] = ode45(@f3,t,y0);
plot(x(:,2),x(:,3))

title('Trajectory in the yz-Plane')
xlabel('Prey y Population')
ylabel('Predator z Population')
```

Figure (5):

- function dxdt = f3(t,x)

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);

end
```

- y0 = [10;10;10];

```
t = linspace(0,50,100);
[t,x] = ode45(@f3,t,y0);
plot3(x(:,1),x(:,2),x(:,3))
xlabel('Prey x Population')
ylabel('Predator/Prey y Population')

zlabel('Predator z Population')

set(gcf,'color','w');

title('Trajectory in the xyz space')
```

Figure (6):

- function dxdt = f3(t,x)

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=0.88;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);

end
```

- y0 = [0.5;1;2];

```
t = linspace(0,50,100);
[t,x] = ode45(@f3,t,y0);
plot3(x(:,1),x(:,2),x(:,3))
xlabel('Prey x Population')
ylabel('Predator/Prey y Population')

zlabel('Predator z Population')

set(gcf,'color','w');

title('Trajectory in the xyz space')
```

Figure (7):

- function dxdt = f3(t,x)

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1.6;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);

end
```

- y0 = [0.5;1;2];

```
t = linspace(0,50,100);
```

```
[t,x] = ode45(@f3,t,y0);
plot3(x(:,1),x(:,2),x(:,3))
xlabel('Prey x Population')
ylabel('Predator/Prey y Population')

zlabel('Predator z Population')

set(gcf,'color','w');

title('Trajectory in the xyz space')
```

Figure (8):

- `function dxdt = f3(t,x)`

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=0.88;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);
```

- `y0 = [0.5;0.5;2];`

```
t = linspace(0,50,100);
[t,x] = ode45(@f3,t,y0);
plot(t,x(:,1),t,x(:,2),t,x(:,3))
xlabel('Time')
ylabel('Population')
set(gcf,'color','w');
title('Three Species Food Chain Equation')
legend('Prey x','Predator y','Predator z','Location','North')
```

Figure (9):

- `function dxdt = f3(t,x)`

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1.6;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);
```

- `y0 = [0.5;1;2];`

```
t = linspace(0,50,100);
[t,x] = ode45(@f3,t,y0);
plot3(t,x(:,1),t,x(:,2),t,x(:,3))
xlabel('Time')
ylabel('Population')
set(gcf,'color','w');
title('Three Species Food Chain Equation')
legend('Prey x','Predator y','Predator z','Location','North')
```

Figure (10):

- function dxdt = f3(t,x)

```
dxdt = [0;0;0];
alpha=1;
beta =1;
e=1;
f=1;
g=1;
dxdt(1) = x(1) - alpha*x(1)*x(2);
dxdt(2) = beta*x(1)*x(2) - x(2)-e*x(2)*x(3);
dxdt(3)=-f*x(3)+g*x(2)*x(3);
```

- y0 = [0.5;1;2];

```
t = linspace(0,50,100);
[t,x] = ode45(@f3,t,y0);
plot3(t,x(:,1),t,x(:,2),t,x(:,3))
xlabel('Time')
ylabel('Population')
set(gcf,'color','w');
title('Three Species Food Chain Equation')
legend('Prey x','Predator y','Predator z','Location','North')
```