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S-Convexity: The Variable Radius Case

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We generalize, to the variable radius case, the main results contained in a recent paper of the authors [*A new class of sets regularity*, J. Convex Analysis 25/4 (2018) 1059–1074], where a new regularity class, called S-*convexity*, is studied and shown to cover well-known regularity properties, including φ_0-convexity, θ_0-exterior sphere condition, and ψ_0-union of closed balls property. Furthermore, we provide using S-convexity, new additional characterizations of the variable and the constant radius form of these three regularity properties.

Keywords: S-convexity, φ-convexity, θ-exterior sphere condition, ψ-union of closed balls property, proximal analysis, nonsmooth analysis.

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1. Introduction

Let A be a nonempty and closed subset of $\mathbb{R}^n$ and let $S \subset \mathbb{R}^n$ be a set containing A. The set A is said to be S-*convex* if and only if no two *normal segments* to A (at two distinct points) contained in S, intersect in S. By normal segment, we mean a closed segment of the form $[a, a+t\zeta]$ where $t > 0$, a belongs to the boundary of A, and ζ is a nonzero *proximal normal* vector to A at a. We recall that for $a \in A$ a boundary point and $\zeta \in \mathbb{R}^n \setminus \{0\}$, ζ is a *proximal normal* vector to A at a if there exists $r > 0$ such that

$$\mathbb{B}\left(a + r\frac{\zeta}{\|\zeta\|}; r\right) \cap A = \emptyset,$$

where $\mathbb{B}(z;\rho)$ denotes the open ball of radius ρ centered at z. In this case, we say that ζ is *realized by an r-sphere.*

The S-convexity is introduced, apparently for the first time, in Nour, Saoud and Takche [11] where *inner regularization* of closed subsets of $\mathbb{R}^n$ is studied. More precisely, under the assumption that S contains an S-convex subset A, the authors proved that S can be approximated, from inside, by sets satisfying an *interior sphere condition.* This result is generalized in Nour and Takche [19] to cover more sets S.

The S-convexity is studied in depth in Nour and Takche [18]. In fact, it was shown in this latter that for *suitable choices* of S, the S-convexity class covers several known regularity properties, including φ_0-*convexity*, θ_0-*exterior sphere condition* and ψ_0-*union of closed balls* property. As a consequence, the authors provided a new

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sufficient condition for the equivalence between φ_0-convexity and the θ_0-exterior sphere condition, an equivalence already established under different conditions in Nour, Stern and Takche [12].

Let us recall the definitions of φ_0-convexity, θ_0-exterior sphere condition and ψ_0-union of closed balls property. For $A \subset \mathbb{R}^n$ a nonempty and closed set:

► We say that A is φ_0*-convex* (for $\varphi_0 > 0$) if and only if for any boundary point a of A and *for all* nonzero proximal normal vector ζ to A at a, ζ is realized by a $\frac{1}{2\varphi_0}$-sphere. We refer the reader to Canino [1], Clarke, Stern and Wolenski [6], Federer [9], Colombo and Marigonda [7], Colombo, Marigonda and Wolenski [8], Poliquin and Rockafellar [21], Poliquin, Rockafellar and Thibault [22], and Shapiro [25], for investigations and applications of φ_0-convexity and related properties such as *positive reach*, *p-convexity*, *prox-regularity* and *proximal smoothness*.

► We say that A satisfies the θ_0*-exterior sphere condition* (for $\theta_0 > 0$) if and only if for any boundary point a of A *there exists* a nonzero proximal normal vector ζ to A at a such that ζ is realized by a $\frac{1}{2\theta_0}$-sphere. The θ_0-exterior sphere condition when applied to the closure of the complement of A, is a well-known condition in control theory, called the θ_0*-interior sphere condition*. In fact, this condition is important in deriving regularity properties for the *minimal time function*, see Cannarsa and Frankowska [2] and Cannarsa and Sinestrari [3, 4].

► We say that A is the ψ_0*-union of closed balls* (for $\psi_0 > 0$) if and only if A can be written as the union of closed balls of radius $\frac{1}{2\psi_0}$. The equivalence between the θ_0-interior sphere condition and the ψ_0-union of closed balls property was a conjecture introduced by Nour, Stern and Takche in [12, 13], and proved by the same authors in [15]. In fact, the authors proved in [15] that if A is a closed set satisfying the θ_0-interior sphere condition then A is $2\theta_0$-union of closed balls (the converse is straightforward and holds for $\theta_0 = \psi_0$).

A more general form of the above three regularity properties is obtained when the radius of the balls, used in their definitions, is taken to be *continuously variable*. This corresponds with the replacement of the constant functions φ_0, θ_0 and ψ_0 by continuous functions. Thus, a natural question arises: Can the main results of [18] be generalized to the variable radius case. A *positive* answer to this question shall be provided in this paper. More precisely, the three equivalence results of [18, Theorem 3.7] will be generalized to the variable radius case. Furthermore, we provide using S-convexity, *new* additional characterizations of the above three regularity properties, namely φ-convexity, θ-exterior sphere condition, and ψ-union of closed balls property. Note that even in the constant radius case, these additional characterizations are new.

The layout of the paper is as follows. Notations and some definitions from nonsmooth analysis will be given in the next section. Section 3 is devoted to the generalization to the variable radius case of the main results of [18], and to our new characterizations of φ-convexity, θ-exterior sphere condition, and ψ-union of closed balls property.

2. Preliminaries

We denote by $\|\cdot\|$, $\langle,\rangle$, $\mathbb{B}$ and $\bar{\mathbb{B}}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^n$, we set $\mathbb{B}(x;\rho) := x + \rho\mathbb{B}$ and $\bar{\mathbb{B}}(x;\rho) := x + \rho\bar{\mathbb{B}}$. For a set $A \subset \mathbb{R}^n$, A^c, $\operatorname{int} A$, $\operatorname{bdry} A$ and $\operatorname{cl} A$ are the complement (with respect to $\mathbb{R}^n$), the interior, the boundary and the closure of A, respectively. We also denote by A' the complement of the interior of A, that is, $A' := (\operatorname{int} A)^c$. The closed segment (resp. open segment) joining two points x and y in $\mathbb{R}^n$ is denoted by $[x,y]$ (resp. (x,y)). On the other hand, for $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $[A,x]$ denotes the union of all segments $[a,x]$ such that $a \in A$. The distance from a point x to a set A is denoted by $d_A(x)$. We also denote by $\operatorname{proj}_A(x)$ the set of closest points in A to x, that is, the set of points a in A satisfying $d_A(x) = \|a-x\|$. For $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$ a unit vector, the directional distance from x to a closed set A in the direction ζ, denoted by $d_A(x,\zeta)$, is defined by $d_A(x,\zeta) := \min\{t \geq 0 : x + t\zeta \in A\}$, where the minimum of an empty set is taken to be ∞.

Now we provide certain geometric definitions from proximal analysis. Our general reference for these constructs is Clarke, Ledyaev, Stern and Wolenski [5]; see also Mordukhovich [10], Penot [20], and Rockafellar and Wets [24]. Let A be a nonempty and closed subset of $\mathbb{R}^n$. For $x \in A$, a vector $\zeta \in \mathbb{R}^n$ is said to be *proximal normal to A at x* provided that there exists $\sigma = \sigma(x,\zeta) \geq 0$ such that

$$\langle \zeta, a - x\rangle \leq \sigma\|a-x\|^2, \ \ \forall a \in A. \tag{1}$$

The relation (1) is commonly referred to as the *proximal normal inequality.* No nonzero ζ satisfying (1) exists if $x \in \operatorname{int} A$, but this may also occur for $x \in \operatorname{bdry} A$. For such points, the only proximal normal is $\zeta = 0$. In view of (1), the set of all proximal normals to A at x is a convex cone, and we denote it by $N_A^P(x)$. Now let $x \in \operatorname{bdry} A$, and suppose that $0 \neq \zeta \in \mathbb{R}^n$ and $r > 0$ are such that

$$\mathbb{B}\Big(x + r\frac{\zeta}{\|\zeta\|}; r\Big) \cap A = \emptyset. \tag{2}$$

Then ζ is a proximal normal to A at x and we say that ζ is *realized by an r-sphere.* Note that ζ is then also realized by an r'-sphere for any $r' \in (0,r]$. One can show that ζ being realized by an r-sphere is equivalent to the proximal normal inequality holding with $\sigma = \frac{1}{2r}$, that is,

$$\Big\langle \frac{\zeta}{\|\zeta\|}, a - x\Big\rangle \leq \frac{1}{2r}\|a-x\|^2, \ \ \forall a \in A. \tag{3}$$

In that case, we have

$$\operatorname{proj}_A(y) = \{x\} \ \text{ for all } y \in \Big[x, x + r\frac{\zeta}{\|\zeta\|}\Big), \ \text{ and } x \in \operatorname{proj}_A\Big(x + r\frac{\zeta}{\|\zeta\|}\Big). \tag{4}$$

On the other hand, for $y \notin A$, $x \in \operatorname{proj}_A(y)$ and $\zeta_x := \frac{y-x}{\|y-x\|}$, we have $\zeta_x \in N_A^P(x)$, and ζ_x is realized by a $\|y-x\|$-sphere.

We proceed to define the φ-convexity property. A detailed analysis of this property can be found in Canino [1] under the same p-convexity. See also Colombo and Marigonda [7] and Nour, Stern and Takche [14].

Definition 2.1. Let $A \subset \mathbb{R}^n$ be a nonempty and closed set. We say that A is *φ-convex* if there exists a continuous function $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x \right\rangle \leq \varphi(x)\|a - x\|^2,$$

for all $x \in \operatorname{bdry} A$, $a \in A$ and $0 \neq \zeta \in N_A^P(x)$. By φ_0-convexity we mean φ-convexity with $\varphi = \varphi_0$ a constant.

An important consequence of φ-convexity is that $N_A^P(x) \neq \{0\}$ for all $x \in \operatorname{bdry} A$. On the other hand, clearly if A is compact and φ-convex then it is φ_0-convex. Indeed, it is sufficient to take φ_0 to be the maximum of φ over $\operatorname{bdry} A$. Now using the equivalence between (2) and (3), we can easily see that A is φ-convex if and only if there exists a continuous function $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that for all $x \in \operatorname{bdry} A$ and for all $0 \neq \zeta_x \in N_A^P(x)$ we have:

- ζ_x is realized by a $\frac{1}{2\varphi(x)}$-sphere, if $\varphi(x) \neq 0$,
- ζ_x is realized by an r-sphere for all $r > 0$, if $\varphi(x) = 0$.

Next, we introduce the θ-exterior sphere condition. We also discuss the equivalence between this property and the φ-convexity proved in Nour, Stern and Takche [12] for constant θ and φ, and generalized to the variable radius case by the same authors in [14].

Definition 2.2. Let $A \subset \mathbb{R}^n$ be a nonempty closed set. We say that A satisfies the *θ-exterior sphere condition* if there exists a continuous function $\theta\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that for all $x \in \operatorname{bdry} A$, one can find $0 \neq \zeta \in N_A^P(x)$ such that

$$\left\langle \frac{\zeta}{\|\zeta\|}, a - x \right\rangle \leq \theta(x)\,\|a - x\|^2, \ \ \forall a \in A.$$

By the θ_0-exterior sphere condition we mean the θ-exterior sphere condition with $\theta = \theta_0$, a constant. On the other hand, we say that A satisfies the *θ-interior sphere condition* if and only if A' satisfies the θ-exterior sphere condition.

Clearly if A is compact and satisfies the θ-exterior sphere condition, then it satisfies the θ_0-exterior sphere condition. On the other hand, as in the φ-convexity case, the equivalence between (2) and (3) implies that the θ-exterior sphere condition coincides with the existence of continuous function $\theta\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that for all $x \in \operatorname{bdry} A$ one can find vector $\zeta_x \neq 0$ satisfying

- ζ_x is realized by a $\frac{1}{2\theta(x)}$-sphere, if $\theta(x) \neq 0$,
- ζ_x is realized by an r-sphere for all $r > 0$, if $\theta(x) = 0$.

The equivalence between φ_0-convexity and the θ_0-exterior sphere condition is studied, apparently for the first time, in Nour, Stern and Takche [12]. After proving, via counterexamples, that the two properties are different, the authors established the equivalence between these two properties under the *epi-Lipschitzness* of the set A and the compactness of $\operatorname{bdry} A$. Recall that a closed set $A \subset \mathbb{R}^n$ is said to be *epi-Lipschitz* if for any point $x \in A$, the set A can be viewed near x, after application of an orthogonal matrix, as the epigraph of a Lipschitz continuous function. This geometric definition was introduced by Rockafellar in [23]. The epi-Lipschitz property is also characterizable in terms of the nonemptiness of the topological interior of the *Clarke tangent cone* which is also equivalent to the *pointedness* of the *Clarke normal cone*; see the monographs [5] and [24]. The equivalence result of [12] is generalized to the variable radius case by the same authors in [14], see also [16]. In fact, in [14] the authors provided, under the epi-Lipschitzness of A, an analytic relation between φ and θ, and as a consequence, they obtained under the compactness of $\operatorname{bdry} A$, an analytic relation between φ_0 and θ_0, a relation not given in [12].

We terminate this section by introducing the ψ-union of closed balls property.

Definition 2.3. A nonempty and closed set $A \subset \mathbb{R}^n$ is said to be the *ψ-union of closed balls* if and only if there exists a function $\psi\colon A \longrightarrow [0,\infty)$ such that:

(i) ψ is upper semicontinuous on A and continuous on $\operatorname{bdry} A$.

(ii) For all $x \in A$, there exists $y_x \in A$ such that:

(1) $x \in \bar{\mathbb{B}}\Big(y_x; \frac{1}{2\psi(x)}\Big) \subset A$, if $\psi(x) > 0$,

(2) $x \in \bar{\mathbb{B}}(x + t(y_x - x); t) \subset A$ for all $t > 0$, if $\psi(x) = 0$.

Note that if $\psi = \psi_0$ is a positive constant, then the ψ_0-union of closed balls property coincides with A being the union of closed balls of radius $\frac{1}{2\psi_0}$. The following analytical characterization of this property is given in Nour and Takche [17].

Proposition 2.4. [17, Proposition 2.1] *A nonempty and closed set $A \subset \mathbb{R}^n$ is the ψ-union of closed balls if and only if there exists a function $\psi\colon A \longrightarrow [0,\infty)$ such that:*

(i) *ψ is upper semicontinuous on A and continuous on* $\operatorname{bdry} A$.

(ii) *For all $x \in A$, one can find a unit vector ζ_x satisfying:*

(1) $\exists t \in \big[0, \frac{1}{2\psi(x)}\big] : \langle \zeta_x, z - x + t\zeta_x\rangle \leq \psi(x)\|z - x + t\zeta_x\|^2,\ \forall z \in A'$, if $\psi(x) > 0$,

(2) $\langle \zeta_x, z - x\rangle \leq 0,\ \ \forall z \in A'$, if $\psi(x) = 0$.

Remark 2.5. From the proof of [17, Proposition 2.1], we deduce that Definition 2.3(ii)(1) and Definition 2.3(ii)(2) are equivalent to Proposition 2.4(ii)(1) and Proposition 2.4(ii)(2), respectively.

One can easily see that if A is the ψ-union of closed balls, then it satisfies the θ-interior sphere condition for $\theta := \psi$. The converse implication, known as the *variable radius form* of the *union of uniform closed balls conjecture*, is proved to be valid in Nour, Stern and Takche [15] and Nour and Takche [17], with a specific relation relating ψ to θ.

3. Main results

We begin by recalling the definition of S-convexity introduced in [11], and studied in [18].

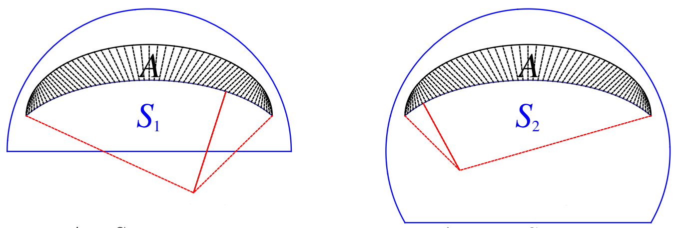
Definition 3.1. Let A be a closed and nonempty subset of $\mathbb{R}^n$ and let S be a set containing A. We say that A is *S-convex* if and only if for all $s \in S \cap A^c$ and

$a \neq a' \in \operatorname{bdry} A$ satisfying

$$\|s-a\| \leq d_{\operatorname{bdry} S}\Big(a, \frac{s-a}{\|s-a\|}\Big) \text{ and } \|s-a'\| \leq d_{\operatorname{bdry} S}\Big(a', \frac{s-a'}{\|s-a'\|}\Big), \tag{5}$$

we have $(s-a) \notin N_A^P(a)$ or $(s-a') \notin N_A^P(a')$.

Note that the two conditions of (5) are equivalent to $[a,s] \subset S$ and $[a',s] \subset S$, respectively. Hence if for $a \in \operatorname{bdry} A$, $\zeta \in N_A^P(a)$ and $t \geq 0$, the segment $[a, a+t\zeta]$ is called *normal segment* to A at a, then the S-convexity of A means that no two normal segments to A, at two distinct points, contained in S, intersect in S (see Figure 3.1).



A is S_1-convex A is not S_2-convex

Figure 3.1: S-convexity

The following elementary properties of S-convexity are provided in Nour and Takche [18].

Proposition 3.2. [18, Proposition 3.3] *Let A and S be two nonempty sets in $\mathbb{R}^n$ such that A is closed and $A \subset S$. Then we have the following:*

(i) *A is convex if and only if A is $\mathbb{R}^n$-convex.*

(ii) *If A is S-convex and $A \subset S_1 \subset S$ then A is S_1-convex.*

(iii) *If A is S-convex and $s \in S$ with $[\operatorname{proj}_A(s), s] \subset S$, then s has a unique projection on A.*

Remark 3.3. It is shown in [18, Example 3.4] that the converse of Propostion 3.2(iii) is not necessary true. More precisely, the authors provided a closed set $A \subset \mathbb{R}^2$ and a set $S \supset A$ such that any point in S has a unique projection on A, but A fails to be S-convex.

In this section, we generalize the main results of [18] to the variable radius case. More precisely, for given nonempty and closed set $A \subset \mathbb{R}^n$ and for *suitable* choices of S, we prove that the three regularity properties of A: φ-convex, θ-exterior sphere condition, and ψ-union of closed balls, *coincide* with S-convexity. This generalizes the equivalences (i), (ii) and (iii) of [18, Theorem 3.7] to the case in which the *constant* $\frac{1}{2r}$ is replaced by the functions φ, θ and ψ. We also provide new additional characterizations of these three regularity properties.

Before proceeding with our results, we introduce the following notations. For $A \subset \mathbb{R}^n$ a nonempty and closed set, $a \in \operatorname{bdry} A$, $\zeta \in N_A^P(a)$ *unit*, and $f \colon \operatorname{bdry} A \longrightarrow [0, \infty)$ a continuous function, we define:

- ► $r(a,\zeta) := \max\{r : \zeta \text{ is realized by an } r\text{-sphere}\}$.
- ► $r(a,\zeta,f) := \max\{r : 2rf(a) \leq 1 \text{ and } \zeta \text{ is realized by an } r\text{-sphere}\}$.
- ► $A^{UP} := \left\{x \notin A : \operatorname{proj}_A(x) = \{a\} \text{ and } \|x-a\| < r\left(a, \frac{x-a}{\|x-a\|}\right)\right\}$.

One can easily verify that

$$r(a,\zeta) \in (0,\infty] \text{ and } \begin{cases} r(a,\zeta,f) = r(a,\zeta) \in (0,\infty], & \text{if } f(a) = 0, \\ r(a,\zeta,f) = \min\left\{r(a,\zeta), \frac{1}{2f(a)}\right\} \in \left(0, \frac{1}{2f(a)}\right], & \text{if } f(a) \neq 0. \end{cases}$$

In the sequel, the following lemma will be used in different places.

Lemma 3.4. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set. Then A is $(A \cup A^{UP})$-convex and*

$$A^{UP} = \bigcup_{\substack{a \in \operatorname{bdry} A \\ \zeta \in N_A^P(a) \text{ unit}}} (a, a + r(a,\zeta)\zeta). \tag{6}$$

Proof. The equality (6) follows directly from the definition of A^{UP} and using (4). Now we prove that A is $(A \cup A^{UP})$-convex. If it is not true, then there exist $x \in A^{UP}$ and two normal segments to A (at two distinct points) such that the two normal segments are in $A \cup A^{UP}$, and intersect at x. By (6), there exists $a \in \operatorname{bdry} A$, $\zeta \in N_A^P(a)$ unit such that $x \in (a, a + r(a,\zeta)\zeta)$. This gives that

$$\operatorname{proj}_A(x) = \{a\}, \;\; \zeta = \zeta_a := \frac{x-a}{\|x-a\|}, \;\text{ and }\; [a,x] \subset A \cup A^{UP}.$$

Since x belongs to two normal segments to A (at two distinct points) that are in $A \cup A^{UP}$, there exist $a' \in \operatorname{bdry} A$ and $\zeta' \in N_A^P(a')$ unit such that

$$a' \neq a, \;\; x \in \{a' + t\zeta' : t > 0\}, \;\text{ and }\; [a',x] \subset A \cup A^{UP}.$$

We consider the following two cases.

Case 1: $r(a',\zeta') = \infty$, or $r(a',\zeta') < \infty$ and $x \in [a', a' + r(a',\zeta')\zeta']$.
Then $a' \in \operatorname{proj}_A(x) = \{a\}$ which contradicts $a' \neq a$.

Case 2: $r(a',\zeta') < \infty$ and $x \notin [a', a' + r(a',\zeta')\zeta']$.
Let $x' := a' + r(a',\zeta')\zeta'$. We have $x' \in A^c$ and $a' \in \operatorname{proj}_A(x')$. Moreover,

$$x' \in [a',x'] \subset [a',x] \subset A \cup A^{UP}.$$

Hence $x' \in A^{UP}$. Then by (6), there exist $a'' \in \operatorname{bdry} A$ and $\zeta'' \in N_A^P(a'')$ unit, such that

$$x' \in (a'', a'' + r(a'',\zeta'')\zeta''). \tag{7}$$

This yields that $\operatorname{proj}_A(x') = \{a''\}$, and hence $a' = a''$. Therefore,

$$\zeta' = \zeta'' = \zeta_{a'} := \frac{x'-a'}{\|x'-a'\|}, \;\text{ and }\; r(a',\zeta') = r(a'',\zeta'') = \|x'-a'\|.$$

Thus, by (7), $x' \in (a', x')$, a contradiction. □

3.1. The φ-convexity case

Let $A \subset \mathbb{R}^n$ be a nonempty and closed set, and let $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ be a continuous function. We define the two sets A_φ and A^φ as the following:

- ► $A_\varphi := \{x \in \mathbb{R}^n : \text{there exists } a \in \operatorname{proj}_A(x) \text{ satisfying } 2\varphi(a)\|x-a\| < 1\}$.
- ► $A^\varphi := A_\varphi \cup A^{UP}$.

Remark 3.5. If $\varphi = \varphi_0$ is a positive constant, then $A_\varphi = A + \frac{1}{2\varphi_0}\mathbb{B}$. Note that even in this case, the set A^{φ_0} does not coincide with A_{φ_0}. Indeed, for $A := \bar{\mathbb{B}}$ and $\varphi := \frac{1}{2}$ constant, we have $A_{\varphi_0} = \bar{\mathbb{B}} + \mathbb{B} = \mathbb{B}(0;2)$, but $A^{\varphi_0} = \mathbb{R}^n$.

Proposition 3.6. *Let $A \subset \mathbb{R}^n$ be a nonempty closed set and $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ a continuous function. Then we have:*

(i) $$A_\varphi = A \cup \bigcup_{\substack{a\in \operatorname{bdry} A,\, \zeta\in N_A^P(a) \text{ unit} \\ r(a,\zeta)=\infty \text{ or } 2\varphi(a)r(a,\zeta)\geq 1}} [a, a+r(a,\zeta,\varphi)\zeta) \cup \bigcup_{\substack{a\in \operatorname{bdry} A,\, \zeta\in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta)<1}} [a, a+r(a,\zeta,\varphi)\zeta].$$

(ii) $$A^\varphi = A \cup \bigcup_{\substack{a\in \operatorname{bdry} A,\, \zeta\in N_A^P(a) \text{ unit} \\ r(a,\zeta)=\infty \text{ or } 2\varphi(a)r(a,\zeta)\geq 1}} [a, a+r(a,\zeta)\zeta) \cup \bigcup_{\substack{a\in \operatorname{bdry} A,\, \zeta\in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta)<1}} [a, a+r(a,\zeta)\zeta].$$

Proof. (i): Let $x \in A_\varphi$. Then there exists $a \in \operatorname{proj}_A(x)$ such that $2\varphi(a)\|x-a\| < 1$. If $x = a$ then $x \in A \subset \text{RHS(i)}$ (the right hand side of (i)). Now we assume that $x \neq a$. Let $\zeta_a := \frac{x-a}{\|x-a\|}$. Since $2\varphi(a)\|x-a\| < 1$, we have that $r(a,\zeta_a,\varphi) \geq \|x-a\|$. We consider the following three cases.

Case 1: $r(a,\zeta_a) = \infty$.
Then, if $\varphi(a) = 0$, we deduce that $r(a,\zeta_a,\varphi) = r(a,\zeta_a) = \infty$. Otherwise, if $\varphi(a) > 0$, we deduce that $r(a,\zeta_a,\varphi) = \frac{1}{2\varphi(a)}$, and hence, $\|x-a\| < \frac{1}{2\varphi(a)} = r(a,\zeta_a,\varphi)$. In both cases, we have $x \in [a, a+r(a,\zeta_a,\varphi)\zeta_a)$ which yields that $x \in \text{RHS(i)}$.

Case 2: $r(a,\zeta_a) < \infty$ and $2\varphi(a)r(a,\zeta_a) \geq 1$.
Then $\varphi(a) > 0$ and $r(a,\zeta_a,\varphi) = \min\left\{r(a,\zeta_a), \frac{1}{2\varphi(a)}\right\} = \frac{1}{2\varphi(a)}$. Hence,

$$\|x-a\| < \frac{1}{2\varphi(a)} = r(a,\zeta_a,\varphi).$$

This gives that $x \in [a, a+r(a,\zeta_a,\varphi)\zeta_a)$ which yields that $x \in \text{RHS(i)}$.

Case 3: $r(a,\zeta_a) < \infty$ and $2\varphi(a)r(a,\zeta_a) < 1$.
Then $\|x-a\| \leq r(a,\zeta_a) = r(a,\zeta_a,\varphi) < \infty$. Thus, $x \in [a, a+r(a,\zeta_a,\varphi)\zeta_a]$ which yields that

$$x \in \bigcup_{\substack{a\in \operatorname{bdry} A,\, \zeta\in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta)<1}} [a, a+r(a,\zeta,\varphi)\zeta] \subset \text{RHS(i)}.$$

Therefore, the first inclusion holds. We proceed to prove the second inclusion. Let $x \in \text{RHS(i)}$. We consider the following three cases.

Case 1: $x \in A$.

Then $x \in \operatorname{proj}_A(x)$ and $2\varphi(x)\|x - x\| = 0 < 1$. This gives that $x \in A_\varphi$.

Case 2: $x \in \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ r(a,\zeta)=\infty \text{ or } 2\varphi(a)r(a,\zeta) \geq 1}} [a, a + r(a,\zeta,\varphi)\zeta)$.

Then there exist $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, such that we have $r(a,\zeta) = \infty$ or $2\varphi(a)r(a,\zeta) \geq 1$, and $x \in [a, a + r(a,\zeta,\varphi)\zeta)$. From the latter inclusion, we deduce that $\|x - a\| < r(a,\zeta,\varphi) \leq r(a,\zeta)$, and hence $\operatorname{proj}_A(x) = \{a\}$.

Case 2.1: $\varphi(a) = 0$.

Then $2\varphi(a)\|x - a\| < 1$ which yields that $x \in A_\varphi$.

Case 2.2: $\varphi(a) > 0$.

Then, since $r(a,\zeta,\varphi) = \min\left\{r(a,\zeta), \frac{1}{2\varphi(a)}\right\}$, and $r(a,\zeta) = \infty$ or $2\varphi(a)r(a,\zeta) \geq 1$, we deduce that $r(a,\zeta,\varphi) = \frac{1}{2\varphi(a)}$. Hence,

$$2\varphi(a)\|x - a\| < 2\varphi(a)r(a,\zeta,\varphi) = 1.$$

This gives that $x \in A_\varphi$.

Case 3: $x \in \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta) < 1}} [a, a + r(a,\zeta,\varphi)\zeta]$.

Then there exist $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, such that

$$r(a,\zeta) < \infty,\ 2\varphi(a)r(a,\zeta) < 1 \text{ and } x \in [a, a + r(a,\zeta,\varphi)\zeta].$$

Hence
$$2\varphi(a)\|x - a\| \leq 2\varphi(a)r(a,\zeta,\varphi) \leq 2\varphi(a)r(a,\zeta) < 1.$$

Therefore, $x \in A_\varphi$. This terminates the proof of (i).

(ii): Let $x \in A^\varphi$. Then $x \in A_\varphi$ or $x \in A^{UP}$.

Case 1: $x \in A^{UP}$.

Then by Lemma 3.4, we get that $x \in \text{RHS(ii)}$.

Case 2: $x \in A_\varphi$.

Then there exists $a \in \operatorname{proj}_A(x)$ such that $2\varphi(a)\|x - a\| < 1$. If $x \in A$, then clearly we have $x \in \text{RHS(ii)}$. If not, then $x \neq a$, and hence for $\zeta_a := \frac{x-a}{\|x-a\|}$, we have $\|x - a\| \leq r(a,\zeta_a)$.

Case 2.1: $\|x - a\| < r(a,\zeta_a)$.

Then $x \in [a, a + r(a,\zeta_a)\zeta_a) \subset \text{RHS(ii)}$.

Case 2.2: $\|x - a\| = r(a,\zeta_a)$.

Then $r(a,\zeta_a) < \infty$ and $2\varphi(a)r(a,\zeta_a) = 2\varphi(a)\|x - a\| < 1$. Hence

$$x \in \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta) < 1}} [a, a + r(a,\zeta)\zeta] \subset \text{RHS(ii)}.$$

This terminates the proof of the first inclusion. For the second one, let $x \in \text{RHS(ii)}$.

Case 1: $x \in A$.
Then $x \in A_\varphi \subset A^\varphi$.

Case 2: $x \notin A$ and $x \in \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ r(a,\zeta)=\infty \text{ or } 2\varphi(a)r(a,\zeta)\geq 1}} [a, a+r(a,\zeta)\zeta)$.

Then there exists $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, such that $x \in (a, a+r(a,\zeta)\zeta)$. From the latter inclusion and using Lemma 3.4, we obtain that $x \in A^{UP} \subset A^\varphi$.

Case 3: $x \notin A$ and $x \in \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ r(a,\zeta)<\infty \text{ and } 2\varphi(a)r(a,\zeta)<1}} [a, a+r(a,\zeta)\zeta]$.

Then there exist $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, such that

$$r(a,\zeta) < \infty, \quad 2\varphi(a)r(a,\zeta) < 1 \quad \text{and} \quad x \in [a, a+r(a,\zeta)\zeta].$$

Hence, $$a \in \operatorname{proj}_A(x) \quad \text{and} \quad 2\varphi(a)\|x-a\| \leq 2\varphi(a)r(a,\zeta) < 1.$$

Therefore, $x \in A_\varphi \subset A^\varphi$. □

The following theorem is the main result of this subsection. It contains a generalization, to the variable radius case, of [18, Theorem 3.7(i)] as we prove in the Corollary 3.8 below.

Theorem 3.7. *Let $A \subset \mathbb{R}^n$ be a nonempty closed set, and let $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ be a continuous function. Then the following assertions are equivalent:*
(i) *A is φ-convex.*
(ii) *A is A^φ-convex.*
(iii) *A is A_φ-convex and A_φ is open.*

Proof. (i)$\Longrightarrow$(ii): Assume that A is φ-convex. For $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, we have the following:

- If $\varphi(a) \neq 0$, then ζ is realized by a $\frac{1}{2\varphi(a)}$-sphere. This gives that $r(a,\zeta) \geq \frac{1}{2\varphi(a)}$, and hence $2\varphi(a)r(a,\zeta) \geq 1$.
- If $\varphi(a) = 0$ then ζ is realized by an r-sphere for all $r > 0$. This gives that $r(a,\zeta) = \infty$.

Hence, by Proposition 3.6(ii) and Lemma 3.4, we obtain that

$$A^\varphi = A \cup \bigcup_{\substack{a \in \operatorname{bdry} A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a+r(a,\zeta)\zeta) = A \cup A^{UP}.$$

Now using Lemma 3.4 again, we deduce that A is A^φ-convex.

(ii)$\Longrightarrow$(iii): We assume that A is A^φ-convex. Since $A_\varphi \subset A^\varphi$, we get that A is A_φ-convex. We proceed to prove that A_φ is open. If not, then there exists a sequence $(x_n)_{n\geq 1}$ such that

$$x_n \notin A_\varphi \text{ for all } n, \text{ and } x_n \longrightarrow x_0 \in A_\varphi.$$

Let $a_n \in \mathrm{proj}_A(x_n) \subset \mathrm{bdry}\, A$. Clearly the sequence $(a_n)_n$ is bounded and hence it admits a subsequence, we do not relabel, that converges to $a_0 \in \mathrm{bdry}\, A$. Using the closedness of the projection map $\mathrm{proj}_A(\cdot)$, we obtain that $a_0 \in \mathrm{proj}_A(x_0)$. Since $x_n \notin A_\varphi$, we obtain

$$2\varphi(a_n)\|x_n - a_n\| \geq 1.$$

Taking $n \longrightarrow \infty$ in the previous inequality, we get that

$$2\varphi(a_0)\|x_0 - a_0\| \geq 1. \tag{8}$$

Since A is A^φ-convex, $x_0 \in A_\varphi \subset A^\varphi$, and

$$\begin{aligned}
[\mathrm{proj}_A(x_0), x_0] &= \bigcup_{a \in \mathrm{proj}_A(x_0)} \left[a, a + \|x_0 - a\| \frac{x_0 - a}{\|x_0 - a\|}\right] \\
&= \{x_0\} \cup \bigcup_{a \in \mathrm{proj}_A(x_0)} \left[a, a + \|x_0 - a\| \frac{x_0 - a}{\|x_0 - a\|}\right) \qquad (9) \\
&\subset \{x_0\} \cup \bigcup_{a \in \mathrm{proj}_A(x_0)} \left[a, a + r\left(a, \frac{x_0 - a}{\|x_0 - a\|}\right) \frac{x_0 - a}{\|x_0 - a\|}\right) \subset A^\varphi,
\end{aligned}$$

we have, by Proposition 3.2(iii), that $\mathrm{proj}_A(x_0)$ is a singleton. This yields that $\mathrm{proj}_A(x_0) = \{a_0\}$. Combining this latter with the definition of A_φ, we conclude that $2\varphi(a_0)\|x_0 - a_0\| < 1$, which contradicts (8).

(iii)$\Longrightarrow$(i): We assume that A is A_φ-convex and that A_φ is open. If A is not φ-convex, then there exist $a \in \mathrm{bdry}\, A$ and $\zeta_a \in N_A^P(a)$ unit such that:

- ζ_a is not realized by a $\frac{1}{2\varphi(a)}$-sphere, if $\varphi(a) \neq 0$.
- ζ_a is not realized by an r_a-sphere for some $r_a > 0$, if $\varphi(a) = 0$.

In both cases we have $r(a, \zeta_a) < \infty$ and $2\varphi(a) r(a, \zeta_a) < 1$.

Let $x_a := a + r(a, \zeta_a)\zeta_a$. For all $x \in [a, x_a]$, we have $a \in \mathrm{proj}_A(x)$ and

$$2\varphi(a)\|x - a\| \leq 2\varphi(a)\|x_a - a\| = 2\varphi(a) r(a, \zeta_a) < 1.$$

Hence, $[a, x_a] \subset A_\varphi$. Since A_φ is open and $x_a \in A_\varphi$, there exists $\rho > 0$ such that $\bar{\mathbb{B}}(x_a, \rho) \subset A_\varphi$. We consider $x_t := x_a + t\zeta_a$, $t \in \mathbb{R}$. Clearly, for all $t \in [0, \rho]$ we have $x_t \in \bar{\mathbb{B}}(x_0; \rho) \subset A_\varphi$. Therefore,

$$[a, x_\rho] = [a, x_a] \cup [x_a, x_\rho] \subset A_\varphi. \tag{10}$$

Since $x_\rho \in A_\varphi$, there exists $a_\rho \in \mathrm{proj}_A(x_\rho)$ such that $2\varphi(a_\rho)\|x_\rho - a_\rho\| < 1$. Note that $a_\rho \neq a$ since $\|x_\rho - a\| = \rho + r(a, \zeta_a) > r(a, \zeta_a)$. Now since $a_\rho \in \mathrm{proj}_A(x_\rho)$, we have for all $x \in [a_\rho, x_\rho]$ that $a_\rho \in \mathrm{proj}_A(x)$ and

$$2\varphi(a_\rho)\|x - a_\rho\| \leq 2\varphi(a_\rho)\|x_\rho - a_\rho\| < 1.$$

This gives that $[a_\rho, x_\rho] \subset A_\varphi$. Combining this latter with (10), we obtain that the two normal segments to A, $[a_\rho, x_\rho]$ and $[a, x_\rho]$, intersect at $x_\rho \in A_\varphi$. This contradicts the A_φ-convexity of A. $\square$

Corollary 3.8. [18, Theorem 3.7(i)] *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set, and let $r > 0$. Then A is $\frac{1}{2r}$-convex if and only if A is $(A + r\mathbb{B})$-convex.*

Proof. It is sufficient to consider φ the constant function $\frac{1}{2r}$, apply the equivalence between (i) and (iii) of Theorem 3.7, and use that A_φ coincides with the *open* set $A + r\mathbb{B}$ (see Remark 3.3). $\square$

In Theorem 3.7, we cannot eliminate the assumption "A_φ is open" from (iii), as we prove in the following example.

Example 3.9. In $\mathbb{R}^2$, we consider $A := \{(x, y) : y \leq 0 \text{ or } y \geq 4\}$, and

$$\varphi(x, y) := \begin{cases} \frac{1}{2} & \text{if } x \in \mathbb{R} \text{ and } y = 0, \\ \frac{1}{6} & \text{if } x \in \mathbb{R} \text{ and } y = 4. \end{cases}$$

We have $A_\varphi = \{(x, y) : y < 1 \text{ and } y \geq 2\}$ is not open, A is A_φ-convex, but A is not φ-convex, see Figure 3.2.

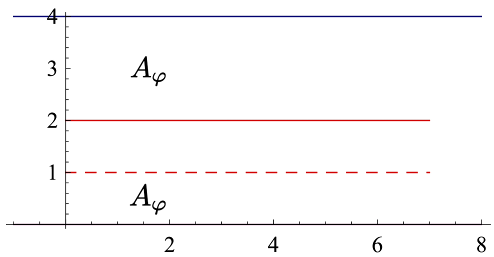


Figure 3.2: Example 3.9

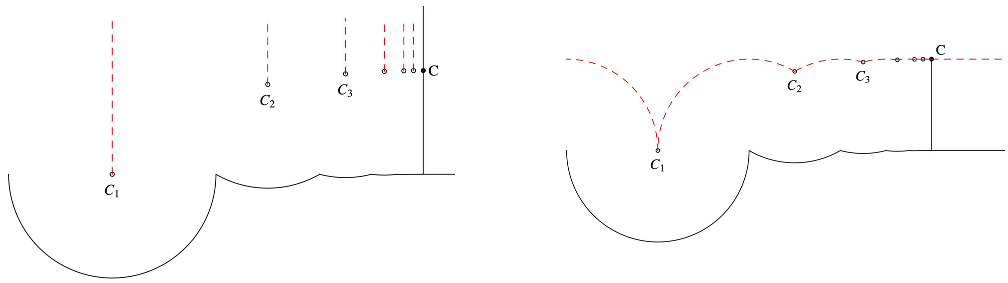
Remark 3.10. From (9), we deduce that if A is A^φ-convex, then each point in A^φ has a unique projection on A. Note that this is not necessarily true for the A_φ-convexity as one can easily see in Example 3.9.

In the following example, we show that under the φ-convexity of A, the set A^φ is not necessarily open.

Example 3.11. In Figure 3.3, A is the closed region below the black curve. The portion of the black curve to the left of the blue line, consists of arcs of circles of radius 1 centered at the points C_n. The portion to the right is a horizontal line. In Figure 3.3a and for $\varphi := \frac{1}{2}$ a constant function, the set A^φ is the set of all points in $\mathbb{R}^2$ except the centers C_n and the red semi-lines above them (the blue semi-line is in A_φ). We have A is φ-convex (and then A^φ-convex), but A^φ is not open. Indeed, the sequence $(C_n)_n$ is outside A^φ with $C_n \longrightarrow C \in A^\varphi$. Note that in this example, A_φ is the region below the red curve in Figure 3.3b which is open.

3.2. The θ-exterior sphere condition case

Let $A \subset \mathbb{R}^n$ be a nonempty and closed set, and let $\theta\colon \operatorname{bdry} A \longrightarrow [0, \infty)$ be a continuous function. We denote by $\operatorname{bdry}_\theta A$ the set of boundary point $a \in \operatorname{bdry} A$ such that $\theta(a) = 0$ and there exists $\zeta \in N_A^P(a)$ unit with $r(a, \zeta) = \infty$, or $\theta(a) > 0$ and there exists $\zeta \in N_A^P(a)$ unit and realized by a $\frac{1}{2\theta(a)}$-sphere.



(a) A^{φ} is not open (b) A_{φ} is open

Figure 3.3: Example 3.11

Thus, we have

$$\text{bdry}_\theta A := \{a \in \text{bdry}\, A : \theta(a) = 0 \text{ and } \exists \zeta \in N_A^P(a) \text{ unit with } r(a,\zeta) = \infty\} \cup$$
$$\{a \in \text{bdry}\, A : \theta(a) > 0 \text{ and } \exists \zeta \in N_A^P(a) \text{ unit with } 2\theta(a) r(a,\zeta) \geq 1\}.$$

One can easily verify that $\text{bdry}_\theta A$ can be written as follows:

$$\text{bdry}_\theta A = \{a \in \text{bdry}\, A : \theta(a) = 0 \text{ and } \exists \zeta \in N_A^P(a) \text{ unit with } r(a,\zeta,\theta) = \infty\} \cup$$
$$\left\{a \in \text{bdry}\, A : \theta(a) > 0 \text{ and } \exists \zeta \in N_A^P(a) \text{ unit with } r(a,\zeta,\theta) = \frac{1}{2\theta(a)}\right\}.$$

The following lemma will play a crucial role in the proof of the main result of this subsection.

Lemma 3.12. $\text{bdry}_\theta A$ *is closed.*

Proof. Let $(a_n)_n$ be a sequence such that $a_n \in \text{bdry}_\theta A$ and $a_n \longrightarrow a_0 \in \text{bdry}\, A$.

Case 1: $(a_n)_n$ has a subsequence, we do not relabel, such that $\theta(a_n) = 0$ for all n. For each n, there exists $\zeta_n \in N_A^P(a_n)$ unit such that $r(a_n,\zeta_n) = \infty$. By the proximal normal inequality, we have

$$\langle \zeta_n, a - a_n \rangle \leq 0, \quad \forall a \in A. \tag{11}$$

Since ζ_n are unit vectors, we can assume that $\zeta_n \longrightarrow \zeta_0$, where ζ_0 is unit. Taking $n \longrightarrow \infty$ in (11) and using the continuity of θ, we get that $\theta(a_0) = 0$, and

$$\langle \zeta_0, a - a_0 \rangle \leq 0, \quad \forall a \in A.$$

This gives that $\zeta_0 \in N_A^P(a_0)$ and $r(a_0,\zeta_0) = \infty$. Therefore, $a_0 \in \text{bdry}_\theta A$.

Case 2: There exists N such that $\theta(a_n) > 0$ for all $n \geq N$.
For each $n \geq N$, we have the existence of $\zeta_n \in N_A^P(a_n)$ such that

$$2\theta(a_n) r(a_n,\zeta_n) \geq 1.$$

Hence, by the proximal normal inequality, we have for all $a \in A$, that

$$\langle \zeta_n, a - a_n \rangle \leq \frac{1}{2r(a_n,\zeta_n)} \|a - a_n\|^2 \leq \theta(a_n) \|a - a_n\|^2.$$

Taking $n \longrightarrow \infty$ and using the continuity of θ, we obtain that

$$\langle \zeta_0, a - a_0 \rangle \leq \theta(a_0)\|a - a_0\|^2, \quad \forall a \in A. \tag{12}$$

Thus, $\zeta_0 \in N_A^P(a_0)$. Now, if $\theta(a_0) = 0$, then (12) yields that $r(a_0, \zeta_0) = \infty$, and hence, $a_0 \in \text{bdry}_\theta A$. If $\theta(a_0) > 0$, then (12) yields that $r(a_0, \zeta_0) \geq \frac{1}{2\theta(a_0)}$, and hence, $a_0 \in \text{bdry}_\theta A$. Therefore, $a_0 \in \text{bdry}_\theta A$. $\square$

We proceed to define the two sets $\bar{A}_\theta$ and $\bar{A}^\theta$ as follows:

- $\bar{A}_\theta := A \cup A_{UP(\theta)} \cup \{x \notin A : \text{proj}_A(x) \not\subset \text{bdry}_\theta A\}$, where
$$A_{UP(\theta)} := \left\{x \notin A : \text{proj}_A(x) = \{a\} \text{ and } \|x - a\| < r\Big(a, \frac{x-a}{\|x-a\|}, \theta\Big)\right\}.$$
- $\bar{A}^\theta := A \cup A^{UP} \cup \{x \notin A : \text{proj}_A(x) \not\subset \text{bdry}_\theta A\}$.

Since $r(a, \zeta, \theta) \leq r(a, \zeta)$ for all $a \in \text{bdry}\, A$ and $\zeta \in N_A^P(a)$ unit, we have $\bar{A}_\theta \subset \bar{A}^\theta$. In fact, we have $\bar{A}^\theta = \bar{A}_\theta \cup A^{UP}$. Furthermore, we have the following.

Proposition 3.13. *Let $A \subset \mathbb{R}^n$ be a nonempty closed set and $\theta\colon \text{bdry}\, A \longrightarrow [0, \infty)$ a continuous function. Then we have:*

(i) $$\bar{A}_\theta = A \cup \bigcup_{\substack{a \in \text{bdry}_\theta A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a, \zeta, \theta)\zeta) \cup \bigcup_{\substack{a \in (\text{bdry}\, A) \cap (\text{bdry}_\theta A)^c \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a, \zeta, \theta)\zeta].$$

(ii) $$\bar{A}^\theta = A \cup \bigcup_{\substack{a \in \text{bdry}_\theta A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a, \zeta)\zeta) \cup \bigcup_{\substack{a \in (\text{bdry}\, A) \cap (\text{bdry}_\theta A)^c \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a, \zeta)\zeta].$$

Proof. (i): Let $x \in \bar{A}_\theta$. If $x \in A$ then $x \in \text{RHS(i)}$. We consider the following two cases.

Case 1: $x \in \{x \notin A : \text{proj}_A(x) \not\subset \text{bdry}_\theta A\}$.
Let $a \in \text{proj}_A(x)$ such that $a \notin \text{bdry}_\theta A$. Then, for $\zeta_a := \frac{x-a}{\|x-a\|}$, we have

$$[\theta(a) = 0 \text{ and } r(a, \zeta_a) < \infty] \text{ or } [\theta(a) > 0 \text{ and } 2\theta(a) r(a, \zeta_a) < 1].$$

Hence, $r(a, \zeta_a, \theta) < \infty$ and $2\theta(a) r(a, \zeta_a, \theta) < 1$. This gives that $r(a, \zeta_a, \theta) = r(a, \zeta_a)$. Then $\|x - a\| \leq r(a, \zeta_a) = r(a, \zeta_a, \theta)$. Therefore,

$$x \in \bigcup_{\substack{a \in (\text{bdry}\, A) \cap (\text{bdry}_\theta A)^c \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a, \zeta, \theta)\zeta] \subset \text{RHS(i)}.$$

Case 2: $x \in A_{UP(\theta)}$.
Then $\text{proj}_A(x) = \{a\}$ and $\|x - a\| < r(a, \zeta_a, \theta)$, where $\zeta_a := \frac{x-a}{\|x-a\|}$. This yields that $x \in [a, a + r(a, \zeta_a, \theta)\zeta_a) \subset \text{RHS(i)}$. Therefore, the first inclusion holds. We proceed to prove the second inclusion. Let $x \in \text{RHS(i)}$. If $x \in A$ then $x \in \bar{A}_\theta$. We assume that $a \notin A$ and we consider the following two cases.

Case 1: $x \in (a, a + r(a, \zeta, \theta)\zeta)$ for some $a \in \text{bdry}\, A$ and $\zeta \in N_A^P(a)$ unit.

Then $\text{proj}_a(x) = \{a\}$ and $\|x - a\| < r(a, \zeta, \theta) = r\Big(a, \frac{x-a}{\|x-a\|}, \theta\Big)$. This gives that $x \in A_{UP(\theta)} \subset \bar{A}_\theta$.

Case 2: $x = a + r(a,\zeta,\theta)\zeta$ for some $a \in (\text{bdry}\, A) \cap (\text{bdry}_\theta A)^c$ and $\zeta \in N_A^P(a)$ unit. Then $a \in \text{proj}_A(x)$, which gives that $\text{proj}_A(x) \not\subset A$. Thus, $x \in \bar{A}_\theta$.

(ii): Follows using arguments similar to those employed in the proof of (i). $\square$

Remark 3.14. In this remark, we prove that $\bar{A}_\theta \subset A_\theta$ and $\bar{A}^\theta \subset A^\theta$. For the first inclusion, let $x \in \bar{A}_\theta$. We have the following:

- If $x \in A$ then $x \in A_\theta$.
- If $x \in [a, a + r(a,\zeta,\theta)\zeta)$ with $a \in \text{bdry}_\theta A$ and $\zeta \in N_A^P(a)$ unit, then using Proposition 3.6(i) and the fact that $[a, a+\zeta(a,\zeta,\theta)\zeta) \subset [a, a+\zeta(a,\zeta,\theta)\zeta]$, we obtain that $x \in A_\theta$.
- If $x \in [a, a + r(a,\zeta,\theta)\zeta]$ with $a \in (\text{bdry}\, A) \cap (\text{bdry}_\theta A)^c$ and $\zeta \in N_A^P(a)$ unit, then

$$[\theta(a) = 0 \text{ and } r(a,\zeta) < \infty] \text{ or } [\theta(a) > 0 \text{ and } 2\theta(a)r(a,\zeta) < 1].$$

  In both cases, we have $r(a,\zeta) < \infty$ and $2\theta(a)r(a,\zeta) < 1$, which gives, using Proposition 3.6(i), that $x \in A_\theta$.

Therefore, $\bar{A}_\theta \subset A_\theta$. For the second inclusion, it is sufficient to remark that

$$\bar{A}^\theta = \bar{A}_\theta \cup A^{UP} \subset A^\theta \cup A^{UP} = A^\theta.$$

The following theorem is the main result of this subsection. It contains a generalization of the equivalence statement [18, Theorem 3.7(ii)] to the variable radius case, see Corollary 3.16 below. Before presenting our main result, we introduce the condition $(\text{UP})_\theta$ as follows:

$\textbf{(UP)}_\theta$ $\quad \forall x \in \bar{A}_\theta \ \left[x \in \text{bdry}\, \bar{A}_\theta \implies \text{proj}_A(x) \text{ is a singleton}\right].$

Theorem 3.15. *Let $A \subset \mathbb{R}^n$ be a nonempty closed set, and let $\theta\colon \text{bdry}\, A \longrightarrow [0,\infty)$ be a continuous function. Then the following assertions are equivalent:*

(i) *A satisfies the θ-exterior sphere condition.*

(ii) *A is $\bar{A}^\theta$-convex.*

(iii) *A is $\bar{A}_\theta$-convex and $(\text{UP})_\theta$ is satisfied.*

Proof. (i)$\Longrightarrow$(ii): Assume that A satisfies the θ-exterior sphere condition. We claim that $\text{bdry}\, A = \text{bdry}_\theta A$. Indeed, for $a \in \text{bdry}\, A$, we have

- If $\theta(a) = 0$, then there exists $\zeta_a \in N_A^P(a)$ unit and realized by an r-sphere for all $r > 0$. This gives that $r(a,\zeta_a) = \infty$, and hence $a \in \text{bdry}_\theta A$.
- If $\theta(a) \neq 0$, then there exists $\zeta_a \in N_A^P(a)$ unit and realized by a $\frac{1}{2\theta(a)}$-sphere. This gives that $2\theta(a)r(a,\zeta) \geq 1$, and hence $a \in \text{bdry}_\theta A$.

Therefore $\text{bdry}\, A = \text{bdry}_\theta A$. This yields, using Proposition 3.13(ii) and Lemma 3.4, that

$$\bar{A}^\theta = A \cup \bigcup_{\substack{a \in \text{bdry}\, A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a,\zeta)\zeta) = A \cup A^{UP}.$$

Now Lemma 3.4 gives that A is $\bar{A}^\theta$-convex.

(ii)$\Longrightarrow$(iii): Since $\bar{A}_\theta \subset \bar{A}^\theta$, we get that A is $\bar{A}_\theta$-convex. Now assume that $(\mathrm{UP})_\theta$ is not satisfied. Then there exists $x_0 \in \bar{A}_\theta \cap \operatorname{bdry} \bar{A}_\theta$ such that $\operatorname{proj}_A(x_0)$ is not a singleton. Let $\{a_1, a_2\} \subset \operatorname{proj}_A(x_0)$ such that $a_0 \neq a_1$. Since $x_0 \in \bar{A}_\theta \subset \bar{A}^\theta$ and $\|x_0 - a_i\| \leq r(a_i, \zeta_i)$ where $\zeta_i := \frac{x_0 - a_i}{\|x_0 - a_i\|}$, $i = 1, 2$, we get by Proposition 3.13(ii) that

$$[a_i, x_0] = \{x_0\} \cup [a_i, a_i + \|x_0 - a_i\|\zeta_i) \subset \{x_0\} \cup [a_i, a_i + r(a_i, \zeta_i)\zeta_i) \subset \bar{A}^\theta, \quad i = 1, 2.$$

Therefore, the two normal segments to A, $[a_1, x_0]$ and $[a_2, x_0]$ are inside $\bar{A}^\theta$ and intersects at x_0. This contradicts the $\bar{A}^\theta$-convexity of A.

(iii)$\Longrightarrow$(i): Assume that A is $\bar{A}_\theta$-convex and $(\mathrm{UP})_\theta$ holds. If A does not satisfy the θ-exterior sphere condition, then there exists $a \in \operatorname{bdry} A$ such that for all $\zeta \in N_A^P(a)$ unit, we have:

- If $\theta(a) = 0$, then one can find $r > 0$ such that $\mathbb{B}(a + r\zeta; r) \cap A \neq \emptyset$. This gives that $r(a, \zeta) < \infty$, and hence $a \notin \operatorname{bdry}_\theta A$. Thus, $[a, a + r(a, \zeta, \theta)\zeta] \subset \bar{A}_\theta$.
- If $\theta(a) \neq 0$, then $\mathbb{B}\left(a + \frac{1}{2\theta(a)}\zeta; \frac{1}{2\theta(a)}\right) \cap A \neq \emptyset$. This gives that $r(a, \zeta) < \frac{1}{2\theta(a)}$, and hence $2\theta(a) r(a, \zeta) < 1$. Then $a \notin \operatorname{bdry}_\theta A$, and thus, $[a, a + r(a, \zeta, \theta)\zeta] \subset \bar{A}_\theta$.

Therefore, there exists $a \in \operatorname{bdry} A \cap (\operatorname{bdry}_\theta A)^c$ such that for all $\zeta \in N_A^P(a)$ unit, we have $r(a, \zeta) < \infty$, $2\theta(a) r(a, \zeta) < 1$ and $[a, a + r(a, \zeta, \theta)\zeta] \subset \bar{A}_\theta$. Since the set of points $x \in \operatorname{bdry} A$ for which $N_A^P(x) \neq \{0\}$ is dense in $\operatorname{bdry} A$, see [6, Corollary 1.6.2], and $\operatorname{bdry}_\theta A$ is closed by Lemma 3.12, we can assume that $N_A^P(a) \neq \{0\}$. We fix $\zeta_a \in N_A^P(a)$ unit, and we denote by $x_a := a + r(a, \zeta_a, \theta)\zeta_a$. We have

$$r(a, \zeta) < \infty, \ \ 2\theta(a) r(a, \zeta) < 1, \ \ [a, x_a] \subset \bar{A}_\theta, \ \text{ and } \ a \in \operatorname{proj}_A(x_a).$$

Case 1: There exists $\varepsilon > 0$ such that $[x_a, x_a + \varepsilon\zeta_a] \subset \bar{A}_\theta$.
We consider $x_\varepsilon := x_a + \varepsilon\zeta_a$, where ε is taken small enough so that $x_\varepsilon \notin A$. We have $[a, x_\varepsilon] = [a, x_a] \cup [x_a, x_\varepsilon] \subset \bar{A}_\theta$. Since $x_\varepsilon \in \bar{A}_\theta$ and using Proposition 3.13(i), there exist $a_\varepsilon \in \operatorname{bdry} A$ and $\zeta_{a_\varepsilon} \in N_A^P(a_\varepsilon)$ unit, such that

$$[a_\varepsilon, x_\varepsilon) \subset [a_\varepsilon, a_\varepsilon + r(a_\varepsilon, \zeta_{a_\varepsilon}, \theta)\zeta_{a_\varepsilon}) \subset \bar{A}_\theta.$$

Since $\|x_\varepsilon - a\| = \varepsilon + r(a, \zeta_a, \theta)$ and $\|x_\varepsilon - a_\varepsilon\| \leq r(a_\varepsilon, \zeta_{a_\varepsilon}, \theta)$, we deduce that $a_\varepsilon \neq a$. Therefore, the two normal segments to A, $[a, x_\varepsilon]$ and $[a_\varepsilon, x_\varepsilon]$ are inside $\bar{A}_\theta$ and intersects at x_ε. This contradicts the $\bar{A}_\theta$-convexity of A.

Case 2: There exists x_n of the form $x_a + t\zeta_a$, $t > 0$: $x_n \notin \bar{A}_\theta$ and $x_n \xrightarrow[n \longrightarrow \infty]{} x_a$.

Then $x_a \in \bar{A}_\theta \cap \operatorname{bdry} \bar{A}_\theta$ which yields by $(\mathrm{UP})_\theta$ that $\operatorname{proj}_A(x_a) = \{a\}$. Now, let $a_n \in \operatorname{proj}_A(x_n)$. Since $x_n \longrightarrow x_a$, we have that $(a_n)_n$ is bounded. Hence, it admits a subsequent, we do not relabel, that converges to a point a_0. Having that the projection map $\operatorname{proj}_A(\cdot)$ is closed and that $\operatorname{proj}_A(x_a) = \{a\}$, we deduce that $a_0 = a$, and then $a_n \longrightarrow a$. Add to this that $\operatorname{bdry}_\theta A$ is closed, we conclude that, for n sufficiently large, $a_n \in (\operatorname{bdry}_\theta A)^c$. This gives that, for n sufficiently large, $\operatorname{proj}_A(x_n) \not\subset \operatorname{bdry}_\theta A$. Hence, using that $x_n \notin A$ (since $x_n \notin \bar{A}_\theta$ and $A \subset \bar{A}_\theta$), the definition of $\bar{A}_\theta$ yields that, for n sufficiently large, $x_n \in \bar{A}_\theta$, a contradiction. $\square$

Corollary 3.16. [18, Theorem 3.7(ii)] *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set, and let $r > 0$. Then A satisfies the $\frac{1}{2r}$-exterior sphere condition if and only if A is $\bar{A}_{\frac{1}{2r}}$-convex.*

Proof. By Theorem 3.15, applied for $\theta = \frac{1}{2r}$ constant, it is sufficient to prove that if A is $\bar{A}_{\frac{1}{2r}}$-convex, then $(\mathrm{UP})_{\frac{1}{2r}}$ is satisfied. If not, then there exists $x_0 \in \bar{A}_{\frac{1}{2r}} \cap \operatorname{bdry} \bar{A}_{\frac{1}{2r}}$ such that $\operatorname{proj}_A(x_0)$ is not a singleton. Let $\{a_1, a_2\} \subset \operatorname{proj}_A(x_0)$ such that $a_1 \neq a_2$. Since $x_0 \in \bar{A}_{\frac{1}{2r}} \subset A_{\frac{1}{2r}}$, we have that $\|x_0 - a_1\| = \|x_0 - a_2\| = d_A(x_0) < r$. Hence, for $\zeta_1 := \frac{x_0 - a_1}{\|x_0 - a_1\|}$ and $\zeta_2 := \frac{x_0 - a_2}{\|x_0 - a_2\|}$, we obtain that

$$\|x_0 - a_i\| \leq r\Big(a, \zeta_0, \frac{1}{2r}\Big) \leq r, \quad i = 1, 2.$$

This gives, using Proposition 3.13(i), that

$$\begin{aligned} [a_i, x_0] &= \{x_0\} \cup [a_i, a_i + \|x_0 - a_i\| \zeta_i) \\ &\subset \{x_0\} \cup \Big[a_i, a_i + r\Big(a_i, \zeta_i, \frac{1}{2r}\Big)\zeta_i\Big) \subset \bar{A}_{\frac{1}{2r}}, \quad i = 1, 2. \end{aligned}$$

Therefore, the two normal segments to A, $[a_1, x_0]$ and $[a_2, x_0]$ are inside $\bar{A}_{\frac{1}{2r}}$ and intersect at x_0. This contradicts the $\bar{A}_{\frac{1}{2r}}$-convexity of A. $\square$

Remark 3.17. As in Remark 3.10, one can easily verify that if A is $\bar{A}^\theta$-convex, then each point in $\bar{A}^\theta$ has a unique projection on A. This does not necessarily hold for $\bar{A}_\theta$, and this explains the addition of the condition $(\mathrm{UP})_\theta$ in Theorem 3.15(iii). Indeed, if we take A and the function $\theta = \varphi$ as in Example 3.9, then we have that $\bar{A}_\theta = \{(x, y) : y < 1 \text{ and } y \geq 2\}$, $(\mathrm{UP})_\theta$ is not satisfied, A is $\bar{A}_\theta$-convex, but A does not satisfy the θ-exterior sphere condition. Note that $(\mathrm{UP})_\theta$ is not satisfied since the points $(x, 2) \in \bar{A}_\theta \cap \operatorname{bdry} \bar{A}_\theta$, $x \in \mathbb{R}$, have two projection points on A.

On the other hand, one can easily verify that if $\bar{A}_\theta$ is open, then the condition $(\mathrm{UP})_\theta$ is satisfied. Indeed, in that case we have $\bar{A}_\theta \cap \operatorname{bdry} \bar{A}_\theta = \emptyset$. The following example proves that the condition $(\mathrm{UP})_\theta$ cannot be replaced by $\bar{A}_\theta$ being open. In $\mathbb{R}^3$, we consider the following:

- ► The points $C_n := \left(\frac{\sqrt{3}}{2}, \frac{3}{2^n}, \sqrt{\frac{1}{4} - \frac{1}{4^n}}\right)$, $n \in \mathbb{N}$.
- ► For each n, S_n is the sphere of center C_n and radius 1.
- ► A is the closed set lying below the spheres S_n, $n \in \mathbb{N}$, and between the planes $x = 0$, $x = \frac{\sqrt{3}}{2}$, $y = 0$ and $y = \frac{3}{2}$, see Figure 3.4.

For any boundary point a of A, there exists $\zeta_a \in N_A^P(a)$ unit realized by a 1-sphere. Indeed, it is sufficient to take ζ_a as the following:

- ► $\zeta_a := i$ for a in the plane $x = \frac{\sqrt{3}}{2}$.
- ► $\zeta_a := -i$ for a in the plane $x = 0$.
- ► $\zeta_a := -j$ for a in the plane $y = 0$.
- ► $\zeta_a := j$ for a in the plane $y = \frac{3}{2}$.
- ► $\zeta_a := \frac{C_n - a}{\|C_n - a\|}$ for $a \in S_n$.

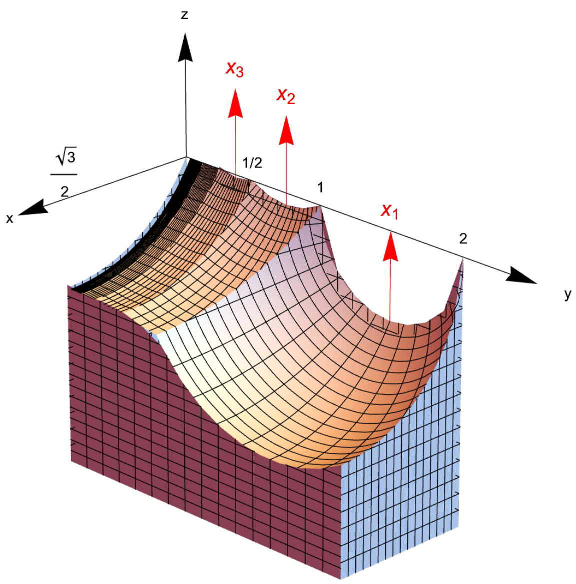


Figure 3.4: The set A of Remark 3.17

Hence, A satisfies the $\frac{1}{2}$-exterior sphere condition. By Theorem 3.15, we deduce that A is $\bar{A}^{\frac{1}{2}}$-convex, and hence, $\bar{A}_{\frac{1}{2}}$-convex since $\bar{A}_{\frac{1}{2}} \subset \bar{A}^{\frac{1}{2}}$. We claim that $\bar{A}_{\frac{1}{2}}$ is not open. Indeed, let x_n be the center of the circle intersection between S_n and the yz-plane. These circles have a radius of $\frac{1}{2}$. We denote by a_n the lowest point in these circles. The unit vector $\frac{x_n - a_n}{\|x_n - a_n\|}$ is normal to A at a_n, and is realized by a $\frac{1}{2}$-ball. This yields that $r\big(a_n, \zeta_{a_n}, \frac{1}{2}\big) = \frac{1}{2}$, and hence, $x_n \notin \bar{A}_{\frac{1}{2}}$. The sequence $x_n \longrightarrow x_0 := \big(0, 0, \frac{1}{2}\big)$. On the other hand, since the vector k is normal to A at the origin and is realized by an r-ball, for all $r \in [0, +\infty)$, we have $r\big(0, k, \frac{1}{2}\big) = 1$. Hence, $x_0 \in [(0,0,0),(0,0,1)) \subset \bar{A}_{\frac{1}{2}}$. Therefore, $\bar{A}_{\frac{1}{2}}$ is not open.

We terminate this subsection by the following corollary in which we provide two new sufficient conditions for the equivalence between φ-convexity and the θ-exterior sphere condition.

Corollary 3.18. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set satisfying the θ-exterior sphere condition. We have the following:*

(i) *If there exists $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that φ is continuous, $\varphi \geq \theta$, and $\bar{A}^{\varphi} = A^{\varphi}$, then A is φ-convex.*

(ii) *If there exists $\varphi\colon \operatorname{bdry} A \longrightarrow [0,\infty)$ such that φ is continuous, $\varphi \geq \theta$, and $\bar{A}_{\varphi} = A_{\varphi}$ is open, then A is φ-convex.*

Proof. (i): Having that A satisfies the θ-exterior sphere condition, we conclude from Theorem 3.15 and its proof that A is $\bar{A}^{\theta}$-convex, $\operatorname{bdry} A = \operatorname{bdry}_{\theta} A$, and

$$\bar{A}^{\theta} = A \cup \bigcup_{\substack{a \in \operatorname{bdry} A \\ \zeta \in N_A^P(a)\,\text{unit}}} [a, a + r(a,\zeta)\zeta). \tag{13}$$

Since $\varphi \geq \theta$, we claim that $\operatorname{bdry}_{\theta} A \subset \operatorname{bdry}_{\varphi} A$. Indeed, let $a \in \operatorname{bdry}_{\theta} A$.

Case 1: $\varphi(a) = 0$.
Then $\theta(a) = 0$. This gives, since $a \in \mathrm{bdry}_\theta A$, the existence of $\zeta \in N_A^P(a)$ such that $r(a,\zeta) = \infty$. Thus, $a \in \mathrm{bdry}_\varphi A$.

Case 2: $\varphi(a) > 0$.
If $\theta(a) = 0$, then $a \in \mathrm{bdry}_\theta A$ gives the existence of $\zeta \in N_A^P(a)$ such that $r(a,\zeta) = \infty$. Hence $\varphi(a)r(a,\zeta) = \infty > 1$. Thus, $a \in \mathrm{bdry}_\varphi A$. Now we assume that $\theta(a) > 0$. Since $a \in \mathrm{bdry}_\theta A$, there exists $\zeta \in N_A^P(a)$ such that $2\theta(a)r(a,\zeta) \geq 1$. Then,

$$2\varphi(a)r(a,\zeta) \geq 2\theta(a)r(a,\zeta) \geq 1.$$

Thus, $a \in \mathrm{bdry}_\varphi A$. The proof of the claim is terminated. Therefore,

$$\mathrm{bdry}\, A = \mathrm{bdry}_\theta A \subset \mathrm{bdry}_\varphi A.$$

This gives that $\mathrm{bdry}_\varphi A = \mathrm{bdry}\, A$. Hence, using (13), Propossition 3.13(ii), and the equality $\bar{A}^\varphi = A^\varphi$, we get that

$$A^\varphi = \bar{A}^\varphi = A \cup \bigcup_{\substack{a \in \mathrm{bdry}\, A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a,\zeta)\zeta) = \bar{A}^\theta.$$

Thus, A is A^φ-convex, and hence by Theorem 3.7, A is φ-convex.

(ii): Follows using arguments similar to those employed in the proof of (i). $\square$

Remark 3.19. As we mentioned in Section 2, Nour and Takche proved in [14, Theorem 7] that the equivalence between φ-convexity and the θ-exterior sphere condition holds if A is epi-Lipschitz. Even if the sufficient conditions given in Corollary 3.18 are weaker than the epi-Lipschitzness of A, what is useful in [14, Theorem 7] is the construction of φ from θ and A, see [14, Remark 12].[1]

3.3. The ψ-union of closed balls property case

Let $A \subset \mathbb{R}^n$ be a nonempty closed set with $A = \mathrm{cl}\,(\mathrm{int}\, A)$, and let $\psi \colon A' \longrightarrow [0,\infty)$ be an upper semicontinuous function which is continuous on $\mathrm{bdry}\, A' = \mathrm{bdry}\, A$. We define the set $A'(\psi)$ as follows:

$$x \in A'(\psi) \iff \exists y_x \in A' : \begin{cases} x \in \bar{\mathbb{B}}\left(y_x; \frac{1}{2\psi(x)}\right) \subset A', & \text{if } \psi(x) > 0, \\ x \in \bar{\mathbb{B}}(x + t(y_x - x); t) \subset A',\ \forall t > 0, & \text{if } \psi(x) = 0. \end{cases}$$

$$\iff \exists \zeta_x \in \mathbb{R}^n \text{ unit satisfying}$$

$$\begin{cases} \exists t \in \left[0, \frac{1}{2\psi(x)}\right] : \langle \zeta_x, z - x + t\zeta_x \rangle \leq \psi(x)\|z - x + t\zeta_x\|^2,\ \forall z \in A, & \text{if } \psi(x) > 0, \\ \langle \zeta_x, z - x \rangle \leq 0,\ \forall z \in A, & \text{if } \psi(x) = 0. \end{cases}$$

Note that the last equivalence is a direct consequence of Proposition 2.4 and Remark 2.5. The following lemma will be essential for reaching the main result of this subsection under minimal assumptions.

[1]In fact, one can easily prove that if A is epi-Lipschitz and satisfies the θ-exterior sphere condition, then the sufficient conditions of Corollary 3.18 are valid.

Lemma 3.20. *$A'(\psi)$ is closed.*

Proof. Let $(a_n)_n$ be a sequence such that $a_n \in A'(\psi)$ for all n, and $a_n \longrightarrow a_0 \in A'$. Denote by $\ell := \limsup_{n\longrightarrow\infty} \psi(a_n)$. By the upper semicontinuity of ψ, we have $\psi(a_0) \geq \ell$.

Case 1: $(a_n)_n$ has a subsequence, we do not relabel, such that $\psi(a_n) = 0$ for all n. Then there exist unit vectors ζ_n such that

$$\langle \zeta_n, a - a_n \rangle \leq 0, \quad \forall a \in A. \tag{14}$$

Since the sequence $(\zeta_n)_n$ is bounded, we can assume that $\zeta_n \longrightarrow \zeta_0$ unit. Taking $n \longrightarrow \infty$ in (14), we obtain that

$$\langle \zeta_0, a - a_0 \rangle \leq 0, \quad \forall a \in A.$$

If $\psi(a_0) = 0$, then clearly we have $a_0 \in A'(\psi)$. Now we assume that $\psi(a_0) > 0$. Hence for $t_0 := 0 \in [0, \frac{1}{2\psi(a_0)}]$, (14) yields that

$$\langle \zeta_0, a - a_0 + t_0\zeta_0 \rangle \leq \psi(a_0)\|a - a_0 + t_0\zeta_0\|^2, \quad \forall a \in A.$$

This gives that $a_0 \in A'(\psi)$.

Case 2: There exists N such that $\psi(a_n) > 0$ for all $n \geq N$.
Then for each $n \geq N$, we have

$$\exists y_n \in A' \,:\, a_n \in \bar{\mathbb{B}}\Big(y_n; \frac{1}{2\psi(a_n)}\Big) \subset A'. \tag{15}$$

This is equivalent to the existence, for each $n \geq N$, of ζ_n unit and $t_n \in \big[0, \frac{1}{2\psi(a_n)}\big]$ such that

$$\langle \zeta_n, z - a_n + t_n\zeta_n \rangle \leq \psi(a_n)\|z - a_n + t_n\zeta_n\|^2, \;\; \forall z \in A. \tag{16}$$

Case 2.1: $\ell = \limsup_{n\longrightarrow\infty} \psi(a_n) > 0$.

Then $\psi(a_0) \geq \ell > 0$, and the sequence $(a_n)_n$ admits a subsequence, we do not relabel, such that $\psi(a_n) \longrightarrow \ell$. Since $t_n \in \big[0, \frac{1}{2\psi(a_n)}\big]$, we deduce that $(t_n)_n$ is bounded, and hence it has a subsequence, we do not relabel, that converges to a $t_0 \in \big[0, \frac{1}{2\psi(a_0)}\big]$. Moreover, having that ζ_n are unit vectors, we can assume that $\zeta_n \longrightarrow \zeta_0$ unit. Now taking $n \longrightarrow \infty$ in (16), we obtain that for all $z \in A$,

$$\langle \zeta_0, z - a_0 + t_0\zeta_0 \rangle \leq \ell\|z - a_0 + t_0\zeta_0\|^2 \leq \psi(a_0)\|z - a_0 + t_0\zeta_0\|^2.$$

Therefore, $a_0 \in A'(\psi)$.

Case 2.2: $\ell = \limsup_{n\longrightarrow\infty} \psi(a_n) = 0$.

We claim that $(\|a_n - y_n\|)_n$ is an unbounded sequence. Indeed, if not, then $(y_n)_n$ is bounded, and hence, we can assume that $y_n \longrightarrow y_0 \in A'$. This gives, since $\psi(a_n) \longrightarrow 0$ and $\bar{\mathbb{B}}\big(y_n; \frac{1}{2\psi(a_n)}\big) \subset A'$, that $\mathbb{R}^n \subset A'$. Thus, $A = \emptyset$, a contradiction. Hence, we can assume that

$$\|a_n - y_n\| \neq 0 \text{ for all } n, \text{ and } \|a_n - y_n\| \longrightarrow \infty.$$

By (15), we have for $\zeta_n := \frac{y_n - a_n}{\|\bar{y}_n - a_n\|}$, that

$$\bar{\mathbb{B}}\big(a_n + \|y_n - a_n\|\zeta_n; \|y_n - a_n\|\big) \subset \bar{\mathbb{B}}\Big(y_n; \frac{1}{2\psi(a_n)}\Big) \subset A'.$$

Hence, by the equivalence between (2) and (3), we deduce that

$$\langle \zeta_n, a - a_n \rangle \leq \frac{1}{\|y_n - a_n\|}\|a - a_n\|^2, \;\; \forall a \in A.$$

Taking $n \longrightarrow \infty$ and assuming that $\zeta_n \longrightarrow \zeta_0$ unit (which can be assumed since ζ_n are unit vectors), we obtain that

$$\langle \zeta_0, a - a_0 \rangle \leq 0, \;\; \forall a \in A.$$

Proceeding as in the last part of Case 1 above, we conclude that $a_0 \in A'(\psi)$. $\square$

Now we define the two sets $\tilde{A}_\psi$ and $\tilde{A}^\psi$ by:

► $$\tilde{A}_\psi := A \cup \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ [a, a + r(a,\zeta,\psi)\zeta] \subset A'(\psi)}} [a, a + r(a,\zeta,\psi)\zeta) \;\cup \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ [a, a + r(a,\zeta,\psi)\zeta] \not\subset A'(\psi)}} [a, a + r(a,\zeta,\psi)\zeta].$$

► $$\tilde{A}^\psi := A \cup \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ [a, a + r(a,\zeta)\zeta] \subset A'(\psi)}} [a, a + r(a,\zeta)\zeta) \;\cup \bigcup_{\substack{a \in \operatorname{bdry} A,\, \zeta \in N_A^P(a) \text{ unit} \\ [a, a + r(a,\zeta)\zeta] \not\subset A'(\psi)}} [a, a + r(a,\zeta)\zeta].$$

Proposition 3.21. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set with $A = \operatorname{cl}(\operatorname{int} A)$, and let $\psi \colon A' \longrightarrow [0, \infty)$ be an upper semicontinuous function which is continuous on $\operatorname{bdry} A' = \operatorname{bdry} A$. Then we have:*

(i) $\tilde{A}_\psi \subset \tilde{A}^\psi$. (ii) $\bar{A}_\psi \subset \tilde{A}_\psi$. (iii) $\bar{A}^\psi \subset \tilde{A}^\psi$.

Proof. (i): Let $x \in \tilde{A}_\psi$. We have the following:

- If $x \in A$ then $x \in \tilde{A}^\psi$.
- If $x \in [a, a + r(a,\zeta,\psi)\zeta)$ with $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, then $x \in \tilde{A}^\psi$.
- If $x = a + r(a,\zeta,\psi)\zeta$ with $a \in \operatorname{bdry} A$, $\zeta \in N_A^P(a)$ unit, and $[a, a + r(a,\zeta,\psi)\zeta] \not\subset A'(\psi)$, then, since $r(a,\zeta,\psi) \leq r(a,\zeta)$, we get that

$$x \in [a, a + r(a,\zeta)\zeta] \;\text{ and }\; [a, a + r(a,\zeta)\zeta] \not\subset A'(\psi).$$

Thus, by the definition of $\tilde{A}^\psi$, we deduce that $x \in \tilde{A}^\psi$.

Therefore, $x \in \tilde{A}^\psi$.

(ii): Let $x \in \bar{A}_\psi$. Using Proposition 3.13(i), we have the following:

- If $x \in A$ then $x \in \tilde{A}_\psi$.
- If $x \in [a, a + r(a,\zeta,\psi)\zeta)$ with $a \in \operatorname{bdry} A$ and $\zeta \in N_A^P(a)$ unit, then $x \in \tilde{A}_\psi$.
- If $x = a + r(a,\zeta,\psi)\zeta$ with $a \in (\operatorname{bdry} A) \cap (\operatorname{bdry}_\psi A)^c$ and $\zeta \in N_A^P(a)$ unit, then:
  - $\psi(a) = 0 \Longrightarrow [\forall \zeta' \in N_A^P(a)$ unit, we have $r(a,\zeta') < \infty]$.
  - $\psi(a) > 0 \Longrightarrow [\forall \zeta' \in N_A^P(a)$ unit, we have $2\psi(a) r(a,\zeta') < 1]$.

  This yields that $a \notin A'(\psi)$, and hence $[a, a + r(a,\zeta,\psi)\zeta] \not\subset A'(\psi)$. Thus, $x \in \tilde{A}_\psi$.

Therefore, $x \in \tilde{A}_\psi$.

(iii): Follows using arguments similar to those employed in the proof of (ii). $\square$

The following theorem is the main result of this subsection. A part of this theorem generalizes the equivalence statement [18, Theorem 3.7(iii)] to the variable radius case, as we will show in Corollary 3.24.

Theorem 3.22. *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set with $A = \operatorname{cl}(\operatorname{int} A)$, and let $\psi \colon A' \longrightarrow [0, \infty)$ be an upper semicontinuous function which is continuous on $\operatorname{bdry} A' = \operatorname{bdry} A$. Then the following assertions are equivalent:*

(i) *A' is the ψ-union of closed balls*

(ii) *A is $\tilde{A}^\psi$-convex.*

(iii) *A is $\tilde{A}_\psi$-convex, $(\mathrm{UP})_\psi$ is satisfied, and $(A_\psi \cap A^c) \subset A'(\psi)$.*

Proof. (i)$\Longrightarrow$(ii): Assume that A' is the ψ-union of closed balls. Then $A'(\psi) = A'$, and hence by the definition of $\tilde{A}^\psi$ and using Lemma 3.4,

$$\tilde{A}^\psi = A \cup \bigcup_{\substack{a \in \operatorname{bdry} A \\ \zeta \in N_A^P(a) \text{ unit}}} [a, a + r(a,\zeta)\zeta) = A \cup A^{UP}.$$

Now by Lemma 3.4, A is $\tilde{A}^\psi$-convex.

(ii)$\Longrightarrow$(iii): Assume that A is $\tilde{A}^\psi$-convex. Since $\tilde{A}_\psi \subset \tilde{A}^\psi$, we deduce that A is $\tilde{A}_\psi$-convex. On the other hand, by Proposition 3.21, we have $\bar{A}^\psi \subset \tilde{A}^\psi$. This gives that A is $\bar{A}^\psi$-convex, and hence by Theorem 3.15, $(\mathrm{UP})_\psi$ is satisfied. We proceed to prove that $(A_\psi \cap A^c) \subset A'(\psi)$. If not, then there exists $x_0 \in (A_\psi \cap A^c)$ such that $x_0 \notin A'(\psi)$. Let $a_0 \in \operatorname{proj}_A(x_0)$. Since $x_0 \notin A$, we have that $a_0 \neq x_0$. Moreover, for $\zeta_0 := \frac{x_0 - a_0}{\|x_0 - a_0\|}$, we have $r(a_0, \zeta_0) \geq \|x_0 - a_0\|$. Then, $x_0 \in [a_0, a_0 + r(a_0,\zeta_0)\zeta_0]$. This latter result, the fact that $x_0 \notin A'(\psi)$, and the definition of $\tilde{A}^\psi$, yield that

$$[a_0, a_0 + r(a_0,\zeta_0)\zeta_0] \subset \tilde{A}^\psi. \tag{17}$$

Let $y_0 := a_0 + r(a_0,\zeta_0)\zeta_0$. Since A is $\tilde{A}^\psi$-convex, $y_0 \in \tilde{A}^\psi$ (by (17)), and

$$\begin{aligned} [\operatorname{proj}_A(y_0), y_0] &= \bigcup_{a \in \operatorname{proj}_A(y_0)} \left[a, a + \|y_0 - a\| \frac{y_0 - a}{\|y_0 - a\|}\right] \\ &= \{y_0\} \cup \bigcup_{a \in \operatorname{proj}_A(y_0)} \left[a, a + \|y_0 - a\| \frac{y_0 - a}{\|y_0 - a\|}\right), \\ &\subset \{y_0\} \cup \bigcup_{a \in \operatorname{proj}_A(y_0)} \left[a, a + r\left(a, \frac{y_0 - a}{\|y_0 - a\|}\right) \frac{y_0 - a}{\|y_0 - a\|}\right) \subset \tilde{A}^\psi, \end{aligned} \tag{18}$$

we have, by Proposition 3.2(iii), that $\operatorname{proj}_A(y_0)$ is a singleton. This yields, since $a_0 \in \operatorname{proj}_A(y_0)$, that $\operatorname{proj}_A(y_0) = \{a_0\}$. For $\varepsilon > 0$, we define y_ε and a_ε as follows:

$$y_\varepsilon := a_0 + (\varepsilon + r(a_0,\zeta_0))\zeta_0 = y_0 + \varepsilon\zeta_0, \text{ and } a_\varepsilon \in \operatorname{proj}_A(y_\varepsilon).$$

Taking ε sufficiently small, we can assume that $y_\varepsilon \notin A$. Moreover, by the definition of $r(a_0, \zeta_0)$, and since $\|y_\varepsilon - a_0\| = \varepsilon + r(a_0, \zeta_0) > r(a_0, \zeta_0)$, we get that $a_\varepsilon \neq a_0$. Now for $\zeta_\varepsilon := \frac{y_\varepsilon - a_\varepsilon}{\|y_\varepsilon - a_\varepsilon\|}$, we have

$$[a_\varepsilon, y_\varepsilon) \subset [a_\varepsilon, a_\varepsilon + r(a_\varepsilon, \zeta_\varepsilon)\zeta_\varepsilon) \subset \tilde{A}^\psi.$$

Since $y_\varepsilon \longrightarrow y_0$ as $\varepsilon \longrightarrow 0$, $\operatorname{proj}_A(y_0) = \{a_0\}$, and using the closedness of the projection map $\operatorname{proj}_A(\cdot)$, we can assume that $a_\varepsilon \longrightarrow a_0$ as $\varepsilon \longrightarrow 0$. Define

$$z_\varepsilon := \lambda_\varepsilon a_\varepsilon + (1 - \lambda_\varepsilon) y_\varepsilon, \text{ where } \lambda_\varepsilon := \frac{\|y_\varepsilon - x_0\|}{\|y_\varepsilon - a_0\|} = \frac{\|y_\varepsilon - x_0\|}{\|y_\varepsilon - x_0\| + \|x_0 - a_0\|} < 1.$$

We have $z_\varepsilon \in [a_\varepsilon, y_\varepsilon]$, and $\|z_\varepsilon - x_0\| = \lambda_\varepsilon \|a_\varepsilon - a_0\|$. Hence, since $A'(\psi)$ is closed (by Lemma 3.20), and $x_0 \in (A'(\psi))^c$, there exists $\varepsilon_0 > 0$ such that $z_\varepsilon \in (A'(\psi))^c$ for all $\varepsilon \in (0, \varepsilon_0]$. This yields that $[a_\varepsilon, y_\varepsilon] \not\subset A'(\psi)$ for all $\varepsilon \in (0, \varepsilon_0]$, and then by the definition of $\tilde{A}^\psi$,

$$y_\varepsilon \in [a_\varepsilon, a_\varepsilon + r(a_\varepsilon, \zeta_\varepsilon)\zeta_\varepsilon] \subset \tilde{A}^\psi \text{ for all } \varepsilon \in (0, \varepsilon_0].$$

Hence, $$[a_0, y_{\varepsilon_0}] = [a_0, y_0] \cup (y_0, y_{\varepsilon_0}] = [a_0, y_0] \bigcup_{\varepsilon \in (0, \varepsilon_0]} \{y_\varepsilon\} \subset \tilde{A}^\psi.$$

Therefore, the two normal segments to A, $[a_{\varepsilon_0}, y_{\varepsilon_0}]$ and $[a_0, y_{\varepsilon_0}]$ are inside $\tilde{A}^\psi$ and intersect at y_{ε_0}. This contradicts the $\tilde{A}^\psi$-convexity of A.

(iii)$\Longrightarrow$(i): Assume that A is $\tilde{A}_\psi$-convex, $(A_\psi \cap A^c) \subset A'(\psi)$, and $(\mathrm{UP})_\psi$ is satisfied. By Proposition 3.21, we have that $\bar{A}_\psi \subset \tilde{A}_\psi$, and hence A is $\bar{A}_\psi$-convex. This yields, using Theorem 3.15, that A satisfies the ψ-exterior sphere condition, and hence

$$\operatorname{bdry}_\psi A = \operatorname{bdry} A = \operatorname{bdry} A'. \tag{19}$$

Now let $x \in \operatorname{int} A' = A^c$.

Case 1: $x \in A_\psi$.
Then $x \in (A_\psi \cap A^c) \subset A'(\psi)$.

Case 2: $x \notin A_\psi$.
Then $2\psi(x) d_A(x) > 1$, which yields that $\psi(x) > 0$ and $d_A(x) > \frac{1}{2\psi(x)}$. Hence,

$$\bar{\mathbb{B}}\Big(x; \frac{1}{2\psi(x)}\Big) \subset A', \text{ and this gives that } x \in A'(\psi).$$

Therefore, $\operatorname{int} A' \subset A'(\psi)$. Then, using (19), we obtain that

$$A' = \operatorname{bdry} A' \cup \operatorname{int} A' = \operatorname{bdry}_\psi A \cup \operatorname{int} A' \subset A'(\psi) \cup A'(\psi) = A'(\psi).$$

Thus, $A' = A'(\psi)$, which yields that A' is the ψ-union of closed balls. □

Remark 3.23. From (18), we deduce that if A is $\tilde{A}^\psi$-convex, then each point in $\tilde{A}^\psi$ has a unique projection on A.

Corollary 3.24. [18, Theorem 3.7(iii)] *Let $A \subset \mathbb{R}^n$ be a nonempty and closed set with $A = \operatorname{cl}(\operatorname{int} A)$, and let $r > 0$. Then A' is the $\frac{1}{2r}$-union of closed balls if and only if A is $\tilde{A}_{\frac{1}{2r}}$-convex.*

Proof. Applying Theorem 3.22 for $\psi = \frac{1}{2r}$ constant, it is sufficient to prove that if A is $\tilde{A}_{\frac{1}{2r}}$-convex, then $(\mathrm{UP})_{\frac{1}{2r}}$ is satisfied and $\big(A_{\frac{1}{2r}} \cap A^c\big) \subset A'\big(\frac{1}{2r}\big)$. Since $\bar{A}_{\frac{1}{2r}} \subset \tilde{A}_{\frac{1}{2r}}$, we deduce that A is $\bar{A}_{\frac{1}{2r}}$-convex. This latter yields, via Corollary 3.16 and Theorem 3.15, that $(\mathrm{UP})_{\frac{1}{2r}}$ is satisfied. Now we prove that $\big(A_{\frac{1}{2r}} \cap A^c\big) \subset A'\big(\frac{1}{2r}\big)$. We consider $x \in \big(A_{\frac{1}{2r}} \cap A^c\big)$ and $a \in \operatorname{proj}_A(x)$. We have $x \neq a$ and $d_A(x) = \|x-a\| < r$. If $x \notin A'\big(\frac{1}{2r}\big)$, then for $\zeta_a := \frac{x-a}{\|x-a\|}$ we have $r\big(a, \zeta_a, \frac{1}{2r}\big) < r$. Since ζ_a is realized by a $\|x-a\|$-sphere and $\|x-a\| < r$, we deduce that $r\big(a, \zeta_a, \frac{1}{2r}\big) \geq \|x-a\|$. Therefore,

$$d_A(x) = \|x-a\| \leq r\Big(a, \zeta_a, \frac{1}{2r}\Big) < r.$$

Let $y := a + r\big(a, \zeta_a, \frac{1}{2r}\big)\zeta_a$. We have $a \in \operatorname{proj}_A(y)$. Moreover, using the definition $\tilde{A}_{\frac{1}{2r}}$ and since $[a, y] \ni x \notin A'\big(\frac{1}{2r}\big)$, we obtain that

$$[a, y] \subset \tilde{A}_{\frac{1}{2r}}. \tag{20}$$

Add to this that A is $\tilde{A}_{\frac{1}{2r}}$-convex and that

$$\begin{aligned}
[\operatorname{proj}_A(y), y] &= \bigcup_{b \in \operatorname{proj}_A(y)} \left[b, b + \|y-b\| \frac{y-b}{\|y-b\|}\right] \\
&= \{y\} \cup \bigcup_{b \in \operatorname{proj}_A(y)} \left[b, b + \|y-b\| \frac{y-b}{\|y-b\|}\right) \\
&\subset \{y_0\} \cup \bigcup_{a \in \operatorname{proj}_A(y)} \left[b, b + r\left(b, \frac{y-b}{\|y-b\|}, \frac{1}{2r}\right) \frac{y-b}{\|y-b\|}\right) \subset \tilde{A}^{\psi},
\end{aligned}$$

we have, via Proposition 3.2(iii), that $\operatorname{proj}_A(y) = \{a\}$. For $\varepsilon > 0$, we define y_ε and a_ε as the following

$$y_\varepsilon := a + \left(\varepsilon + r\Big(a, \zeta_a, \frac{1}{2r}\Big)\right)\zeta_a = y + \varepsilon\zeta_a, \text{ and } a_\varepsilon \in \operatorname{proj}_A(y_\varepsilon).$$

Taking ε sufficiently small, we can assume that $y_\varepsilon \notin A$ and that $\varepsilon + r\big(a, \zeta_a, \frac{1}{2r}\big) < r$. Hence, since $\|y_\varepsilon - a\| = \varepsilon + r\big(a, \zeta_a, \frac{1}{2r}\big) > r\big(a, \zeta_a, \frac{1}{2r}\big)$, we get that $a_\varepsilon \neq a$. Now for $\zeta_\varepsilon := \frac{y_\varepsilon - a_\varepsilon}{\|y_\varepsilon - a_\varepsilon\|}$, we have due to $\|y_\varepsilon - a_\varepsilon\| < \|y_\varepsilon - a\| = \varepsilon + r\big(a, \zeta_a, \frac{1}{2r}\big) < r$, that

$$[a_\varepsilon, y_\varepsilon) \subset \left[a_\varepsilon, a_\varepsilon + r\Big(a_\varepsilon, \zeta_\varepsilon, \frac{1}{2r}\Big)\zeta_\varepsilon\right) \subset \tilde{A}^{\psi}.$$

Since $y_\varepsilon \longrightarrow y$ as $\varepsilon \longrightarrow 0$, $\operatorname{proj}_A(y) = \{a\}$, and using the closedness of the projection map $\operatorname{proj}_A(\cdot)$, we can assume that $a_\varepsilon \longrightarrow a$ as $\varepsilon \longrightarrow 0$. Define

$$z_\varepsilon := \lambda_\varepsilon a_\varepsilon + (1-\lambda_\varepsilon) y_\varepsilon, \text{ where } \lambda_\varepsilon := \frac{\|y_\varepsilon - x\|}{\|y_\varepsilon - a\|} = \frac{\|y_\varepsilon - x\|}{\|y_\varepsilon - x\| + \|x - a\|} < 1.$$

We have $z_\varepsilon \in [a_\varepsilon, y_\varepsilon]$, and $\|z_\varepsilon - x\| = \lambda_\varepsilon \|a_\varepsilon - a\|$. Hence, since $A'\big(\frac{1}{2r}\big)$ is closed (by Lemma 3.20), and $x \in \big(A'\big(\frac{1}{2r}\big)\big)^c$, there exists $\varepsilon_0 > 0$ such that $z_\varepsilon \in \big(A'\big(\frac{1}{2r}\big)\big)^c$ for all $\varepsilon \in (0, \varepsilon_0]$. This yields that $[a_\varepsilon, y_\varepsilon] \not\subset A'\big(\frac{1}{2r}\big)$ for all $\varepsilon \in (0, \varepsilon_0]$, and then by the definition of $\tilde{A}_{\frac{1}{2r}}$,

$$y_\varepsilon \in \Big[a_\varepsilon, a_\varepsilon + r\Big(a_\varepsilon, \zeta_\varepsilon, \frac{1}{2r}\Big)\zeta_\varepsilon\Big] \subset \tilde{A}_{\frac{1}{2r}} \text{ for all } \varepsilon \in (0, \varepsilon_0].$$

Hence, using (20),

$$[a, y_{\varepsilon_0}] = [a, y] \cup (y, y_{\varepsilon_0}] = [a, y] \bigcup_{\varepsilon \in (0, \varepsilon_0]} \{y_\varepsilon\} \subset \tilde{A}_{\frac{1}{2r}}.$$

Therefore, the two normal segments to A, $[a_{\varepsilon_0}, y_{\varepsilon_0}]$ and $[a, y_{\varepsilon_0}]$ are inside $\tilde{A}_{\frac{1}{2r}}$ and intersect at y_{ε_0}. This contradicts the $\tilde{A}_{\frac{1}{2r}}$-convexity of A. □

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