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CHARACTERIZATION OF THE STATE CONSTRAINED BILATERAL MINIMAL TIME FUNCTION

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We characterize the state constrained bilateral minimal time function as the unique proximal solution of a partial Hamilton-Jacobi equations with certain boundary conditions. This generalizes [13, Theorem 3.4] where Stern studied the unilateral case.

Keywords: Bilateral and unilateral minimal time function; State constraint, Hamilton-Jacobi equations; Proximal analysis; Nonsmooth analysis

1. Introduction

Let F be a multifunction mapping points x in \mathbb{R}^n to subsets $F(x)$ of \mathbb{R}^n and let $S \subset \mathbb{R}^n$ a closed set. Associated with F is the differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad a.e. \quad t \in [0, T], \quad x(0) = x_0. \quad (1)$$

A solution to (1) is an absolutely continuous function $x(\cdot)$ defined on the interval $[0, T]$ with initial value $x(0) = x_0$, in which case we say that $x(\cdot)$ is a trajectory of F that originates from x_0 . The notation $\dot{x}(t)$ refers to the derivative of $x(\cdot)$ at t and is the right derivative if $t = 0$.

We assume throughout this paper that F satisfies the *standing hypotheses*; that is, F takes nonempty compact convex values, has closed graph, and satisfies a linear growth condition: for some positive constants γ and c , and for all $x \in \mathbb{R}^n$,

$$v \in F(x) \implies \|v\| \leq \gamma\|x\| + c.$$

The multivalued function F is also taken to be locally Lipschitz: every $x \in \mathbb{R}^n$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$

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such that

$$x_1, x_2 \in U \implies F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

We associate with F the following function h , the *lower Hamiltonian*:

$$h(x, p) := \min\{\langle p, v \rangle : v \in F(x)\}.$$

Now let $S \subset \mathbb{R}^n$ a closed set. A trajectory $x(t)$ of F over $[0, T]$ which satisfies the state constraint $x(t) \in S$ for all $t \in [0, T]$ is called an S -trajectory of F . For $\Sigma \subset S$, a closed set, the S -constrained (unilateral) minimal time function (associated with Σ) $T_S(\cdot, \Sigma) : S \rightarrow \mathbb{R}^n \cup \{+\infty\}$ is defined as follows: For $\alpha \in S$, $T_S(\alpha, \Sigma)$ is the minimum time taken by an S -trajectory to go from α to Σ (when no such trajectory exists, $T_S(\alpha, \Sigma)$ is taken to be $+\infty$). We assume throughout this note that $T_S(\cdot, \Sigma)$ is extended to be $+\infty$ on S^c , the complement of S . When $S = \mathbb{R}^n$ (unconstrained case) then $T_S(\cdot, \Sigma)$ coincides with the well known minimal time function (denoted by $T(\cdot, \Sigma)$) which has a large literature; see for example Refs. 4,10,11,14,15 for the regularity study of $T(\cdot, \Sigma)$, and Refs. 1,3,6,7,12,15 for its Hamilton-Jacobi characterization. We also invite the reader to see [2, Chapter 4] for a thorough history of such results.

The Hamilton-Jacobi characterization of the state constrained (unilateral) minimal time function is studied (apparently for the first time) in Stern¹³ where a characterization of this function is given as the unique proximal solution of Hamilton-Jacobi equations with certain boundary conditions. Let us recall the main result of this paper. We begin with some definitions. Our reference will be the book Clarke *et al.*⁵ and the paper Stern¹³. For a lower semicontinuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom } f := \{x' : f(x') < +\infty\}$, we denote by $\partial_P f(x)$ the *proximal subdifferential* of f at x . We recall that $\zeta \in \partial_P f(x)$ if and only if there exists $\sigma = \sigma(x, \zeta) \geq 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x \rangle,$$

for all y in a neighborhood of x .

We say that S is *wedged* provided that at each boundary point x one has pointedness of $N_S^C(x)$; that is, $N_S^C(x) \cap \{-N_S^C(x)\} = \{0\}$, where $N_S^C(x)$ denotes the *Clarke normal cone* to S at x . We note that wedgedness implies S is the closure of its interior, and that any closed convex body is wedged. We also say that *strict inwardness* condition is satisfied if

$$h(x, \zeta) < 0 \quad \forall \zeta \in N_S^C(x), \quad \forall x \in \text{bdry } S.$$

A function $\varphi : \mathbb{R}^n \rightarrow]-\infty, +\infty]$ is said to be (Σ, S) -continuous if there exist $\gamma_\varphi > 0$ and a function $\omega_\varphi : [0, \gamma] \rightarrow [0, +\infty[$ such that

$$\lim_{s \downarrow 0} \omega(s) = 0 \text{ and } \varphi(x) \leq \omega_\varphi(d_\Sigma(x)) \quad \forall x \in S \cup \{\Sigma + \gamma_\varphi B\},$$

where $d_\Sigma(\cdot)$ is the Euclidean distance function and B is the unit open ball. If $T_S(\cdot, \Sigma)$ is (Σ, S) -continuous then we say that F is S -constrained small time controllable. It is well known that the S -constrained small time controllability condition is equivalent to the continuity of $T_S(\cdot, \Sigma)$ on $S \cap \{\Sigma + \nu B\}$ for some $\nu > 0$. Now we can state the main result of Stern¹³.

Theorem 1.1. *Assume that S is compact and wedged, and that the strict inwardness condition holds. Assume also that F is S -constrained small time controllable. Then the function $T_S(\cdot, \Sigma)$ is the unique lower semicontinuous function $\varphi : \mathbb{R}^n \rightarrow]-\infty, +\infty]$ which is (Σ, S) -continuous, bounded below on \mathbb{R}^n , identically 0 on Σ , identically $+\infty$ on S^c and satisfies*

- $h(x, \partial_P \varphi(x)) + 1 = 0$ for all $x \in \{\text{int } S\} \setminus \Sigma$,
- $h(x, \partial_P \varphi(x)) + 1 \geq 0$ for all $x \in \{\text{int } S\} \cap \Sigma$, and
- $h(x, \partial_P \varphi(x)) + 1 \leq 0$ for all $x \in \{\text{bdry } S\} \setminus \Sigma$,

The purpose of this note is to generalize the preceding result to the bilateral case, that is, find a Hamilton-Jacobi characterization for the S -constrained bilateral minimal time function $T_S : S \times S \rightarrow [0, +\infty]$ defined by the following: For $(\alpha, \beta) \in S \times S$, $T_S(\alpha, \beta)$ is the minimum time taken by a S -trajectory to go from α to β (when no such trajectory exists, $T_S(\alpha, \beta)$ is taken to be $+\infty$). We note that the unconstrained bilateral case is studied in Nour⁸ where we give a Hamilton-Jacobi characterization and some regularity results.

In the next section we present our main result and then we sketch its proof. More details can be found in the forthcoming paper Nour & Stern⁹.

2. Main result

We begin with the following definition. We say that a function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow]-\infty, +\infty]$ is *bilateral S -continuous* if there exist $\gamma_\varphi > 0$ and a function $\omega_\varphi : [0, \gamma_\varphi] \rightarrow [0, +\infty[$ such that $\lim_{s \downarrow 0} \omega_\varphi(s) = 0$ and

$$\varphi(x, y) \leq \omega_\varphi(\|x - y\|) \quad \forall \alpha \in S, \quad \forall x, y \in (\alpha + \gamma_\varphi B) \cap S.$$

If the function $T_S(\cdot, \cdot)$ is bilateral S -continuous then we say that F is *bilateral S -constrained small time controllable*.

Let $D : \{(\alpha, \alpha) : \alpha \in \mathbb{R}^n\}$ be the diagonal set. The following is our main result (here the function $T_S(\cdot, \cdot)$ is also assumed to be $+\infty$ outside $S \times S$).

Theorem 2.1. *Assume that S is compact and wedged, and that the strict inwardness condition holds. Assume also that F is bilateral S -constrained small time controllable. Then the function $T_S(\cdot, \cdot)$ is the unique lower semi-continuous function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow]-\infty, +\infty]$ which is bilateral S -continuous, bounded below on $\mathbb{R}^n \times \mathbb{R}^n$, identically 0 on $D \cap (S \times S)$, identically $+\infty$ on $(S \times S)^c$ and satisfies*

- $\forall \alpha \neq \beta \in \text{int } S$ and $\forall (\zeta, \theta) \in \partial_P \varphi(\alpha, \beta)$ we have:

$$1 + h(\alpha, \zeta) = 0,$$

- $\forall \alpha \in \text{int } S$ and $\forall (\zeta, \theta) \in \partial_P \varphi(\alpha, \alpha)$ we have:

$$1 + h(\alpha, \zeta) \geq 0,$$

- $\forall \alpha \in \text{bdry } S$ and $\forall \beta \in S$ such that $\alpha \neq \beta$ we have:

$$\begin{aligned} \forall (\zeta, \theta) \in \partial_P \varphi(\alpha, \beta), \quad 1 + h(\alpha, \zeta) \leq 0, \text{ and} \\ \forall (\zeta, \theta) \in \partial_P \varphi(\beta, \alpha), \quad 1 + h(\beta, \zeta) \leq 0. \end{aligned}$$

Sketch of the proof. We consider the multifunction \tilde{F} defined over $\mathbb{R}^n \times \mathbb{R}^n$ by $\tilde{F}(x, y) := F(x) \times \{0\}$. For $\Sigma := D \cap (S \times S)$ we can verify that the $S \times S$ -constrained (unilateral) minimal time function for the new dynamic \tilde{F} associated to Σ (denoted by $\tilde{T}_{S \times S}(\cdot, \Sigma)$) coincides with the S -constrained bilateral minimal time function $T_S(\cdot, \cdot)$. Then the result of the theorem follows by applying Theorem 1.1 to $\tilde{T}_{S \times S}(\cdot, \Sigma)$ (which is $T_S(\cdot, \cdot)$) after verifying that:

- We have an $S \times S$ -constrained trajectory tracking result for \tilde{F} (this follows since we already have an S -constrained trajectory tracking result for F and since a trajectory of \tilde{F} is of the form $(x(t), \beta)$ where $x(t)$ is a trajectory of F).
- A function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow]-\infty, +\infty]$ is bilateral S -continuous if and only if it is (Σ, S) -continuous (for \tilde{F}). \square

References

1. M. Bardi, *A boundary value problem for the minimum-time function*, SIAM J. Control Optim., **27** (1989), pp. 776–785.

2. M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, With appendices by Maurizio Falcone and Pierpaolo Soravia. Birkhäuser Boston, Inc., Boston, MA, 1997.
3. M. Bardi and V. Staicu, *The Bellman equation for time-optimal control of noncontrollable, nonlinear systems*, Acta Appl. Math., **31** (1993), pp. 201–223.
4. P. Cannarsa and C. Sinestrari, *Convexity properties of the minimum time function*, Calc. Var. **3** (1995), pp. 273–298.
5. F. H. Clarke, Yu. Ledyev, R. Stern and P. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, 178. Springer-Verlag, New York (1998).
6. F. H. Clarke and C. Nour *The Hamilton-Jacobi equation of minimal time control*, J. Convex Anal., **11** (2004), no. 2, 413–436.
7. C. Nour, *Semigeodesics and the minimal time function*, ESAIM Control Optim. Calc. Var., **12** (2006), 120–138.
8. C. Nour, *The bilateral minimal time function*, J. Convex Anal., **3** (2006), no. 1, 61–80.
9. C. Nour and R. J. Stern, *The state constrained bilateral minimal time function*, in progress.
10. N. N. Petrov, *On the Bellman function for the time process problem*, J. Appl. Math. Mech., **34** (1970), pp. 785–791.
11. P. Soravia, *Pursuit-evasion problems and viscosity solutions of Isaacs equations*, SIAM J. Control Optim., **31** (1993), pp. 604–623.
12. P. Soravia, *Discontinuous viscosity solutions to Dirichlet problems for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations, **18** (1993), pp. 1493–1514.
13. R. J. Stern, *Characterization of the state constrained minimal time function*, SIAM J. Control Optim., **43** (2004), 697–707.
14. V. M. Veliov, *Lipschitz continuity of the value function in optimal control*, Journal of Optimization Theory and Applications., **94** (1997), pp. 335–363.
15. P. Wolenski, Y. Zhuang, *Proximal analysis and the minimal time function*, SIAM J. Control Optim., **36** (1998), pp. 1048–1072.