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Semi-Discrete Shocks for a Microscopic Pedestrian Model

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Dedications

To my Mom and Dad without whom none of my success would be possible...

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ABSTRACT

In this thesis, we explore in the framework of viscosity solutions the existence and uniqueness of traveling solutions using a pedestrian microscopic model. We consider that the velocity of the pedestrian at position $u(y)$ depends on the velocity of two pedestrians ahead.

$$u'(y) = V(u(y+1) - u(y)) + V(u(y+2) - u(y))$$

Throughout this work, we give the definition of a viscosity solution first then we prove the exponential behavior of the solution at $\pm\infty$. After that, using the monotony of the pedestrians interdistance which we achieve by the strong comparison principle, we derive necessarily conditions for the existence of such solutions. With this established, we proceed by constructing a traveling solution considering an approximate non-local operator on a bounded domain and using Perron's method.

Keywords: Pedestrian traffic, Semi-discrete shocks, Social force model, Strong comparison principle, Viscosity solution

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Chapter 1

Introduction

Pedestrian traffic has definitely grown. Transportation hubs, such as airports, railway stations, and subways, as well as theaters, stadiums, and other venues, can be extremely overcrowded [1]. From another side, walking is now recognized as a significant means of transport, friendly, highly flexible, and healthy, especially in conjunction with public transportation, and it should really be promoted in order to reduce the traffic system's ecological consequences [21]. Therefore, pedestrian simulation is now widely used in practice. It has the potential to save not only money but also peoples lives, since overcrowding has resulted in a number of catastrophic crushing accidents around the world. For example, in 1989, when police opened gates to relieve crowding at the Hillsborough football fields in Sheffield, ninety-six Liverpool fans were killed [3] and during the annual Hajj trip, a stampede killed 251 Muslims in the Mena Valley disaster [4, 5].

Psychologist Gustave Le Bon started the studies of pedestrian groups in the early 1900s where he examined crowds and multitudes from a psychological standpoint [6]. He argued that in a crowd, individual personalities are hidden and a collective crowd consciousness rules. In the twentieth century, the initial pedestrian flow studies that were conducted, focused on various social conditions in which people assemble in groups, primarily in urban areas as stated in [11, 12, 13, 14, 15, 16]. In the seventies, researchers focused on various issues, such as building evacuation as in [7] and the connection between pedestrians and architectural spaces as in [8, 9, 10].

Although it might seem simple, explaining pedestrian dynamics is really tough. To begin with, each pedestrian has its own preferences, goals, and destinations and walking alone differs a lot from walking in a group. Moreover, the underlying infrastructure is widely diverse (sidewalks, stairs, elevators, crossings, retail malls, etc.) and walking is not associated with a car on a lane, unlike other displacement models. As a result, the ideal method for describing pedestrian dynamics is extremely dependent on the model's purpose and the specific situation to be portrayed. For instance, the logit-type choice model was chosen by the authors in [17] where they attempted to cover the pedestrian route selection between escalators and stairways in mass transport systems and in [18] where they studied the wayfinding problem of people in the Hong Kong international airport. Another example is the Hughes model used in [20] for pedestrian flow under various types of congested conditions, with a goal to measure the flow of a massive pedestrian group attempting to reach a destination as quickly as possible. A model for airport passenger terminals has been created as well in [19]. Hence, in the rise of emerging pedestrian behavior models, a very remarkable model has been introduced: the social force model (SFM).

Consequently, one has to distinguish between three types of models: macroscopic, microscopic and mesoscopic. Each has its advantages and disadvantages [1]. Macroscopic models model pedestrian flow as a whole without taking into account the motion of single people and can be applied in settings where human interaction isn't well understood. It has also been shown that macroscopic models are successful in modeling car traffic, therefore there is reason to believe it will be successful in modeling pedestrian traffic as well [21]. Microscopic models are extremely detailed, allowing pedestrians to discern between social contacts: other walkers and the environment. So unlike the macroscopic approach, no averaging process occurs that leads to data loss and the model can explicitly account for population heterogeneity. Mesoscopic models are characterized by the simplification of dynamics while using fewer data. They are like a compromise between macroscopic and microscopic models: seeking macroscopic models' computing efficiency and analytical tractability but at the same time capturing some of the variety and uniqueness of actual pedestrians [21].

With this in mind, we turn to a microscopic pedestrian model introduced two decades ago by Helbing and Moln´ar in [22] and has been extensively researched in the literature since then e.g. [23]. SFM is one of the many force-based pedestrian dynamics models with a history that dates back to the work of [24]. The SFM's original uses were mostly focused on replicating building emergency evacuations [25]. The purpose of pedestrians in such applications is to get to an exit as rapidly as possible. However, since its inception, the model has been expanded to accommodate a variety of nontrivial phenomena like people preferring to move in groups [27] and dynamic route choice [26]. There have also been many modifications of the social force model and here are a few examples that demonstrate the diversity of the variations: in [28], A. Johansson explored the social field's shape, allowing it to be determined by the relative speeds of the involved pedestrians and conducted a detailed comparison of the suggested specifications with the field's customary specifications then in [29], Parisi et al. modified the model to prevent people with a high desired velocity to push other pedestrians ahead or to the side when attempting to pass by them. To achieve this, if the space directly in front of the pedestrian is occupied, the optimal speed is temporarily set to zero. Moreover, in [30] Lakoba et al. presented a relatively efficient and stable methodology to establish a minimum distance for pedestrians and suggested that the work be expanded to also include memory effects. The form of the social force model chosen in our work is presented as follows:

$$u'(y) = V(u(y+1) - u(y)) + V(u(y+2) - u(y)) \tag{1.1}$$

Here $u(y)$ denotes the position of the pedestrian y, $u'(y)$ its velocity and $V : \mathbb{R} \rightarrow$

$\mathbb{R}$ represents the optimal velocity function that satisfies the non strict inequality

$$V(p) + V(2p) \geq p \qquad \text{for any } p \in [a, b] \tag{1.2}$$

which we explore later in this paper.

To get to equation (1.1), we started by the origin of the SFM which is a 2D model [34] of the form

$$U_i''(t) = \frac{V_0 - U_i'(t)}{\tau} + A \sum_{j<i} \lambda F(U_i(t) - U_j(t)) - A \sum_{j>i} F(U_j(t) - U_i(t))$$

with V_0: desired velocity, τ: relaxation time (the time to perceive and process traffic situations and the resulting actions), $A > 0$ parameter and $\lambda \in [0, 1]$ and $F(x) = e^{-Bx}$ with $B > 0$ parameter. In this work we are considering a steady state which means $U_i''(t) = 0$ and that the traffic flow is influenced by the traffic state in front and not from the back which means $\lambda = 0$.

$$U_i'(t) = \frac{V_0}{\tau} - A \sum_{j=i+1}^{i+m} F(U_j(t) - U_i(t))$$

We let $-F = \tilde{V} \Rightarrow \tilde{V} = -e^{-Bx}$

Moreover, we consider that the velocity of the pedestrian at position U_i depends on the force of interaction between him and two pedestrians ahead:

$$U_i'(t) = V_0 + A\tau \left(\tilde{V}(U_{i+1}(t) - U_i(t)) + \tilde{V}(U_{i+2}(t) - U_i(t)) \right)$$

We end up finally with

$$U_i' = V(U_{i+1} - U_i) + V(U_{i+2} - U_i)$$

with $V(p) = \frac{1}{2}[V_0 - A\tau e^{-Bp}]$

The unmodified social force model seen assumes the velocity of the pedestrian at position $u(y)$ was assumed to be dependent on the force of interaction between him and of all people behind and in front of him. In this work, we assume that the traffic flow is only dependent on the traffic state in front and not from the back; the velocity of the pedestrian at position $u(y)$ depends on the force of interaction between him and two people ahead.

Before we give a definition of our particular type of solutions called semi-discrete shocks, we start by showing the motivation of our work.

Consider the microscopic pedestrian model:

$$U_i' = V(U_{i+2} - U_i) + V(U_{i+1} - U_i) \tag{1.3}$$

and consider the macroscopic model as follows:

$$\chi_t = V(2\chi_y) + V(\chi_y) = F(\chi_y) \qquad \text{for } t > 0,\ y \in \mathbb{R} \tag{1.4}$$

It was shown that (1.4) can be rigorously derived from the microscopic model (1.3) (see [33], Homogenization of fully overdamped Frenkel-Kontorova models). Let $a < b$, we define the constants

$$\frac{1}{T} = \frac{V(a) - V(b)}{a - b} \quad \text{and } c = \frac{aV(b) - bV(a)}{a - b}$$

The function

$$\chi(t, y) = \min(ay + tV(a), by + tV(b)) = \begin{cases} ay + tV(a) & \text{if } y > -\frac{t}{T} \\ by + tV(b) & \text{if } y < -\frac{t}{T} \\ ct & \text{if } y = -\frac{t}{T} \end{cases} \tag{1.5}$$

is a viscosity solution of (1.4) if and only if

$$c + \frac{p}{T} \leq V(2p) + V(p) = F(p) \tag{1.6}$$

The function χ can be interpreted as a "shock" since its left and right derivatives at the point $-\frac{t}{T}$ are different. We recall that $\chi_y = \frac{1}{\rho}$ where ρ is the pedestrians density and is the solution of the scalar conservation law

$$\rho_t + (f(\rho))_x = 0 \tag{1.7}$$

where the flux is $f(p) = pF(\frac{1}{p})$. This means that the spacing after the shock (resp. before the shock) is a (resp. b) and the speed of propagation of the shock is c. Finally, let us remark that χ is a traveling wave since we can write

$$\chi(t,y) = \chi\left(0, y + \frac{t}{T}\right) + ct \tag{1.8}$$

with $\chi(0,y) = \min(ay, by)$. The idea is to construct solutions of (1.3) which can be seen as the discrete analogue of the function χ. We look for particular shock solutions U_i of (1.3) satisfying

$$\begin{cases} U_{i+1}(t) - U_i(t) \to b & \text{as } i \to -\infty \\ U_{i+1}(t) - U_i(t) \to a & \text{as } i \to +\infty \end{cases} \tag{1.9}$$

Moreover, we will prove that the interdistance $U_{i+1} - U_i$ is decreasing (see Theorem 4.1.1). The traffic interpretation of (1.9) is that a shock occurred at the microscopic level and the interdistance is b far before the shock, and is a far after it.

Following the definition of χ in (1.8),

$$U_i(t) = u(i + \frac{t}{T}) + ct$$

with

$$\begin{cases} u(y+1) - u(y) \to b \quad \text{as } y \to -\infty \\ u(y+1) - u(y) \to a \quad \text{as } y \to +\infty \end{cases} \tag{1.10}$$

We recall that at the macroscopic scale, (1.6) is a necessarily condition to prove that χ is a solution. At the microscopic scale, we provide a necessarily condition to ensure the existence of traveling solution satisfying (1.10) (see Theorem 4.1.1).

For the rest of the work, we define the viscosity solutions of (1.1) and give strong comparison principle results. Then, we prove the exponential behavior of the solution at $\pm\infty$. Once we have done these, we prove our first main result (Theorem 4.1.1) which is the classification of the solution theorem. To do that, we prove the monotonicity of the interdistances using the strong comparison principle. Finally, using the stability of viscosity solutions, we obtain the second result (Theorem 5.0.1) which is the existence and uniqueness of the solution by considering a non-local operator G_R for which we can construct sub and super solutions and where we use Perron's method to construct a solution.

Chapter 2

Viscosity Solutions

In this chapter, we give first the definition of viscosity solution of (1.1). Then we prove the strong maximum principle which will constitute a very important step to get the first result seen later in chapter 4.

2.1 Definition of a viscosity solution

Definition 2.1.1 *Let u be a function such that $u \in L^{\infty}_{loc}(\mathbb{R})$.*

1. *We say that u is a viscosity sub-solution (resp. super-solution) of (1.1) if u is upper-semi continuous (resp. lower-semi continuous) and if for all test function $\phi \in C^1(\mathbb{R})$ such that $u - \phi$ attains a local maximum (resp. local minimum) at some point x_0, we have*

$$\phi'(x_0) \leq V(u(x_0 + 1) - u(x_0)) + V(u(x_0 + 2) - u(x_0))$$

$$(resp.\ \phi'(x_0) \geq V(u(x_0 + 1) - u(x_0)) + V(u(x_0 + 2) - u(x_0)))$$

2. *We say that u is a viscosity solution of (1.1) if $u \in C(\mathbb{R})$ and u is a subsolution and supersolution of (1.1).*

2.2 Assumptions on the Optimal Velocity Function V

We look for a particular shock solution that solves:

$$u'(y) = V(u(y+1) - u(y)) + V(u(y+2) - u(y))$$

and satisfies

$$\begin{cases} u(y+1) - u(y) \to b & \text{as } y \to -\infty \\ u(y+1) - u(y) \to a & \text{as } y \to +\infty \end{cases}$$

To obtain our results, we need the following assumptions on the optimal velocity function V.

Assumptions (A) on V

- *(A1) (Regularity)*

$$V \in C^1(\mathbb{R}), V' \in L^\infty(\mathbb{R})$$

- *(A2) (Monotonicity)*

$$V' > 0 \text{ on } \mathbb{R}$$

- *(A3) (Strict chord inequality) There exists $a, b \in \mathbb{R}$ such that*

$$V(p) + V(2p) \geq p \qquad \text{for any } p \in [a, b]$$

  with equality if and only if $p \in \{a, b\}$.

- *(A4) (Non degeneracy)*

$$V'(a) > 1 > V'(b)$$

2.3 Strong Comparison Principle

Theorem 2.3.1 *(Strong Maximum Principle) Let V satisfying assumptions (A1) and (A2), and let u_1, u_2 be respectively a viscosity sub and super solution of (1.1).*

Assume that $\begin{cases} u_2 \geq u_1 & \text{on } \mathbb{R} \\ u_2(y_0) = u_1(y_0) \end{cases}$ *then* $u_1 = u_2$ *for all* $y \leq y_0$

Proof. Step 1: Let $w(y) = u_2(y) - u_1(y)$

Since u_2 is a super solution and u_1 is a sub solution of (1.1), then

$u_2'(y) \geq V(u_2(y+1) - u_2(y)) + V(u_2(y+2) - u_2(y))$ *and*

$u_1'(y) \leq V(u_1(y+1) - u_1(y)) + V(u_1(y+2) - u_1(y))$

But since $w'(y) = u_2'(y) - u_1'(y)$ and using the doubling of variable method, we can show that w is a viscosity super solution of

$$\begin{aligned} w'(y) &= u_2'(y) - u_1'(y) \\ &\geq V(u_2(y+1) - u_2(y)) + V(u_2(y+2) - u_2(y)) \\ &- V(u_1(y+1) - u_1(y)) - V(u_1(y+2) - u_1(y)) \quad \text{on } \mathbb{R} \end{aligned}$$

But $u_2(y+1) \geq u_1(y+1)$

so $u_2(y+1) - u_2(y) \geq u_1(y+1) - u_2(y)$

Using the fact that V is increasing, we get

$$V(u_2(y+1) - u_2(y)) \geq V(u_1(y+1) - u_2(y))$$

Same way we get:

$$V(u_2(y+2) - u_2(y)) \geq V(u_1(y+2) - u_2(y))$$

Then

$$w'(y) \geq V(u_1(y+1) - u_2(y)) + V(u_1(y+2) - u_2(y))$$
$$- V(u_1(y+1) - u_1(y)) - V(u_1(y+2) - u_1(y))$$

But since V is lipschitz

$V(u_1(y+1) - u_2(y)) - V(u_1(y+1) - u_1(y)) \geq ||V'||_\infty |u_2(y) - u_1(y)|$

and

$V(u_1(y+2) - u_2(y)) - V(u_1(y+2) - u_1(y)) \geq ||V'||_\infty |u_2(y) - u_1(y)|$

Hence, $w'(y) \geq -2||V'||_\infty |u_2(y) - u_1(y)|$

but as $u_2(y) \geq u_1(y)$

then

$$w'(y) \geq -2||V'||_\infty w(y) \quad \forall y \in \mathbb{R} \tag{2.1}$$

Now we notice that $h(y) = w(y_1)e^{-2||V'||_\infty (y-y_1)}$ *satisfies the last inequality for any* $y_1 \in \mathbb{R}$. *As* $h(y_1) = w(y_1)$, *then using the comparison principle for the "ode" (2.1),*

we deduce that

$$w(y) \geq w(y_1)e^{-2||V'||_\infty (y-y_1)}$$

for all $y \geq y_1$.

Now let $y_1 < y_0$. *If* $w(y_1) > 0$ *then* $w(y_0) > 0$ *and this implies* $u_2(y_0) > u_1(y_0)$ *which is a contradiction as* $u_2(y_0) = u_1(y_0)$.

This implies that $w(y) \leq 0 \qquad \forall y \leq y_0$.

Using the fact that $w(y) \geq 0 \quad \forall y \leq y_0$.

We finally get $w(y) = 0 \qquad \forall y \leq y_0$

and that is

$$u_2(y) = u_1(y) \qquad \forall y \leq y_0$$

Step 2: Since we have already proved that $u_2 \leq u_1 \quad \forall y \leq y_0$, we will prove now that $u_2 \leq u_1 \quad \forall y \geq y_0$.

Let u_1, u_2 be respectively a viscosity sub and super solution of (1.1).
Then as we did before and using the doubling of variable method, we can show that $w = u_2 - u_1$ satisfies

$$\begin{aligned} w'(y) \geq\; & V(u_2(y+1) - u_2(y)) + V(u_2(y+2) - u_2(y)) \\ & - V(u_1(y+1) - u_1(y)) - V(u_1(y+2) - u_1(y)) \end{aligned}$$

on $\mathbb{R}$ in the viscosity sense.
As w is a viscosity super-solution of this last inequality, $w(y_0) = 0$ and $w \geq 0$ on $\mathbb{R}$, we deduce that w has a minimum at y_0 so $w'(y_0) = u_2'(y_0) - u_1'(y_0) = 0$.
Hence,

$$\begin{aligned} 0 \geq\; & V(u_2(y_0+1) - u_2(y_0)) + V(u_2(y_0+2) - u_2(y_0)) \\ & - V(u_1(y_0+1) - u_1(y_0)) - V(u_1(y_0+2) - u_1(y_0)) \end{aligned}$$

but $u_1(y_0) = u_2(y_0)$ then

$$\begin{aligned} 0 \geq\; & V(u_2(y_0+1) - u_1(y_0)) + V(u_2(y_0+2) - u_1(y_0)) \\ & - V(u_1(y_0+1) - u_1(y_0)) - V(u_1(y_0+2) - u_1(y_0)) \end{aligned}$$

Also, having $u_2(y_0+2) \geq u_1(y_0+2)$ and $u_2(y_0+1) \geq u_1(y_0+1)$ and V increasing then

$$\begin{aligned} 0 \geq\; & V(u_1(y_0+1) - u_1(y_0)) + V(u_1(y_0+2) - u_1(y_0)) \\ & - V(u_1(y_0+1) - u_1(y_0)) - V(u_1(y_0+2) - u_1(y_0)) \\ & = 0 \end{aligned}$$

which gives us

$$\begin{aligned} 0 &\le V(u_2(y_0+1)-u_2(y_0))+V(u_2(y_0+2)-u_2(y_0)) \\ &-V(u_1(y_0+1)-u_1(y_0))-V(u_1(y_0+2)-u_1(y_0)) \\ &\le 0 \end{aligned}$$

$\Rightarrow V(u_2(y_0+1)-u_2(y_0))+V(u_2(y_0+2)-u_2(y_0))-V(u_1(y_0+1)-u_1(y_0))-V(u_1(y_0+2)-u_1(y_0))=0$
$\Rightarrow V(u_2(y_0+1)-u_2(y_0))+V(u_2(y_0+2)-u_2(y_0))=V(u_1(y_0+1)-u_1(y_0))+V(u_1(y_0+2)-u_1(y_0))$

But $u_2(y_0+2)-u_1(y_0)\ge u_1(y_0+2)-u_1(y_0)$
and $u_2(y_0+1)-u_1(y_0)\ge u_1(y_0+1)-u_1(y_0)$ *and* V *is increasing then*

$$u_2(y_0+2)=u_1(y_0+2) \quad or \quad u_2(y_0+1)=u_1(y_0+1)$$

because otherwise, assume by contradiction that
$u_2(y_0+2)-u_1(y_0)>u_1(y_0+2)-u_1(y_0)$ *and* $u_2(y_0+1)-u_1(y_0)>u_1(y_0+1)-u_1(y_0)$
then using that V *is increasing we get:*
$V(u_2(y_0+2)-u_1(y_0))+V(u_2(y_0+1)-u_1(y_0))>V(u_1(y_0+2)-u_1(y_0))+V(u_1(y_0+1)-u_1(y_0))$
Contradiction.
Hence either

$$u_2(y_0+2)=u_1(y_0+2) \quad or \quad u_2(y_0+1)=u_1(y_0+1)$$

Suppose now that $u_2(y_0+2)=u_1(y_0+2)$
Using step 1, this implies $u_2(y)=u_1(y) \quad \forall\ y\le y_0+2$
and if $u_2(y_0+1)=u_1(y_0+1)$ *then also by step 1,* $u_2(y)=u_1(y) \quad \forall\ y\le y_0+1$

We deduce that $u_2(y) = u_1(y) \quad \forall\ y \leq y_0 + 1$

Repeating the above argument, we get $u_2(y) = u_1(y) \quad \forall\ y \leq y_0 + k\ ,\ k \in \mathbb{N}$.

Finally, this implies that $w(y) = 0$ *on* $\mathbb{R}$

That is,

$$u_2(y) = u_1(y) \qquad on\ \mathbb{R}$$

□

Proposition 2.3.2 *Assume (A1) and (A2). Let* u^1 *and* u^2 *be two solutions of (1.1). We assume that*

$$\lim_{|y| \to +\infty} (u^1(y) - u^2(y)) \leq 0 \tag{2.2}$$

then we have, $u^1 \leq u^2$ *on* $\mathbb{R}$.

Proof. *We define* $M = \sup_{y \in \mathbb{R}} \{u^1(y) - u^2(y)\}$
We need to prove that $M \leq 0$.
Assume by contradiction that $M > 0$.
Using (2.2), we deduce that $\exists\ y_0$ *such that* $M \leq 0$, *so* M *is reached at some point* y_0 *and*

$$M = u^1(y_0) - u^2(y_0)$$

We define $\bar{u}^2(y) = u^2(y) + M$
For all $y \in \mathbb{R}$, *we have* $\bar{u}^2(y) = u^2(y) + M \geq u^1(y)$ *and*
$\bar{u}^2(y_0) = u^2(y_0) + u^1(y_0) - u^2(y_0) = u^1(y_0)$

Using that $\bar{u}^2$ *is a solution of (1.1) and the strong comparison principle, we get that*

$$\bar{u}^2(y) = u^1(y) \quad \forall\ y \leq y_0$$

i.e $u^2(y) + M = u^1(y) \quad \forall\ y \leq y_0$

$\Rightarrow M = u^1(y) - u^2(y) \quad \forall\ y \leq y_0$

Taking $y \to -\infty$, *we get that* $M \leq 0$. *Contradiction.*

Hence $u^1 \leq u^2$ *on* $\mathbb{R}$.

$\square$

Chapter 3

Asymptotics

In this chapter, we study the asymptotic behavior of the solution of (1.1).

Proposition 3.0.1 *Assume (A). Let* u *be a solution of (1.1) and let* $G(y) = u(y+1) - u(y)$.

We assume that

$$\begin{cases} u'(+\infty) = G(+\infty) = a \\ u'(-\infty) = G(-\infty) = b \end{cases} \tag{3.1}$$

and that for $y \in \mathbb{R}$,

$$a \leq G(y) \leq b \tag{3.2}$$

Then there exists $k, \gamma > 0$ *and* $c_1, c_2 \in \mathbb{R}$ *such that*

$$\begin{cases} |u(y) - ay - c_1| \leq ke^{-\gamma y} & \text{for } y \geq 0 \\ |u(y) - by - c_2| \leq ke^{\gamma y} & \text{for } y \leq 0 \end{cases} \tag{3.3}$$

The proof of this proposition is a direct consequence of the following lemma.

Lemma 3.0.2 *Assume (A). Let u be a solution of (1.1) satisfying (3.1) and let $G(y) = u(y+1) - u(y)$ satisfying (3.2).*

We remark that G satisfies:

$$G'(y) = V(G(y+1)) + V(G(y+2) + G(y+1)) - V(G(y)) - V(G(y+1) + G(y)) \quad (3.4)$$

Recalling that $V'(a) > 1 > V'(b)$, let $\epsilon > 0$ small enough such that

$$\begin{cases} V'(p) + V'(q) > 1 & \text{if } p \in [2a - \epsilon, 2a + 2\epsilon] \text{ and } q \in [a - \epsilon, a + \epsilon] \\ V'(p) + V'(q) < 1 & \text{if } p \in [2b - 3\epsilon, 2b + 2\epsilon] \text{ and } q \in [b - 2\epsilon, b + \epsilon] \end{cases}$$

We have the following:

1) Let γ be small enough such that

$$V'(p) + V'(q) > 1 > \frac{\gamma}{1 - e^{-\gamma}} \qquad \text{for } p \in [2a - \epsilon, 2a + 2\epsilon] \text{ and } q \in [a - \epsilon, a + \epsilon]$$

There exists a constant $C > 0$ such that for all $y \geq 0$,

$$G(y) \leq a + Ce^{-\gamma y} \quad (3.5)$$

2) Let γ be small enough such that

$$V'(p) + V'(q) < 1 < \frac{\gamma}{e^{2\gamma} - 1} \qquad \text{for } p \in [2b - 3\epsilon, 2b + 2\epsilon] \text{ and } q \in [b - 2\epsilon, b + \epsilon]$$

There exists a constant $C > 0$ such that for all $y \leq 0$,

$$G(y) \geq b - Ce^{\gamma y} \quad (3.6)$$

Proof.

1) Since $u'(+\infty) = G(+\infty) = a$, let $y_0 > 0$ such that for all $y \geq y_0$,

$$a + \epsilon \geq G(y)$$

We will prove that for $y \geq y_0$*,*

$$G(y) \leq a + Ce^{-\gamma(y-y_0)} \tag{3.7}$$

We define $\varphi(y) = G(y) - a - Ce^{-\gamma(y-y_0)}$

along with $M = \sup\limits_{y \geq y_0}(\varphi(y))$

We will prove that $M \leq 0$*.*

Assume by contradiction that $M > 0$*.*

Using the fact that $G(y) \to a$ *as* $y \to +\infty$*, we deduce that* M *is reached at some point* x*, so* $M = \sup\limits_{y \geq y_0}(\varphi(y)) = \varphi(x)$ *(otherwise we get* $-Ce^{-\gamma y} > 0$*)*

First we will prove that $x \neq y_0$*.*

If $x = y_0$*, we get* $0 < G(x) - a - C \leq \epsilon - C < 0$ *for* ϵ *small.*

We deduce that $x > y_0$*.*

Writing the viscosity inequality, we get:

$$\begin{aligned} -C\gamma e^{-\gamma(x-y_0)} &\leq V(G(x+1)) + V(G(x+2) + G(x+1)) \\ &\quad - V(G(x)) - V(G(x+1) + G(x)) \end{aligned} \tag{3.8}$$

Now since $x > y_0$ *then* $x + 1 > y_0$ *and* $x + 2 > y_0$*.*

So $\varphi(x) \geq \varphi(x+1)$

$G(x) - a - Ce^{-\gamma(x-y_0)} \geq G(x+1) - a - Ce^{-\gamma(x+1-y_0)}$

$G(x+1) \leq G(x) - Ce^{-\gamma(x-y_0)} + Ce^{-\gamma(x+1-y_0)}$

$G(x+1) \leq G(x) + Ce^{-\gamma(x-y_0)}(e^{-\gamma} - 1)$

and $\varphi(x) \geq \varphi(x+2)$ *gives that* $G(x+2) \leq G(x) + Ce^{-\gamma(x-y_0)}(e^{-2\gamma} - 1)$

Replacing these 2 inequalities in (3.8), we get

$$\begin{aligned} -C\gamma e^{-\gamma(x-y_0)} &\leq V(G(x) + Ce^{-\gamma(x-y_0)}(e^{-2\gamma} - 1) + G(x+1)) - V(G(x+1) + G(x)) \\ &\quad + V(G(x) + Ce^{-\gamma(x-y_0)}(e^{-\gamma} - 1)) - V(G(x)) \end{aligned}$$

Now

$V(G(x)+Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1)+G(x+1))-V(G(x+1)+G(x)) = V'(p)Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1)$

and $V(G(x)+Ce^{-\gamma(x-y_0)}(e^{-\gamma}-1)) - V(G(x)) = V'(q)Ce^{-\gamma(x-y_0)}(e^{-\gamma}-1)$

with $p \in [G(x)+G(x+1)+Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1), G(x)+G(x+1)]$
and $q \in [G(x)+Ce^{-\gamma(x-y_0)}(e^{-\gamma}-1), G(x)]$

$$\begin{aligned}\Rightarrow -C\gamma e^{-\gamma(x-y_0)} &\leq V'(p)Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1) \\ &+ V'(q)Ce^{-\gamma(x-y_0)}(e^{-\gamma}-1)\end{aligned}$$

To get a contradiction, we need

$$V'(p)Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1) + V'(q)Ce^{-\gamma(x-y_0)}(e^{-\gamma}-1) < -C\gamma e^{-\gamma(x-y_0)}$$

i.e $V'(p)(1-e^{-2\gamma}) + V'(q)(1-e^{-\gamma}) > \gamma$

But $V'(p)(1-e^{-2\gamma}) + V'(q)(1-e^{-\gamma}) > (V'(p)+V'(q))(1-e^{-\gamma}) > \gamma$
Since $V'(p)+V'(q) > \frac{\gamma}{1-e^{-\gamma}}$ *by assumption.*

because $p \in [G(x)+G(x+1)+Ce^{-\gamma(x-y_0)}(e^{-2\gamma}-1), G(x)+G(x+1)]$

$$\begin{aligned}p &\leq G(x)+G(x+1) \\ &\leq a+\epsilon+a+\epsilon \\ &= 2a+2\epsilon \quad (G(y) \leq a+\epsilon \quad \forall y \geq y_0)\end{aligned}$$

and

$$\begin{aligned}
p &\geq G(x) + G(x+1) + Ce^{-\gamma(x-y_0)}(e^{-2\gamma} - 1) \\
&\geq a + a + Ce^{-\gamma(x-y_0)}(e^{-2\gamma} - 1) \\
&= 2a + Ce^{-\gamma(x-y_0)}(e^{-2\gamma} - 1) \\
&\geq 2a + \epsilon(e^{-2\gamma} - 1) \\
&\geq 2a - \epsilon
\end{aligned}$$

Since $M > 0 \Rightarrow a + Ce^{-\gamma(x-y_0)} < G(x) \leq a + \epsilon) \Rightarrow Ce^{-\gamma(x-y_0)} < \epsilon$

So

$$2a - \epsilon \leq p \leq 2a + 2\epsilon$$

Same way $q \in [G(x) + Ce^{-\gamma(x-y_0)}(e^{-\gamma} - 1), G(x)]$

$q \leq G(x) \leq a + \epsilon$

and

$$\begin{aligned}
q &\geq G(x) + Ce^{-\gamma(x-y_0)}(e^{-\gamma} - 1) \\
&\geq a + Ce^{-\gamma(x-y_0)}(e^{-\gamma} - 1) \\
&\geq a + \epsilon(e^{-\gamma} - 1) \\
&\geq a - \epsilon
\end{aligned}$$

So

$$a - \epsilon \leq q \leq a + \epsilon$$

Hence we get the desired result by contradiction.

We deduce that $M \leq 0$ *and in particular (3.5).*

2) Since $u'(-\infty) = G(-\infty) = b$*, let* $y_0 < 0$ *such that for all* $y \leq y_0$*,*

$$b - \epsilon \leq G(y+2)$$

But G *decreasing means that we also have* $b - \epsilon \leq G(y)$ *and* $b - \epsilon \leq G(y+1)$*.*
We will prove that $y \leq y_0$*,*

$$G(y) \geq b - Ce^{\gamma(y-y_0)} \tag{3.9}$$

with $C > b - a$*. We define* $\varphi(y) = b - Ce^{\gamma(y-y_0)} - G(y)$
along with $M = \sup_{y \leq y_0}(\varphi(y))$
We will prove that $M \leq 0$*.*
Assume by contradiction that $M > 0$*.*

Using the fact that $G(y) \to b$ *as* $y \to -\infty$*, we deduce that* M *is reached at some point* x*, so* $M = \sup_{y \leq y_0}(\varphi(y)) = \varphi(x)$ *(otherwise we get* $-Ce^{\gamma y} > 0$*)*
First we will prove that $x \neq y_0$*.*
If $x = y_0$*, we get* $0 < b - C - G(x) \leq b - C - a < 0$*.*
We deduce that $x < y_0$*.*
Writing the viscosity inequality, we get:

$$\begin{aligned} -C\gamma e^{\gamma(x-y_0)} &\geq V(G(x+1)) + V(G(x+2) + G(x+1)) \\ &\quad - V(G(x)) - V(G(x+1) + G(x)) \end{aligned} \tag{3.10}$$

We claim that $x + 2 < y_0$*.*

If $x+2 \geq y_0$, *we get*

$$\begin{aligned}
\varphi(x) &= b - Ce^{\gamma(x-y_0)} - G(x) \\
&\leq b + (a-b)e^{-2\gamma} - a \qquad \text{for } C > \frac{b-a}{e^{-2\gamma}} \\
&\leq b + a - b - a \\
&= 0
\end{aligned}$$

Contradiction as $M > 0$.

So we proceed similarly as above using $\varphi(x) \geq \varphi(x+1)$ *and* $\varphi(x) \geq \varphi(x+2)$

$\varphi(x) \geq \varphi(x+1)$

$b - Ce^{\gamma(x-y_0)} - G(x) \geq b - Ce^{\gamma(x+1-y_0)} - G(x+1)$

$G(x+1) \geq G(x) + Ce^{\gamma(x-y_0)}(1-e^{\gamma})$

and $\varphi(x) \geq \varphi(x+2)$ *gives that* $G(x+2) \geq G(x) + Ce^{\gamma(x-y_0)}(1-e^{2\gamma})$

Replacing these 2 inequalities in (3.10), we get

$$\begin{aligned}
-C\gamma e^{\gamma(x-y_0)} \leq\ & V(G(x) + Ce^{\gamma(x-y_0)}(1-e^{2\gamma}) + G(x+1)) - V(G(x+1) + G(x)) \\
& + V(G(x) + Ce^{\gamma(x-y_0)}(1-e^{\gamma})) - V(G(x))
\end{aligned}$$

Now

$V(G(x)+Ce^{\gamma(x-y_0)}(1-e^{2\gamma})+G(x+1)) - V(G(x+1)+G(x)) = V'(p)Ce^{\gamma(x-y_0)}(1-e^{2\gamma})$

and $V(G(x) + Ce^{\gamma(x-y_0)}(1-e^{\gamma})) - V(G(x)) = V'(q)Ce^{\gamma(x-y_0)}(1-e^{\gamma})$

with $p \in [G(x) + G(x+1) + Ce^{\gamma(x-y_0)}(1-e^{2\gamma}), G(x) + G(x+1)]$

and $q \in [G(x) + Ce^{\gamma(x-y_0)}(1-e^{\gamma}), G(x)]$

$$\Rightarrow -C\gamma e^{\gamma(x-y_0)} \leq V'(p)Ce^{\gamma(x-y_0)}(1-e^{2\gamma}) + V'(q)Ce^{\gamma(x-y_0)}(1-e^{\gamma})$$

To get a contradiction, we need

$$V'(p)Ce^{\gamma(x-y_0)}(1-e^{2\gamma}) + V'(q)Ce^{\gamma(x-y_0)}(1-e^{\gamma}) > -C\gamma e^{\gamma(x-y_0)}$$

i.e $V'(p)(e^{2\gamma}-1) + V'(q)(e^{-\gamma}-1) < \gamma$

But $V'(p)Ce^{\gamma(x-y_0)}(1-e^{2\gamma}) + V'(q)Ce^{\gamma(x-y_0)}(1-e^{\gamma}) < (V'(p)+V'(q))(e^{2\gamma}-1) < \gamma$
Since $V'(p) + V'(q) < \frac{\gamma}{e^{2\gamma}-1}$ *by assumption.*

because $p \in [G(x) + G(x+1) + Ce^{\gamma(x-y_0)}(1-e^{2\gamma}), G(x) + G(x+1)]$

$$\begin{aligned} p &\leq G(x) + G(x+1) \\ &\leq b + \epsilon + b + \epsilon \\ &= 2b + 2\epsilon \quad (G(x) \leq b + \epsilon \text{ and } G(x+1) \leq b + \epsilon \text{ using } G(-\infty) = b) \end{aligned}$$

and

$$\begin{aligned} p &\geq G(x) + G(x+1) + Ce^{\gamma(x-y_0)}(1-e^{2\gamma}) \\ &\geq b - \epsilon + b - \epsilon + \epsilon(1-e^{2\gamma}) \quad (G(y) \geq b - \epsilon \ \ \forall y \leq y_0) \\ &= b - \epsilon + b - \epsilon + \epsilon - \epsilon e^{2\gamma} \quad (M > 0 \Rightarrow Ce^{\gamma(x-y_0)} < \epsilon) \\ &\geq 2b - 3\epsilon \quad (\text{for } \gamma \text{ small } \epsilon e^{2\gamma} < 2\epsilon) \end{aligned}$$

So

$$2b - 3\epsilon \leq p \leq 2b + 2\epsilon$$

Same way $q \in [G(x) + Ce^{\gamma(x-y_0)}(1 - e^{\gamma}), G(x)]$

$q \leq G(x) \leq b + \epsilon$

and

$$\begin{aligned} q &\geq G(x) + Ce^{\gamma(x-y_0)}(1 - e^{\gamma}) \\ &\geq b - \epsilon + \epsilon(1 - e^{\gamma}) \\ &\geq b - \epsilon + \epsilon - \epsilon e^{\gamma} \\ &= b - 2\epsilon \qquad \textit{for } \gamma \textit{ small } \epsilon e^{\gamma} < \epsilon e^{2\gamma} < 2\epsilon \end{aligned}$$

So

$$b - 2\epsilon \leq q \leq b + \epsilon$$

Hence we get the desired result by contradiction.

We deduce that $M \leq 0$ *and in particular (3.6).*

□

We know go back to the proof of proposition 3.0.1.

Proof. *Let's prove that* $|u(y) - ay - c_1| \leq ke^{-\gamma y}$ *for* $y \geq 0$.

$$\begin{aligned}
u(x) - u(0) &= \int_0^x u'(s)\ ds \\
&= \int_0^x V(u(s+2) - u(s)) + V(u(s+1) - u(s))\ ds \\
&= \int_0^x V(u(s+2) - u(s+1) + u(s+1) - u(s)) + V(u(s+1) - u(s))\ ds \\
&= \int_0^x V(G(s+1) + G(s)) + V(G(s))\ ds \\
&\leq \int_0^x V(a + Ce^{-\gamma(s+1)} + a + Ce^{-\gamma s}) + V(a + Ce^{-\gamma s})\ ds \\
&= \int_0^x V(2a + Ce^{-\gamma s}(1 + e^{-\gamma})) + V(a + Ce^{-\gamma s})\ ds \\
&= \int_0^x V(2a + Ce^{-\gamma s}(1 + e^{-\gamma})) - V(2a) + V(2a) + V(a + Ce^{-\gamma s}) - V(a) \\
&+ V(a)\ ds \\
&\leq ||V'||_\infty C(1 + e^{-\gamma}) \int_0^x e^{-\gamma s}\ ds + ||V'||_\infty C \int_0^x e^{-\gamma s}\ ds + ax \\
&= (2 + e^{-\gamma})||V'||_\infty C \int_0^x e^{-\gamma s}\ ds + ax \\
&= ||V'||_\infty \frac{C}{\gamma}(1 - e^{-\gamma x})(2 + e^{-\gamma}) + ax
\end{aligned}$$

Now

$u(x) - ax - u(0) - ||V'||_\infty \frac{C}{\gamma}(2 + e^{-\gamma}) \leq -||V'||_\infty \frac{C}{\gamma}(2 + e^{-\gamma})e^{-\gamma x}$

We get $|u(y) - ay - C_1| \leq ke^{-\gamma y}$ *for* $y \geq 0$

We proceed same way to prove that $|u(y) - by - C_2| \leq ke^{\gamma y}$ *for* $y \leq 0$ □

Proposition 3.0.3 *Let* u *be a solution of (1.1) such that* u *satisfies*

$$|u(y) - \bar{u}(y)| \leq C$$

with $\bar{u}(y) = \min(ay, by)$ *with* $(a < b)$.

Assume that u *is concave and that* $u'(-\infty) = b$ *and* $u'(+\infty) = a$

There exists constants $\alpha,\ \beta$ *such that* $\tilde{u}$ *is a solution of (1.1) with* $\lim_{|y| \to +\infty} (\tilde{u}(y) - \bar{u}(y)) = 0$ *and* $\tilde{u}(y) = \alpha + u(y + \beta)$.

Proof. *We want to define* $\tilde{u}(y) = \alpha + u(y+\beta)$ *such that* $\tilde{u}$ *is a solution and* $\tilde{u}(y) - \bar{u}(y) \to 0$ *as* $y \to \pm\infty$.

First, let's prove that $\tilde{u}$ *is a solution.*

$$\begin{aligned}\tilde{u}' &= u'(y+\beta)\\ &= V(u(y+\beta+2) - u(y+\beta)) + V(u(y+\beta+1) - u(y+\beta))\\ &= V(\tilde{u}(y+2) - \tilde{u}(y)) + V(\tilde{u}(y+1) - \tilde{u}(y))\end{aligned}$$

Second, let's prove that $\bar{u}(y) - \tilde{u}(y) \to 0$ *as* $y \to \pm\infty$.

Let $\phi(y) = u(y) - \bar{u}(y)$ *with* $\bar{u}(y) = \begin{cases} ay & y > 0 \\ by & y < 0 \end{cases}$

We have $|\phi(y)| = |u(y) - \bar{u}(y)| \leq C$

This implies that the limit of ϕ *is not infinity as* $y \to \pm\infty$ *but we still need to prove that the limit exists.*

Case 1: $y \to +\infty$

If $y > 0$, $\phi''(y) = u''(y) - 0 \leq 0$

1) Assume $\exists\ y_0$ *s.t* ϕ *has a local maximum at* y_0. *We have that* ϕ *is decreasing for* $y \geq y_0$.

By contradiction, assume that $\exists y_1 > y_0$ *s.t* ϕ *has a local minimum at* y_1 *with* $\phi(y) > \phi(y_1)$, $y \geq y_1$.

This gives that $\phi'(y) \leq 0$ *for* $y \leq y_1$ *and* $\phi'(y) > 0$ *for* $y > y_1$.

$\Rightarrow \phi'' > 0$ *on a neighborhood of* y_1. *Contradiction since* $\phi'' < 0$

then $\phi(y) = \phi(y_0) \qquad \forall y \geq y_0$

2) Similarly, assume $\exists\ y_0$ *s.t* ϕ *has a local minimum at* y_0*. We get also that* $\phi(y) = \phi(y_0) \qquad \forall y \geq y_0$.

1) and 2) implies that ϕ *is decreasing after a point* $y_0 > 0$.

$\Rightarrow \phi$ *decreasing and bounded implies that* $\phi(+\infty)$ *exists.*

$\Rightarrow \lim_{y\to+\infty} \phi(y) = \lim_{y\to+\infty} (u(y) - \bar{u}(y)) = c^+$

Case 2: Arguing as above, we get that $\lim_{y\to-\infty} (u(y) - \bar{u}(y)) = c^-$

Now

$$\begin{aligned}
&\lim_{y\to+\infty} (\tilde{u}(y) - \bar{u}(y)) = \\
&\lim_{y\to+\infty} (\alpha + h(y+\beta) - ay) = \\
&\lim_{y\to+\infty} (\alpha + h(y+\beta) - a(y+\beta) = a\beta) = \\
&\alpha + c^+ + a\beta = \\
&0
\end{aligned}$$

Similarly, $\alpha + c^- + b\beta = 0$ *where* (α, β) *solves the system* $\begin{cases} \alpha + a\beta = -c^+ \\ \alpha + b\beta = -c^- \end{cases}$

□

In this last proposition, we show how to construct a particular solution of (1.1). This result will be used to prove Theorem 4.3.1.

Chapter 4

Classification of the Solution

In this chapter, we prove our first main result, Theorem 4.3.1. First we prove that the bounded interdistance is monotone.
Let u be a solution of (1.1). For $y \in \mathbb{R}$, we define the function

$$G(y) = u(y+1) - u(y) \tag{4.1}$$

The function G satisfies

$$G'(y) = V(G(y+1)) + V(G(y+2) + G(y+1)) - V(G(y)) - V(G(y+1) + G(y)) \tag{4.2}$$

4.1 Monotonicity of the Interdistance

Lemma 4.1.1 *Let $y_0 \in \mathbb{R}$,*
We have

$$\begin{cases} G(y) \leq G(y_0) & \forall y \geq y_0 \quad \text{or} \\ G(y) \geq G(y_0) & \forall y \geq y_0 \end{cases}$$

Proof. *We will prove the first inequality, the second one can be done similarly.*
Let $y_0,\ y_1 \in \mathbb{R}$ such that $y_1 > y_0$ and

$$G(y_1) \leq G(y_0) \tag{4.3}$$

We need $\forall y > y_0,\ G(y) \le G(y_0)$.

Let $M = \sup\limits_{y \ge y_0}\{G(y) - G(y_0) - \eta\}$ *for* $\eta > 0$

Need $M \le 0$

Assume that $M > 0$.

First,

$$\begin{aligned}G'(y) &= u'(y+1) - u'(y)\\ &= V(u(y+2) - u(y+1)) + V(u(y+3) - u(y+1)) - V(u(y+1) - u(y))\\ &- V(u(y+2) - u(y))\\ &= V(G(y+1)) + V(u(y+3) - u(y+2) + u(y+2) - u(y+1)) - V(G(y))\\ &- V(u(y+2) - u(y+1) + u(y+1) - u(y))\\ &= V(G(y+1)) + V(G(y+2) + G(y+1)) - V(G(y)) - V(G(y+1) + G(y))\end{aligned}$$

Case 1: Assume that the supremum M *is reached at a certain* $\bar{y} > y_0$.

Then $M = G(\bar{y}) - G(y_0) - \eta$

Now writing the viscosity inequality, we get that:

$0 \le V(G(\bar{y}+1)) + V(G(\bar{y}+2) + G(\bar{y}+1)) - V(G(\bar{y})) - V(G(\bar{y}+1) + G(\bar{y}))$

But since $M = \sup\limits_{y \ge y_0}\{G(y) - G(y_0) - \eta\} = G(\bar{y}) - G(y_0) - \eta$

$\Rightarrow G(\bar{y}) - G(y_0) - \eta \ge G(y) - G(y_0) - \eta \quad \forall y \ge y_0$ *in particular for* $y = \bar{y}+1$,

$\Rightarrow G(\bar{y}) - G(y_0) \ge G(\bar{y}+1) - G(y_0)$

$\Rightarrow G(\bar{y}) \ge G(\bar{y}+1) \Rightarrow V(G(\bar{y}+1)) - V(G(\bar{y})) \le 0$

Then $0 \le V(G(\bar{y}+2) + G(\bar{y}+1)) - V(G(\bar{y}+1) + G(\bar{y}))$

$\Rightarrow V(G(\bar{y}+1) + G(\bar{y})) \le V(G(\bar{y}+2) + G(\bar{y}+1))$

But since V *is increasing,* $G(\bar{y}+1) + G(\bar{y}) \le G(\bar{y}+2) + G(\bar{y}+1)$

$\Rightarrow G(\bar{y}) \le G(\bar{y}+2)$ *but* $G(\bar{y}+2) \le G(\bar{y})$ *By definition of* M,

then $G(\bar{y}) = G(\bar{y}+2) \Rightarrow M = G(\bar{y}+2) - G(y_0) - \eta$

Continuing in the same way, we construct a sequence $y_n = \bar{y} + 2n$ *such that* $M = G(y_n) - m - \eta$

We define the following function

$$G_n(y) = G(y + y_n) - m$$

Using the fact that G is bounded lipschitz continuous function, then (up to passing to the limit on a subsequence),

$$G_n \to G_\infty$$

The stability of viscosity solutions imply that G_∞ solves (3.2). In addition, using the definition of M, we also have for $y \in \mathbb{R}$, $G_\infty(0) \geq G_\infty(y)$.

Since let $\bar{y} \in \mathbb{R}$. We will prove that $\exists\ n_0 \in \mathbb{N}$ s.t $\forall\ n \geq n_0,\ G_n(0) \geq G_n(\bar{y})$.

$G_n(0) = G(y_n) - G(y_0)$ *and* $G_n(\bar{y}) = G(\bar{y} + y_n) - G(y_0)$

Now since $M = \sup_{y \geq y_0}\{G(y) - G(y_0) - \eta\}$ and since $\bar{y} + y_n > y_0$ (as y_n is increasing)

So $G(y_n) - G(y_0) - \eta \geq G(\bar{y} + y_n) - G(y_0) - \eta$

then $G_n(0) \geq G_n(\bar{y})$

Taking $n \to +\infty$, $G_\infty(0) \geq G_\infty(y)$.

Using the strict comparison principle, we get for all $y \in \mathbb{R}$,

$$G_\infty(y) = G_\infty(0) \geq \eta > 0$$

But (4.3) implies that $G(y_1) \leq G(y_0)$

$\Rightarrow G(y_1 - y_n + y_n) - G(y_0) \leq 0$

$\Rightarrow G_n(y_1 - y_n) \leq 0$

Taking $n \to +\infty$, we get $G_\infty(0) \leq 0$ a contradiction.

We deduce that $M \leq 0$. Sending η to 0, we get the desired result.

Case 2: M is not reached. In this case, there exists a sequence $y_n \to +\infty$ such *that*

$$G(y_n) - G(y_0) - \eta \to M \tag{4.4}$$

We define $G_n(y) = G(y + y_n) - G(y_0) - \eta$

Up to a subsequence, we have $G_n \to G_\infty$. *Assume that (4.4) holds. This implies for all* $y \in \mathbb{R}$, $G_\infty(0) \geq G_\infty(y)$.

Using the strict comparison principle,

$$G_\infty(x) = G_\infty(0) > 0$$

This contradicts (4.3) because $G(y_1) - G(y_0) < 0$

$\Rightarrow G(y_1 - y_n + y_n) - G(y_0) < 0$

$\Rightarrow G_n(y_1 - y_n) < 0$

Taking $n \to +\infty$, $G_\infty(0) < 0$.

We deduce that $M \leq 0$.

The second case is treated similarly and hence we get that G is monotone. □

4.2 Strict Monotony of the Interdistance

Proposition 4.2.1 *Let* $G \in C^1(\mathbb{R})$ *be bounded function such that G solution of (3.2). We have that*

$$\begin{cases} G' > 0 \text{ on } \mathbb{R} \text{ or} \\ G' < 0 \text{ on } \mathbb{R} \text{ or} \\ G' = 0 \text{ on } \mathbb{R} \end{cases}$$

Proof. We have already proved that G is monotone on $\mathbb{R}$ *which means* $G' \geq 0$ *on* $\mathbb{R}$ *or* $G' \leq 0$ *on* $\mathbb{R}$ *or* $G' = 0$ *on* $\mathbb{R}$.

Now we need strict monotony.

Assume that $G' \geq 0$ *on* $\mathbb{R}$*. We will show that* $G' > 0$ *on* $\mathbb{R}$ *or* $G' = 0$ *on* $\mathbb{R}$*.*

Step 1: Constant left implies constant right.

Assume that there exists $r > 0$ *and* $y_0 \in \mathbb{R}$ *such that* $G(y) = G(y_0)$ *for* $y \in]y_0 - r, y_0[$*. We will prove that* $G(y) = G(y_0)$ *for all* $y \geq y_0$*.*

Let $y_1 \in]y_0 - r, y_0[$*. We have* $G(y_1) = G(y_0)$ *which can be written as* $G(y_0) \leq G(y_1)$*.*

Now using the previous lemma, we obtain for all $y \geq y_1$*,* $G(y) \leq G(y_1)$

which implies $G(y) \leq G(y_0)$*.*

In addition, we have $G(y) \geq G(y_0)$ *for* $y \geq y_0$ *since* G *is assumed to be increasing and thus* $G(y) = G(y_0)$ *for* $y \geq y_0$*.*

Step 2: Conclusion: We need to prove that G *is strictly increasing or* G *is constant. So let us assume by contradiction that* G *is not strictly increasing and* G *is not constant.*

By step 1, this implies that G *has a global maximum at some point* y_0 *with*

$$\begin{cases} G(y) = G(y_0) & \text{if } y \geq y_0 \\ G(y) < G(y_0) & \text{if } y < y_0 \end{cases}$$

Now using the strict comparison principle, since $G(y) \leq G(y_0)$ $\forall y \in \mathbb{R}$*,*

we get $G(y) = G(y_0)$ *for all* $y \in \mathbb{R}$

which gives a contradiction.

Hence $G' > 0$ *or* $G' = 0$*.*

We proceed similarly to prove that $G' < 0$ *or* $G' = 0$*.* □

4.3 First Main Result

Theorem 4.3.1 (Classification of the solution) *Assume (A1) and (A2). Let $u \in C^1(\mathbb{R})$ be a solution of (1.1). If $G(y) = u(y+1) - u(y)$ is bounded, then there exists $a, b \in \mathbb{R}$ such that*

$$\begin{cases} u'(+\infty) = a \\ u'(-\infty) = b \end{cases}$$

Moreover, if $a < b$, then $u''(y) < 0$ and $p \leq V(2p) + V(p),\ \forall\ p \in [a, b]$ with equality if $p = a, b$.

If $a > b$, then $u''(y) > 0$ and $p \geq V(2p) + V(p),\ \forall\ p \in [a, b]$ with equality if $p = a, b$.

If $a = b$, then $u''(y) = 0,\ \forall\ y \in \mathbb{R}$.

Proof. Using proposition 4.2.1 (the strict monotony of G), we deduce that the limit of G at $\pm\infty$ exist, $G(+\infty) = a$ and $G(-\infty) = b$. (Since G is bounded and monotone)
Case A: If $G' < 0$. In this case, $a < b$.
$u'(y) = V(u(y+1) - u(y)) + V(u(y+2) - u(y))$

$$\begin{aligned} u''(y) &= (u'(y+1) - u'(y))V'(u(y+1) - u(y)) + (u'(y+2) - u'(y))V'(u(y+2) - u(y)) \\ &= (u'(y+1) - u'(y))V'(u(y+1) - u(y)) + (u'(y+2) - u'(y+1) + u'(y+1) \\ &- u'(y))V'(u(y+2) - u(y)) \\ &= G'(y)V'(u(y+1) - u(y)) + (G'(y+1) + G'(y))V'(u(y+2) - u(y)) \end{aligned}$$

Since $G' < 0\ \ \forall\ y \in \mathbb{R}$ and V increasing then $u''(y) < 0\ \ \forall\ y \in \mathbb{R}$

Therefore $u'(\pm\infty)$ *exist and* $\begin{cases} u'(+\infty) = G(+\infty) = a \\ u'(-\infty) = G(-\infty) = b \end{cases}$

Moreover, we have

$$\begin{aligned} u'(y) &= V(u(y+1) - u(y)) + V(u(y+2) - u(y)) \\ &= V(u(y+1) - u(y)) + V(u(y+2) - u(y+1) + u(y+1) - u(y)) \\ &= V(G(y)) + V(G(y+1) + G(y)) \end{aligned}$$

Taking y *to* $+\infty$,

$u'(+\infty) = V(G(+\infty)) + V(G(+\infty) + G(+\infty))$

$a = V(a) + V(2a)$

It remains to prove that $p < V(p) + V(2p)$ *if* $p \in (a, b)$.

We define the function $\hat{u}(t, y) = u(t + y)$

So

$$\begin{aligned} \hat{u}_t(t, y) &= u_t(t + y) \\ &= V(u(t + y + 1) - u(t + y)) + V(u(t + y + 2) - u(t + y)) \\ &= V(\hat{u}(t, y + 1) - \hat{u}(t, y)) + V(\hat{u}(t, y + 2) - \hat{u}(t, y)) \end{aligned}$$

We rescale $\hat{u}$ *to get:* $\hat{u}^\varepsilon(t, y) = \varepsilon \hat{u}\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right)$

As ε *goes to zero, we have that* $\hat{u}^\varepsilon \to u^0$ *with* $u^0(t, y) = \begin{cases} a(y + t) & \text{if } y + t \geq 0 \\ b(y + t) & \text{if } y + t < 0 \end{cases}$

because $\hat{u}^\varepsilon_t(t, y) = \hat{u}_t\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right) = u'\left(\frac{t+y}{\varepsilon}\right)$ *and* $\hat{u}^\varepsilon_y(t, y) = \hat{u}_y\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right) = u'\left(\frac{t+y}{\varepsilon}\right)$

Using that u' *is bounded* $\Rightarrow$ $\hat{u}^\varepsilon_t, \hat{u}^\varepsilon_y$ *are bounded. By the Ascoli-Arzela theorem, up to a subsequence,* $\hat{u}^\varepsilon$ *converges locally uniformly towards* u^0.

It remains to show that $\hat{u}^\varepsilon \to u^0$ *with*

$$u^0(t,y) = \begin{cases} a(y+t) & \text{if } y+t \geq 0 \\ b(y+t) & \text{if } y+t < 0 \end{cases}$$

$\hat{u}^\varepsilon(t,y) = \varepsilon \hat{u}\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right) = \varepsilon u\left(\frac{t+y}{\varepsilon}\right)$

We know that $u'(+\infty) = a$ *then for* h *small and* C *big, if* $y > C$, *we have*

$$\left| \frac{u\left(\frac{t+y}{\varepsilon} + \frac{h}{\varepsilon}\right) - u\left(\frac{t+y}{\varepsilon}\right)}{\frac{h}{\varepsilon}} - a \right| < o\left(\frac{h}{\varepsilon}\right)$$

$$\Rightarrow \left| \frac{\varepsilon u\left(\frac{t+y}{\varepsilon} + \frac{h}{\varepsilon}\right) - \varepsilon u\left(\frac{t+y}{\varepsilon}\right)}{\frac{h}{\varepsilon}} - a\varepsilon \right| < o\left(\frac{h}{\varepsilon}\right)\varepsilon$$

$$\Rightarrow \left| \left(\frac{\hat{u}^\varepsilon(t,y+h) - \hat{u}^\varepsilon(t,y)}{h} - a \right) \varepsilon \right| < o\left(\frac{h}{\varepsilon}\right)\varepsilon$$

As $\varepsilon \to 0$, *we get* $\left| \frac{u^0(t,y+h) - u^0(t,y)}{h} - a \right| = 0$ *if* $y+t \geq 0$

$\Rightarrow \forall\ h > 0,\ u^0(t,y+h) - u^0(t,y) = ah$ *so* $u^0(t,y) = a(y+t)$ *if* $y+t \geq 0$

Similarly, $u^0(t,y) = b(y+t)$ *if* $y+t \leq 0$.

Moreover, by stability of viscosity solutions u^0 *is a solution of* $u^0_t = V(u^0_y) + V(2u^0_y)$ *by [31].*

u^0 *solution* $\Rightarrow u^0$ *is a sub-solution and super-solution.*

Using that u^0 *is a sub-solution, if* $u^0(t_0,y_0) = \phi(t_0,y_0)$ *and* $u^0(t,y) \leq \phi(t,y)$, *we have* $\phi_t(t_0,y_0) \leq V(\phi_y(t_0,y_0)) + V(2\phi_y(t_0,y_0))$.

Case 1: $t_0 + y_0 > 0 \Rightarrow u^0(t_0,y_0) = a(y_0+t_0)$ *and* $u^0(t,y) \in C^1$ *if* $t+y > 0$

$\Rightarrow \phi_t(t_0,y_0) = u^0_t(t_0,y_0) = a$ *and* $\phi_y(t_0,y_0) = u^0_y(t_0,y_0) = a$

We obtain $a \leq V(a) + V(2a)$

Case 2: $t_0 + y_0 < 0$. *Similarly, we get* $b \leq V(b) + V(2b)$.

Case 3: $t_0 + y_0 = 0$

For $t + y = 0 \Rightarrow y = -t$

We have $u(t, y) = u(t, -t) = 0$

So $\phi(t, -t) = 0 \Rightarrow \frac{d}{dt}\phi(t, -t) = 0 \Rightarrow \phi_t(t, -t) - \phi_y(t, -t) = 0$

Take $t = t_0 \Rightarrow \phi_t(t_0, y_0) - \phi_y(t_0, y_0) = 0 \Rightarrow \phi_t(t_0, y_0) = \phi_y(t_0, y_0) = p$

then $p \leq V(p) + V(2p)$

We still need to prove that $p \in [a, b]$.

Case 3.1: $h > 0$ *As* $\phi(t_0, y_0) = u(t_0, y_0)$ *then*

$\frac{\phi(t_0,y_0+h)-\phi(t_0,y_0)}{h} \geq \frac{u^0(t_0,y_0+h)-u^0(t_0,y_0)}{h}$

and since $y_0 + t_0 = 0$ *then* $y_0 + t_0 + h > 0$.

This gives us $u^0(t_0, y_0 + h) = a(t_0 + y_0 + h) = ah$

Hence $\frac{\phi(t_0,y_0+h)-\phi(t_0,y_0)}{h} \geq a$

Case 3.2: $h < 0$

$\phi(t_0, y_0 + h) - \phi(t_0, y_0) \geq u^0(t_0, y_0 + h) - u^0(t_0, y_0)$

and since $y_0 + t_0 = 0$ *then* $y_0 + t_0 + h < 0$.

$\phi(t_0, y_0 + h) - \phi(t_0, y_0) \geq b(y_0 + h + t_0) - 0 = bh$

$\frac{\phi(t_0,y_0+h)-\phi(t_0,y_0)}{h} \leq \frac{bh}{h} = b$

Hence $a \leq p \leq b$

And we deduce that $p \leq V(p) + V(2p)$.

Now assume by contradiction that there exists $c \in [a, b]$ *such that* $c = V(2c)+V(c)$.

Using proposition 3.0.3, there exists a solution $\tilde{u}$ *of (1.1) such that*

$$\lim_{|y|\to+\infty} (\tilde{u} - \bar{u}) = 0$$

Using that $V(2c) + V(c) = c$, *we define the following solution of (1.1):* $v(y) = cy$
Moreover, we have $\lim_{|y| \to +\infty} (\tilde{u} - v) = -\infty$.
Also using proposition 2.3.2, we obtain

$$\tilde{u} \le v \quad on \quad \mathbb{R}$$

We also remark using $\lim_{|y| \to +\infty} (\tilde{u} - v) = -\infty$ *that* $m = \min_{\mathbb{R}}(v - \tilde{u})$ *exist.*
We have $\forall\ y \in \mathbb{R}$, $\tilde{u} + m \le v$ *with equality at some point* y_0.
Assume that $\tilde{u}(y_0) + m_1 = v(y_0)$, *then using the strong comparison principle we get that* $\tilde{u}(y) + m_1 = v(y)\ \forall\ y \le y_0$.
Taking $y \to -\infty$ *we get a contradiction.*

Case B: If $G' > 0$ *we proceed same as above.*

□

Chapter 5

Existence and Uniqueness of the Solution

In this chapter, we prove the second main result of this paper, Theorem 5.0.1. The proof will be split into four sections and the idea is to construct the solution for a suitable non-local operator and then to pass to the limit.

Theorem 5.0.1 *i) (Existence). Assume that (A) holds for some* $a, b \in \mathbb{R}$*. There exists a solution* u *of (1.1) satisfying for some constant* $C > 0$,
$|u(y) - \bar{u}(y)| \leq C$ *with*

$$\bar{u}(y) = \begin{cases} ay & \text{if} \quad y \geq 0 \\ by & \text{if} \quad y < 0 \end{cases}$$

Moreover, we have
$u'(+\infty) = a \leq u(y+1) - u(y) \leq b = u'(-\infty)$

ii) (Uniqueness). The solution u *is unique (up to translation and addition of constants) among the solutions* $v \in C^1(\mathbb{R})$ *such that* $|v - \bar{u}| \leq C$ *for some constant* $C > 0$

Proof. *We will be constructing a viscosity solution u of (1.1) such that $G(y) = u(y+1) - u(y)$ is a bounded function on $\mathbb{R}$.*
Using the classification of the solution theorem and assumption (A3)
$(p \leq v(p) + v(2p)$ *for* $p \in \mathbb{R} \Leftrightarrow p \in [a,b]$, *with equality* $\Leftrightarrow p \in \{a,b\})$
We get that G is strictly decreasing and $\begin{cases} G(y) \to a & \text{as } y \to +\infty \\ G(y) \to b & \text{as } y \to -\infty \end{cases}$

with $a = \inf_{\mathbb{R}} G$ *and* $b = \sup_{\mathbb{R}} G$ *so*

$$a \leq G(y) \leq b$$

5.1 Construction of the approximated solution

First, we define the function $\hat{V}$ which is a modification of the function V as

$$\hat{V}(p) = \begin{cases} a & \text{if } p < a \\ V(p) & \text{if } p \in [a,b] \\ b & \text{if } p > b \end{cases}$$

In the rest of the proof, denote $\hat{V}$ by V for simplicity.
We also consider for $R > 0$ the following operator,

$$G_R(x,u,v,w,p) = \psi_R(y)(V(v-u) + V(w-u)) + (1 - \psi_R(y))(V(p) + V(2p))$$

with $\psi_R \in C^\infty(\mathbb{R})$ defined by

$$\psi_R(x) = \begin{cases} 1 \text{ if } |x| < R \\ 0 \text{ if } |x| > R \end{cases}$$

We consider the following equation for $y \in \mathbb{R}$

$$u_R'(y) = G_R(y, u_R(y), u_R(y+1), u_R(y+2), u_R'(y)) \tag{5.1}$$

Proposition 5.1.1 *There exists a viscosity solution u_R of (5.1). Moreover, u_R is lipschitz continuous.*

Proof. Let l be a big positive number greater than R.

We define:

$$u_R^+(y) = \begin{cases} ay & \text{if } y \geq 0 \\ by & \text{if } y < 0 \end{cases} \quad \text{and} \quad u_R^-(y) = \begin{cases} a(y+l) - bl & \text{if } -l < y < 0 \\ by + (a-b)l & \text{if } 0 < y < l \\ by & \text{if } y \leq -l \\ ay & \text{if } y \geq l \end{cases}$$

It is easy to show that u_R^+ is a super solution of (5.1) and that u_R^- is a sub solution of (5.1). Let us show first that u_R^+ is a super solution of (5.1).

Using the definition of viscosity solutions, let $\varphi \in C^1(\mathbb{R})$ and assume that $u_R^+ - \varphi$ has a local minimum at x_0.

We should prove that

$$\varphi'(x_0) \geq \psi_R(x_0)[V(u_R^+(x_0+2) - u_R^+(x_0)) + V(u_R^+(x_0+1) - u_R^+(x_0))]$$
$$+ (1 - \psi_R)(x_0))[V(2\varphi'(x_0) + V(\varphi'(x_0))]$$

***Remark**: If f is differentiable at some point x_1 and f has a local minimum or maximum at x_1 then $f'(x_1) = 0$.*

$$u_R^+(x) = \begin{cases} ax & x \geq 0 \\ ax & x < 0 \end{cases}$$ *and $\varphi - u_R^+$ is differentiable since both are differentiable for $x \neq 0$.*

Case 1: If $x_0 > 0$

$(\varphi - u_R^+)'(x_0) = 0 \Rightarrow \varphi'(x_0) = u_R^{+\prime}(x_0) = a$

and $a \geq \psi_R(x_0)(V(2a)+V(a)) + (1-\psi_R(x_0))(V(2a)+V(a)) = V(2a)+V(a)$ *is satisfied since* $a = V(2a)+V(a)$

Case 2: If $x_0 < -2$

$(\varphi - u_R^+)'(x_0) = 0 \Rightarrow \varphi'(x_0) = {u_R^+}'(x_0) = b$

and $b \geq \psi_R(x_0)(V(2b)+V(b)) + (1-\psi_R(x_0))(V(2b)+V(b)) = V(2b)+V(b)$

is satisfied since $b = V(2b)+V(b)$.

Case 3: $-2 \leq x_0 < 0$

$(\varphi - u_R^+)'(x_0) = 0 \Rightarrow \varphi'(x_0) = {u_R^+}'(x_0) = b$.

and $b \geq \psi_R(x_0)[V(u_R^+(x_0+2)-bx_0)+V(u_R^+(x_0+1)-bx_0)] + (1-\psi_R(x_0))[V(2b)+V(b)]$

Now $x_0+2 \geq 0 \Rightarrow u_R^+(x_0+2) = a(x_0+2)$

but $-1 \leq x_0+1 < 1$

Case 3.1: $-1 \leq x_0+1 < 0$

$b \geq \psi_R(x_0)[V(a-b)x_0+2a)+V(b)] + (1-\psi_R(x_0))[V(2b)+V(b)]$

$\geq \psi_R(x_0)[V(a-b)(-2)+2a)+V(b)] + (1-\psi_R(x_0))[V(2b)+V(b)]$

$\geq \psi_R(x_0)[V(2b)+V(b)] + (1-\psi_R(x_0))[V(2b)+V(b)]$

$= V(2b)+V(b)$ *which is satisfied as* $b = V(2b)+V(b)$

where we used that V *is increasing and* $x_0 \geq -2$.

Case 3.2: $0 < x_0 + 1 < 1$

$$b \geq \psi_R(x_0)[V((a-b)x_0 + 2a) + V((a-b)x_0 + a)] + (1 - \psi_R(x_0))[V(2b) + V(b)]$$
$$\geq \psi_R(x_0)[V(2b) + V(a-b)(-2) + a)] + (1 - \psi_R(x_0))[V(2b) + V(b)]$$
$$\geq \psi_R(x_0)[V(2b) + V(b)] + (1 - \psi_R(x_0))[V(2b) + V(b)]$$
$$= V(2b) + V(b)$$

which is satisfied as $b = V(2b) + V(b)$

where we used that $-a > -b$ *and* $x_0 \geq -2$.

Case 4: $x_0 = 0$

In this case $u_R^{+\prime}(0)$ *does not exist since* u_R^+ *is not differentiable at* 0.

We need to check if:

$\varphi'(0) \geq \psi_R(0)[V(u_R^+(2) - u_R^+(0)) + V(u_R^+(1) - u_R^+(0))] + (1 - \psi_R(0))[V(2\varphi'(0) + V(\varphi'(0))]$

Now using the fact that $u_R^+ - \varphi$ *has a local minimum at* $x_0 = 0$.

Then $(u_R^+ - \varphi)(0) \leq (u_R^+ - \varphi)(x)$ *if* x *close to* 0

$\Rightarrow \varphi(x) - \varphi(0) \leq u_R^+(x) - u_R^+(0)$

$\Rightarrow \varphi(0) - \varphi(x) \leq u_R^+(0) - u_R^+(x)$

But by definition, $\varphi(0) = \lim\limits_{x \to 0} \frac{\varphi(x) - \varphi(0)}{x}$

i) $x \to 0^+$

$\varphi(0) - \varphi(x) \geq u_R^+(0) - u_R^+(x) = ax$

$\Rightarrow \frac{\varphi(x) - \varphi(0)}{x} \leq a$

ii) $x \to 0^-$

$\varphi(x) - \varphi(0) \geq u_R^+(x) - u_R^+(0) = bx$

$\Rightarrow \frac{\varphi(x) - \varphi(0)}{x} \geq b$

We can't find a function φ *of class* C^1 *such that* $u_R^+ - \varphi$ *has a local minimum at*

0.

Then u_R^+ *is a super solution of (3).*

Now let's show that u_R^- *is a sub solution of (3).*

Same way, let $\phi \in C^1(\mathbb{R})$ *and assume that* $u_R^- - \varphi$ *has a local maximum at* x_0*.*

We should prove that:

$$\varphi'(x_0) \leq \psi_R(x_0)[V(u_R^-(x_0+2) - u_R^-(x_0)) + V(u_R^-(x_0+1) - u_R^-(x_0))]$$
$$+ (1 - \psi_R(x_0))[V(2\varphi'(x_0)) + V(\varphi'(x_0))]$$

$$\text{Now } u_R^-(x) = \begin{cases} a(x+l) - bl & \text{if } -l < x < 0 \\ bx + (a-b)l & \text{if } 0 < x < l \\ bx & \text{if } x \leq -l \\ ax & \text{if } x \geq l \end{cases}$$

Case 1: $x_0 > l \Rightarrow \psi_R(x_0) = 0$

$(\varphi - u_R^-)'(x_0) = 0 \Rightarrow \varphi'(x_0) = {u'}_R^-(x_0) = a$

then we need to prove that $a \leq V(2a) + V(a)$

which is satisfied as $a = V(2a) + V(a)$

Case 2: $x_0 < -l \Rightarrow \psi_R(x_0) = 0$

$(\varphi - u_R^-)'(x_0) = 0 \Rightarrow \varphi'(x_0) = {u_R^-}'(x_0) = b$

then we need to prove that $b \leq V(2b) + V(b)$

which is satisfied as $b = V(2b) + V(b)$

Case 3: $x_0 = l$

${u_R^-}'(l)$ *doesn't exist since* u_R^- *is not differentiable at* $-l$*.*

We need to check if:

$$\varphi'(-l) \leq \psi_R(-l)[V(u_R^-(-l+2) - u_R^-(-l)) + V(u_R^-(-l+1) - u_R^-(-l))]$$
$$+ (1 - \psi_R(-l))[V(2\varphi'(-l)) + V(\varphi'(-l))]$$

but as $l >> R$ *then* $\psi_R(-l) = 0$

so it suffices to prove that $\varphi'(l) \leq V(2\varphi'(-l)) + V(\varphi'(-l))$

Now using the fact that $u_R^- - \varphi$ *has a local maximum at* $x_0 = -l$.

then $(u_R^- - \varphi)(-l) \geq (u_R^- - \varphi)(x)$ *if* x *close to* $-l$.

$\Rightarrow \varphi(-l) - \varphi(x) \leq u_R^-(-l) - u_R^-(x)$

$\Rightarrow \varphi(x) - \varphi(-l) \geq u_R^-(x) - u_R^-(-l)$

i) $x \to -l^+$

$\varphi(x) - \varphi(-l) \geq u_R^-(x) - u_R^-(-l) = ax + al - bl + bl = a(x+l)$

$$\Rightarrow \frac{\varphi(x) - \varphi(-l)}{x+l} \leq a$$

ii) $x \to -l^-$

$\varphi(x) - \varphi(-l) \geq u_R^-(x) - u_R^-(-l) = bx + bl = b(x+l)$

$$\Rightarrow \frac{\varphi(x) - \varphi(-l)}{x+l} \geq b$$

We can't find a function φ *of class* C^1 *such that* $u_R^- - \varphi$ *has a local maximum at* $-l$.

Case 4: $x_0 = 0$

Also in this case ${u_R^-}'(0)$ *doesn't exist since* u_R^- *is not differentiable at* 0.

We need to check if:

$$\varphi'(0) \le \psi_R(0)[V(u_R^-(2) - u_R^-(0)) + V(u_R^-(1) - u_R^-(0))]$$
$$+ (1 - \psi_R(0))[V(2\varphi'(0)) + V(\varphi'(0))]$$

but as $-R < 0 < R$ *then* $\psi_R(0) = 1$

so it suffices to prove that $\varphi'(0) \le V(2b) + V(b)$

Now using the fact that $u_R^- - \varphi$ *has a local maximum at* 0.

then $(u_R^- - \varphi)(0) \ge (u_R^- - \varphi)(x)$ *if* x *close to 0*

$\Rightarrow \varphi(x) - \varphi(0) \ge u_R^-(x) - u_R^-(0)$

But by definition, $\varphi'(0) = \lim_{x\to 0} \frac{\phi(x)-\phi(0)}{x}$

i) $x \to 0^+$

$\varphi(x) - \varphi(0) \ge u_R^-(x) - u_R^-(0) = bx + al - bl - (a-b)l = bx$

$$\Rightarrow \frac{\varphi(x) - \varphi(0)}{x} \ge b$$

ii) $x \to 0^-$

$\varphi(x) - \varphi(0) \ge u_R^-(x) - u_R^-(0) = ax + (a-b)l - (a-b)l = ax$

$$\Rightarrow \frac{\varphi(x) - \varphi(0)}{x} \le a$$

We can't find a function φ *of class* C^1 *such that* $u_R^- - \varphi$ *has a local maximum at* 0.

Case 5: $x_0 = l \Rightarrow \psi_R(x_0) = 0$

We just need to check if:

$$\varphi'(l) \leq V(2\varphi'(l)) + V(\varphi'(l))$$

$(u_R^- - \varphi)(l) \geq (u_R^- - \varphi)(x) \qquad \forall x \in \mathbb{R}$

$\Rightarrow \varphi(x) - \varphi(l) \geq u_R^-(x) - u_R^-(l)$

i) $x \to l^+$

$$\Rightarrow \frac{\varphi(x) - \varphi(l)}{x - l} \geq a$$

ii) $x \to 0^-$

$$\Rightarrow \frac{\varphi(x) - \varphi(0)}{x - l} \leq b$$

$\Rightarrow a \leq \varphi'(l) \leq b$

Then

$$\varphi'(l) \leq V(2\varphi'(l)) + V(\varphi'(l))$$

.

Case 6: $0 < x_0 < l$

Case 6.1: $0 < x_0 < R + 1$

$(\varphi - u_R^-)'(x_0) = 0 \Rightarrow \varphi'(x_0) = 0 = u_R^{-\prime}(x_0) = b$

and $u_R^-(x_0) = bx_0 + (a - b)l,\ u_R^-(x_0 + 1) = b(x - 0 + 1) + (a - b)l,\ u_R^-(x_0 + 2) = b(x_0 + 2) + (a - b)l$

This is because $l >> R$ *so* $x_0 < R + 1 < l$ *and* $x_0 + 1 < R + 2 < l$ *and* $x_0 + 2 < R + 3 < l$.

We need

$$\varphi'(x_0) \leq \psi_R(x_0)[V(2b) + V(b)] + (1 - \psi_R(x_0))[V(2b) + V(b)]$$

which is satisfied since $V(2b) + V(b) = b$

Case 6.2: $R + 1 \leq x_0 < l \Rightarrow \psi_R(x_0) = 0$

$\varphi'(x_0) \leq V(2\varphi'(x_0)) + V(\varphi'(x_0))$

$\Rightarrow b \leq V(2b) + V(b)$

Since also here $\varphi'(x_0) = u_R^{-\prime}(x_0) = b$

Case 7: $-l < x_0 < 0 \Rightarrow \varphi'(x_0) = u_R^{-}(x_0) = a$

Case 7.1: $-l < x_0 < -2$

Here $u_R^-(x_0) = a(x_0 + l) - bl,\ u_R^-(x_0 + 1) = a(x_0 + 1 + l) - bl,\ u_R^-(x_0 + 2) = a(x_0 + 2 + l) - bl$

Need $a \leq \psi_R(x_0)(V(2a) + V(a)) + (1 - \psi_R(x_0)(V(2a) + V(a))$

$\Rightarrow a \leq V(2a) + V(a)$ *Since* $a = V(2a) + V(a)$.

Case 7.2 $-2 \leq x_0 < 0 \quad \Rightarrow \psi_R(x_0) = 1$

Case 7.2.1: $-2 \leq x_0 \leq -1 \Rightarrow -1 \leq x_0 + 1 \leq 0$ *and* $0 \leq x_0 + 2 \leq 2$

Here $u_R^-(x_0 + 1) = a(x_0 + 1 + l) - bl$ *and* $u_R^-(x_0 + 2) = b(x_0 + 2 + l) - bl$

Need $a \leq V(b(x_0+2)+(a-b)l-a(x_0+l)+bl)+V(a(x_0+1+l)-bl-a(x_0+l)+bl)$

$\Rightarrow a \leq V((b - a)x_0 + 2b) + V(a)$

but $(b - a)x_0 + 2b \geq -2(b - a) + 2b = 2a$.

Using that V *is increasing we get:* $a \leq V(2a) + V(a)$

Case 7.2.2: $0 \leq x_0 + 1 < 1$ *and* $1 \leq x_0 + 2 < 2$

Here $u_R^-(x_0+1) = b(x_0+1) + (a-b)l$ *and* $u_R^-(x_0+2) = b(x_0+2) + (a-b)l$

Need $a \leq V(b(x_0+2)+(a-b)l-a(x_0+l)+bl)+V(b(x_0+1)+(a-b)l-a(x_0+l)+bl)$

But

$$\begin{aligned} b(x_0+2) + (a-b)l - a(x_0+l) + bl &= (b-a)x_0 + 2b \\ &\geq -b + a + 2b \\ &= b + a \\ &\geq 2a \end{aligned}$$

$b(x_0+1) + (a-b)l - a(x_0+l) + bl \geq -b + a + b = a$

Then $a \leq V(2a) + V(a)$

And u_R^- *is a sub solution.* □

Next by Perron's method [32], we construct a viscosity solution u_R *of (5.1) satisfying for* $y \in \mathbb{R}$,

$$u_R^- \leq u_R \leq u_R^+$$

In particular, from the definition of the sub and super solution, we remark that

$$u_R(y) = \begin{cases} ay & \textit{if } y \geq l \\ by & \textit{if } y \leq -l \end{cases}$$

5.2 Oscillations

Proposition 5.2.1 *Let* u_R *be the solution of (5.1) provided by the previous proposition. Let* $y_0 \in (-1,0)$ *and let* $\gamma, \delta \in (0,1)$ *small enough such that*

$$\begin{cases} \delta < (a-b)y_0 \qquad V'(p) + V'(q) > 1 \textit{ for } p \in [2a, 2a+\delta] \textit{ and } q \in [a, a+\delta] \\ \\ \frac{\gamma(1+\tanh(\gamma))}{\tanh(\gamma)} < 1 \textit{ and } \frac{\gamma(1+\tanh(2\gamma))}{\tanh(2\gamma)} < 1 \end{cases}$$

We have

$$u_R(y) - u_R(y_0) \geq a(y - y_0) + \delta \tanh(\gamma y) \qquad \forall\ y \geq y_0$$

Proof. *We need* $u_R(y) - u_R(y_0) - a(y - y_0) - \delta \tanh(\gamma y) \geq 0$

i.e $u_R(y_0) - u_R(y) + a(y - y_0) + \delta \tanh(\gamma y) \leq 0 \qquad \forall\ y \geq y_0$

Let $\varphi(y) = u_R(y_0) - u_R(y) + a(y - y_0) + \delta \tanh(\gamma y)$

We define $M = \sup_{y \geq y_0} \{u_R(y_0) - u_R(y) + a(y - y_0) + \delta \tanh(\gamma y)\}$

We need $M \leq 0$.

Assume by contradiction that $M > 0$.

Step 1: M *is reached at some point* x.

If $y \geq l,\ u_R(y) = ay$

then

$$\begin{aligned}
&u_R(y_0) - u_R(y) + a(y - y_0) + \delta \tanh(\gamma y) \\
&= u_R(y_0) - ay_0 + \delta \tanh(\gamma y) \\
&\leq u_R^+(y_0) - ay_0 + \delta \\
&= by_0 - ay_0 + \delta \quad \textit{as } y_0 < 0 \\
&= (b - a)y_0 + \delta \\
&< (b - a)y_0 + (a - b)y_0 \quad \textit{as } \delta < (a - b)y_0 \\
&= 0
\end{aligned}$$

Since $u_R(y_0) - u_R(y) + a(y - y_0) + \delta \tanh(\gamma y) < 0$ *for* y *big, then* M *is reached at some point* x *and* $M = \sup_{y \geq y_0}(\varphi(y)) = \varphi(x)$.

Step 2: We need to show that $x \neq y_0$.

If $x = y_0,\ \varphi(y_0) = \delta \tanh(\gamma y_0) = M$

But $\delta\tanh(\gamma y_0) \leq 0$ *as* $y_0 < 0$ *and* $M > 0$ *by assumption so contradiction.*

Step 3: We now have $u_R(y_0) - u_R(y) + a(y - y_0) + \delta\tanh(\gamma y)$ *has a maximum at some* $x > y_0$ *i.e* $u_R(y) - (u_R(y_0) + a(y - y_0) + \delta\tanh(\gamma y))$ *has a minimum at* x*.*

Writing the viscosity super solution inequality, we get:

$$\begin{aligned} a + \delta\gamma(1 - tanh^2(\gamma x)) \geq \psi_R(x)\,&(V(u_R(x+2) - u_R(x)) + V(u_R(x+1) - u_R(x))) \\ &+ (1 - \psi_R(x))\,\big(V(2(a + \delta\gamma(1 - tanh^2(\gamma x)))) \\ &+ V(a + \delta\gamma(1 - tanh^2(\gamma x)))\big) \end{aligned}$$

We have $M = \varphi(x) \geq \varphi(y) \quad \forall\ y \geq y_0$

$\Rightarrow \varphi(x) \geq \varphi(x+1)$ *and* $\varphi(x) \geq \varphi(x+2)$ *which give us*

$u_R(x+1) - u_R(x) \geq a + \delta(\tanh(\gamma(x+1)) - \tanh(\gamma x))$

$u_R(x+2) - u_R(x) \geq 2a + \delta(\tanh(\gamma(x+2)) - \tanh(\gamma x))$

Injecting these 2 inequalities in the one above and using that V *is increasing, we get:*

$$\begin{aligned} a + f(\delta,\gamma) \geq \psi_R(x)\,&(V(a + h(\delta,\gamma) + V(2a + g(\delta,\gamma))) \\ &+ (1 - \psi_R(x))\,(V(2(a + f(\delta,\gamma))) + V(a + f(\delta,\gamma))) \end{aligned}$$

with

$$\begin{aligned} h(\delta,\gamma) &= \delta(\tanh(\gamma(x+1)) - \tanh(\gamma x)) \\ g(\delta,\gamma) &= \delta(\tanh(\gamma(x+2)) - \tanh(\gamma x)) \\ f(\delta,\gamma) &= \delta\gamma(1 - tanh^2(\gamma x)) \end{aligned}$$

To get a contradiction we need:

$$V(a + h(\delta, \gamma) + V(2a + g(\delta, \gamma)) > a + f(\delta, \gamma) \quad (5.2)$$

and

$$V(2(a + f(\delta, \gamma))) + V(a + f(\delta, \gamma)) \quad (5.3)$$

To prove (5.3), it suffices to prove that $a + f(\delta, \gamma) > a$ *because we have* $V(p) + V(2p) > p$ *for* $p \in (a, b)$

but $a + f(\delta, \gamma) = a + \delta\gamma(1 - tanh^2(\gamma x)) > a$ *as* $f(\delta, \gamma) > 0$

then (5.3) is proved.

To prove (5.2), we have:

$V(2a + g(\delta, \gamma)) + V(a + h(\delta, \gamma)$

$= V(2a) + g(\delta, \gamma)V'(p_0) + V(a) + h(\delta, \gamma)V'(q_0)$

$= V(2a) + V(a) + g(\delta, \gamma)V'(p_0) + h(\delta, \gamma)V'(q_0)$

$= a + g(\delta, \gamma)V'(p_0) + h(\delta, \gamma)V'(q_0)$

with $p_0 \in [2a, 2a + g(\delta, \gamma)]$ *and* $q_0 \in [a, a + h(\delta, \gamma)]$

So now we need to get that:

$a + g(\delta, \gamma)V'(p_0) + h(\delta, \gamma)V'(q_0) > a + f(\delta, \gamma)$

i.e

$$\frac{g(\delta, \gamma)}{f(\delta, \gamma)}V'(p_0) + \frac{h(\delta, \gamma)}{f(\delta, \gamma)}V'(q_0) > 1 \quad (5.4)$$

Remark 1 $\tanh(A + B) = \frac{\tanh(A)+\tanh(B)}{1+\tanh(A)\tanh(B)}$

then

$$g(\delta,\gamma) = \delta\left(\frac{\tanh(\gamma x)+\tanh(2\gamma)}{1+\tanh(\gamma x)\tanh(2\gamma)} - \tanh(\gamma x)\right)$$
$$= \delta\frac{\tanh(2\gamma)(1-\tanh^2(\gamma x))}{1+\tanh(\gamma x)\tanh(2\gamma)}$$

Similarly, $h(\delta,\gamma) = \delta\frac{\tanh(\gamma)(1-\tanh^2(\gamma x))}{1+\tanh(\gamma x)\tanh(\gamma)}$

Replacing in (5.4), we deduce that we need $mV'(p_0) + nV'(q_0) > 1$

with $m = \frac{\tanh(2\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(2\gamma))}$ *and* $n = \frac{\tanh(\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(\gamma))}$

that is $V'(p_0) + V'(q_0) > \frac{1}{\min(m,n)}$

Now using the fact that $V'(p) + V'(2p) > 1$ *if* $p > a$ *and* p *is close to* a*, then*

$V'(p_0) + V'(q_0) > 1$ *if* $p_0 > a,\ q_0 > 2a$ *and* $p_0 \sim a,\ q_0 \sim 2a$

$2a \leq p_0 \leq 2a + \delta(\tanh(\gamma(x+2)) - \tanh(\gamma x)) \leq 2a + \delta$

and $a \leq q_0 \leq a + \delta(\tanh(\gamma(x+1)) - \tanh(\gamma x)) \leq a + \delta$

We get by assumption that $V'(p_0) + V'(q_0) > 1$ *since* δ *very small.*

Final thing needed is $1 > \frac{1}{\min(m,n)}$ *that is* $\min(m,n) > 1$

For that it suffices to prove that $m > 1$ *and* $n > 1$*.*

but $\frac{\gamma(1+\tanh(\gamma x)\tanh(\gamma))}{\tanh(\gamma)} \leq \frac{\gamma(1+\tanh(\gamma))}{\tanh(\gamma)} < 1$

and $\frac{\gamma(1+\tanh(\gamma x)\tanh(2\gamma))}{\tanh(2\gamma)} \leq \frac{\gamma(1+\tanh(2\gamma))}{\tanh(2\gamma)} < 1$

Hence going back and replacing we get that:

$$a + \delta\gamma(1 - tanh^2(\gamma x)) > \psi_R(x)\left(a + \delta\gamma(1 - tanh^2(\gamma x))\right)$$
$$+ (1 - \psi_R(x))\left(a + \delta\gamma(1 - tanh^2(\gamma x))\right)$$

$$\Rightarrow a + \delta\gamma(1 - tanh^2(\gamma x)) > a + \delta\gamma(1 - tanh^2(\gamma x)) \qquad Contradiction$$

Hence, we get the desired result. □

Proposition 5.2.2 *Let u_R be the solution of (5.1) provided by the previous proposition. Let $y_0 \in (0,1)$ and let $\gamma, \delta \in (0,1)$ small enough such that*

$$\begin{cases} \delta < b - a \qquad V'(p) + V'(q) < 1 \text{ for } p \in [2b-\delta, 2b] \text{ and } q \in [b-\delta, b] \\ \\ \frac{\tanh(\gamma)}{\gamma(1-\tanh(\gamma))} < 1 \text{ and } \frac{\tanh(2\gamma)}{\gamma(1-\tanh(2\gamma))} < 1 \end{cases}$$

We have the following:

i) Either there exists $y \in (-1,1)$ such that

$$u_R(y+1) - u_R(y) \le b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+1)))$$

or

ii) There exists $y \in (-2,1)$ such that

$$u_R(y+2) - u_R(y) \le b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+2)))$$

or

iii) For all $y_0 \in (0,1)$, we have

$$u_R(y_0) - u_R(y) \le b(y_0 - y) + \delta\tanh(\gamma y) \qquad \forall\, y \le y_0$$

Proof. Assume that i) and ii) are not true i.e

$$\forall\ y \in (-1,1), \quad u_R(y+1) - u_R(y) > b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+1))) \qquad (5.5)$$

and

$$\forall\ y \in (-2,1), \quad u_R(y+2) - u_R(y) > b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+2))) \qquad (5.6)$$

We need to prove that $u_R(y_0) - u_R(y) \leq b(y_0 - y) + \delta \tanh(\gamma y) \qquad \forall\ y \leq y_0$

Let $\varphi(y) = u_R(y_0) - u_R(y) - b(y_0 - y) - \delta \tanh(\gamma y)$

We define $M = \sup_{y \leq y_0} \{\varphi(y)\}$

We need $M \leq 0$.

Assume by contradiction that $M > 0$.

Step 1: M is reached at some point x.

If $y \leq -l,\ u_R(y) = by$

then

$$\begin{aligned}
& u_R(y_0) - u_R(y) - b(y_0 - y) - \delta \tanh(\gamma y) \\
& = u_R(y_0) - by_0 - \delta \tanh(\gamma y) \\
& \leq u_R^+(y_0) - by_0 + \delta \\
& = ay_0 - by_0 + \delta \quad \textit{as } y_0 > 0 \\
& = (a-b)y_0 + \delta \\
& < (a-b) + (b-a) \quad \textit{as } \delta < b - a \textit{ and } y_0 < 1 \\
& = 0
\end{aligned}$$

Since $u_R(y_0) - u_R(y) - b(y_0 - y) - \delta \tanh(\gamma y) < 0$ *for y small, then M is reached* *at some point* x *and* $M = \sup_{y \le y_0}(\varphi(y)) = \varphi(x)$.

Step 2: We need to show that $x \neq y_0$.

If $x = y_0,\ \varphi(y_0) = -\delta \tanh(\gamma y_0) = M$

But $-\delta \tanh(\gamma y_0) \le 0$ *as* $y_0 > 0$ *and* $M > 0$ *by assumption so contradiction.*

Step 3: We now have $u_R(y_0) - u_R(y) - b(y_0 - y) - \delta \tanh(\gamma y)$ *has a maximum at some* $x < y_0$ *i.e* $u_R(y) - (u_R(y_0) - b(y_0 - y) - \delta \tanh(\gamma y)$ *has a minimum at* x. *Writing the viscosity super solution inequality, we get:*

$$
\begin{aligned}
b - \delta\gamma(1 - tanh^2(\gamma x)) \ge \psi_R(x)\left(V(u_R(x+2) - u_R(x)) + V(u_R(x+1) - u_R(x))\right) \\
+ (1 - \psi_R(x))\big(V(2(b - \delta\gamma(1 - tanh^2(\gamma x)))) \\
+ V(b - \delta\gamma(1 - tanh^2(\gamma x))) \quad (6)
\end{aligned}
\tag{5.7}
$$

Case 1: $x + 2 \le y_0 \Rightarrow x + 1 \le y_0 - 1 \le y_0$

In this case, since $M = \varphi(x) \ge \varphi(y) \quad \forall\ y \le y_0$

$\Rightarrow \varphi(x) \ge \varphi(x+1)$ *and* $\varphi(x) \ge \varphi(x+2)$ *which give us*

$u_R(x+1) - u_R(x) \ge b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1)))$

$u_R(x+2) - u_R(x) \ge 2b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+2)))$

Injecting these 2 inequalities in the one above and using that V *is increasing we get:*

$$
\begin{aligned}
b - f(\delta,\gamma) \ge \psi_R(x)\left(V(b - h(\delta,\gamma) + V(2b - g(\delta,\gamma))\right) \\
+ (1 - \psi_R(x))\left(V(2(b - f(\delta,\gamma))) + V(b - f(\delta,\gamma))\right)
\end{aligned}
$$

with

$$h(\delta,\gamma) = \delta(\tanh(\gamma(x+1)) - \tanh(\gamma x))$$
$$g(\delta,\gamma) = \delta(\tanh(\gamma(x+2)) - \tanh(\gamma x))$$
$$f(\delta,\gamma) = \delta\gamma(1 - tanh^2(\gamma x))$$

To get a contradiction we need:

$$V(b - h(\delta,\gamma) + V(2b - g(\delta,\gamma)) > b - f(\delta,\gamma) \tag{5.8}$$

and

$$V(2(b - f(\delta,\gamma))) + V(b - f(\delta,\gamma)) > b - f(\delta,\gamma) \tag{5.9}$$

To prove (5.9), it suffices to prove that $b - f(\delta,\gamma) < b$ *because we have* $V(p) + V(2p) > p$ *for* $p \in (a, b)$
but $b - f(\delta,\gamma) = b - \delta\gamma(1 - tanh^2(\gamma x)) < b$ *as* $f(\delta,\gamma) > 0$
then (5.9) is proved.

To prove (5.8), we have:
$V(2b - g(\delta,\gamma)) + V(b - h(\delta,\gamma)$
$= V(2b) - g(\delta,\gamma)V'(p_0) + V(b) - h(\delta,\gamma)V'(q_0)$
$= V(2b) + V(b) - g(\delta,\gamma)V'(p_0) - h(\delta,\gamma)V'(q_0)$
$= b - g(\delta,\gamma)V'(p_0) - h(\delta,\gamma)V'(q_0)$

with $p_0 \in [2b - g(\delta,\gamma), 2b]$ *and* $q_0 \in [b - h(\delta,\gamma), b]$

So now we need to get that:

$b - g(\delta,\gamma)V'(p_0) - h(\delta,\gamma)V'(q_0) > b - f(\delta,\gamma)$

i.e

$$\frac{g(\delta,\gamma)}{f(\delta,\gamma)}V'(p_0) + \frac{h(\delta,\gamma)}{f(\delta,\gamma)}V'(q_0) < 1 \tag{5.10}$$

Using the expressions of $g(\delta,\gamma)$ *and* $h(\delta,\gamma)$ *from the previous proposition and replacing them in (5.10), we deduce that we need* $mV'(p_0) + nV'(q_0) < 1$

with $m = \frac{\tanh(2\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(2\gamma))}$ *and* $n = \frac{\tanh(\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(\gamma))}$

that is $V'(p_0) + V'(q_0) < \frac{1}{\max(m,n)}$

Now $2b - \delta \leq 2b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+2))) \leq p_0 \leq 2b \Rightarrow p_0 \in [2b-\delta, 2b]$
and $b - \delta \leq b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1))) \leq q_0 \leq b \Rightarrow q_0 \in [b-\delta, b]$

We get by assumption that $V'(p_0) + V'(q_0) < 1$ *since* δ *very small.*

Final thing needed is $1 < \frac{1}{\max(m,n)}$ *that is* $\max(m,n) < 1$
For that it suffices to prove that $m < 1$ *and* $n < 1$.
but $\frac{\tanh(\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(\gamma))} \leq \frac{\tanh(\gamma)}{\gamma(1-\tanh(\gamma))} < 1$
and $\frac{\tanh(2\gamma)}{\gamma(1+\tanh(\gamma x)\tanh(2\gamma))} \leq \frac{\tanh(2\gamma)}{\gamma(1-\tanh(2\gamma))} < 1$

Hence going back and replacing we get that:

$$\begin{aligned} b - \delta\gamma(1 - tanh^2(\gamma x)) &> \psi_R(x)\left(b - \delta\gamma(1 - tanh^2(\gamma x))\right) \\ &\quad + (1 - \psi_R(x))\left(b - \delta\gamma(1 - tanh^2(\gamma x))\right) \end{aligned}$$

$\Rightarrow b - \delta\gamma(1 - tanh^2(\gamma x)) > b - \delta\gamma(1 - tanh^2(\gamma x))$ *Contradiction*

Hence, we get the desired result.

Case 2: $x + 1 > y_0$

In this case $-2 < -1 < y_0 - 1 < x < y_0 < 1$, *so we replace in (4.7) (6)*

$$u_R(x+1) - u_R(x) > b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1)))$$

and

$$u_R(x+2) - u_R(x) > b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+2)))$$

and then we proceed similarly to what is done in case 1.

Case 3: $x + 1 \leq y_0 < x + 2$

In this case $-2 < y_0 - 2 < x < y_0 < 1$, *so we replace in (5.7)*

$$u_R(x+2) - u_R(x) > b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+2)))$$

and since $x + 1 \leq y_0$, *we use* $\varphi(x) \geq \varphi(x+1)$ *which implies*

$$u_R(x+1) - u_R(x) \geq b + \delta(\tanh(\gamma x) - \tanh(\gamma(x+1)))$$

that we also replace in (5.7) and we proceed similarly to what is done in case 1. □

5.3 Passing to the limit

We define the following function:

$$\bar{u}_R = u_R(x) - u_R(0)$$

Using Ascoli-Arzela Theorem, and up to a sub-sequence we have locally uniformly $\bar{u}_R \to u$ *as* $R \to +\infty$. *The stability of viscosity solutions implies that* u *is a solution of (1.1).*

Using step 1, we have

$$a \leq G(x) = u(x+1) - u(x) \leq b$$

and using step 2, we have

$$\text{for } y_0 \in (-1,0), \qquad u(y_0+1) - u(y_0) \geq a + \delta \tanh(\gamma(y_0+1)) > a$$

and

$$i) \text{ for all } y \in (-1,1), \qquad u(y+1) - u(y) \leq b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+1))) < b$$

or

$$ii) \text{ for all } y \in (-2,1), \qquad u(y+2) - u(y) \leq b + \delta(\tanh(\gamma y) - \tanh(\gamma(y+2))) < b$$

or

$$iii) \text{ for } y_0 \in (0,1), \qquad u(y_0) - u(y_0-1) \leq b + \delta \tanh(\gamma(y_0-1)) < b$$

Using the classification of the solution theorem, we have that

$$\tilde{a} = u'(+\infty) = G(+\infty) \leq \tilde{b} = u'(-\infty) = G(-\infty) \text{ with } \tilde{a},\ \tilde{b} \in \{a, b\} \tag{5.11}$$

If we have equality in (5.11), we deduce using the proposition of the strict monotony of G *that* G *is constant which contradicts the inequalities deduced above.*
This implies that $a = u'(+\infty) = G(+\infty) < b = u'(-\infty) = G(-\infty)$.

Finally, using proposition 3.0.1, there exists a constant C *such that*

$$|u - \bar{u}| \leq C$$

5.4 Uniqueness

The uniqueness is a direct consequence of the following proposition.

Proposition 5.4.1 *Assume that* $V \in C^1(\mathbb{R}),\ V' \in L^\infty(\mathbb{R})$ *and* $V' > 0$ *on* $\mathbb{R}$.
Let $a < b$ *and* u *solution of (1.1) with* $u \in C^1(\mathbb{R})$ *such that* u *is concave and satisfying* $|u - \bar{u}| \leq C$ *then* u *is unique up to translation and addition of constants on* $\mathbb{R}$.

Proof. By proposition 3.0.3, we construct a solution $\tilde{u}$ *of (1.1) satisfying*

$$\lim_{|y| \to +\infty} (\tilde{u}(y) - \bar{u}(y)) = 0$$

Hence, $\tilde{u}(y) = \bar{u}(y)$ *on* $\mathbb{R}$. □

Hence, we have proved the existence and the uniqueness of the solution.

□

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