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Front dynamics of elliptical gravity currents on a uniform slope

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Abstract

In the present investigation, we report data from direct numerical simulations of elliptical, finite release, Boussinesq gravity currents propagating down a uniform slope. The study comprises a series of simulations of elliptical gravity currents on a range of slope angles. The shape parameters are varied to study the effects of the initial cross-sectional aspect ratio ($\Lambda_0$) and mean height to lock radius ratio ($\Gamma$) on the dynamics of the gravity current. It is found that the long-time development of the current spatial mass distribution is influenced by its initial shape at smaller slope angles ($\theta = 5^\circ$ and $10^\circ$) whereas the long-time motion of the gravity current is relatively insensitive to its initial shape but is sensitive to the slope angle. The switching of axes are observed for all the non-circular releases studied in the present work. Multiple acceleration phases are observed for the current centre of mass in the case of the current with a small or moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.1, 0.2, 0.5, 1$ and $2$) whereas one single acceleration phase exists for the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and $10$). The Froude numbers ($Fr$) for the currents released with the same slope angle but different initial shapes are observed to attain a similar constant value after the second acceleration phase. The mean Froude number ($\overline{Fr}$) is seen to increase with increasing slope angles. The mean height to lock radius ratio is found to only affect the early development of the current with little influence on the long-time evolution.

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I. INTRODUCTION

Gravity currents, also known as density currents, are buoyancy driven flows caused by a density difference between a light fluid and a heavy fluid. The density difference may be due to variations of temperature, dissolved materials, or inhomogeneous distribution of suspended particles. The flows are driven by gravity acting on the density difference between the two fluids. The behaviour of a gravity current can be quite complex while the driving force behind the flow is rather simple. Gravity currents often develop within the atmosphere as in the case of thunderstorms, dust storms and sea breeze fronts. Gravity currents can also be found in oceanic and river systems, for example, warm fresh water from a river draining into cold saline oceanic water forms a river plume or a salt wedge estuary; large suspensions of sediment in water leads to turbidity currents. Furthermore, many gravity currents exist in the form of devastating destructive currents, including basaltic lava streams, pyroclastic flows of hot volcanic tephra and snow avalanches. Gravity currents are also important in many industrial applications such as dense gas dispersion, smoke and heat ventilation in buildings and hazardous chemical spillages. These examples show the variety of gravity currents and the importance of studying such flows. A wide range of applications and laboratory studies of gravity currents can be found in [1].

Much of the existing work has concentrated on the problem of a gravity current on a horizontal boundary through one of two canonical configurations, namely planar or axisymmetric release. During a planar release, the heavy fluid is confined between parallel lateral walls resulting in a statistically two-dimensional planar current. As for the axisymmetric case, the heavy fluid is initially confined within a vertical hollow circular cylinder and spreads radially outwards over the entire horizontal boundary forming a statistically axisymmetric current. Therefore, the planform area of a planar current increases linearly with front location as opposed to quadratically for a axisymmetric current, resulting in fundamental difference in the spreading and front velocity decay rates. Many experiments have been performed to study the various aspects of these types of flows, including spreading pattern, rate of spreading and flow structures [2–8]. [9] described the spreading of a gravity current in three phases. There is a first slumping phase, during which the current develops and moves at a constant or nearly constant velocity. This may be followed by a self-similar inertial phase, wherein the buoyancy force is balanced by the inertial force. Finally, a second self-similar
phase, viscous phase, is on-sight when the viscous force becomes significant compared to the buoyancy force. During the inertial and viscous phases, the front Froude number \((Fr)\) was found to be constant. Experiments by [10] and [11] showed that the Froude number is Reynolds number \((Re)\) dependent for \(Re \lesssim O(10^3)\) but essentially constant for \(Re \gtrsim O(10^3)\).

Various analytical models have been developed to predict the bulk motion of the current through detailed theoretical approaches [see 2, 9]. An integral model, also known as box model, was introduced by [9] to describe the current evolution through a series of equal area rectangles and an empirical formula describes the Froude number in terms of the fractional depth \(\phi = h/H\), where the height of the ambient fluid is \(H\), and that of the current is \(h\). Experimentally [see 5, 9], it is found that the Froude number may be expressed as \(Fr = (1/2)\phi^{-1/3}\) when \(0.075 \leq \phi \leq 1\) and \(Fr = 1.19\) for smaller fractional depths \(\phi < 0.075\). [12] and [2] suggested that there exist power law self-similar solutions using the depth averaged shallow water equations, which can describe the front velocity of a gravity current. [13, 14] confirmed these similarity solutions are the large time limits of a class of initial value problems associated with the shallow water equations, so the solutions are valid only sufficiently long after the initial release. The similarity solutions make better predictions for the current height profile compared to the integral models. These mathematical models, despite their simplicity, can often capture the physics of the problem and provide a good estimate for the front position and height of the current.

The study of the dynamics of gravity currents using high resolution numerical simulations has gained considerable momentum with the recent advancement in computational fluid dynamics (CFD) methodologies. Effort to use numerical computations in the investigation of gravity currents can date back to the late 1970s [15]. However, those early numerical studies usually adopted coarse computation grids with lower order numerical schemes due to the limitations of computational power at the time. Direct numerical simulations (DNS) resolve the entire range of spatial and temporal scales of the fluid motion, which provides new insights into the structure and dynamics of gravity currents. It is demonstrated that DNS is capable of not only reproducing the global flow properties observed in the experiments [16–18] but also capturing the detailed flow structure and dynamics of the currents [19–22]. However, many studies found that two-dimensional simulations underpredict the front location and velocity of the current during the inertial and viscous phases as a result of their inherent inability to model the pronounced three-dimensional instabilities [see 16, 17, 21].
In many real situations, the effects of topography can play an important role. A number of studies have investigated gravity currents on a sloping boundary. Earlier investigations of this type of flows were focused on a gravity current confined in a sloping channel. [23] carried out experiments on gravity currents from a continuous source in a sloping channel and found that entrainment was a function of the Richardson number. [24] considered the motion of a dense current flowing down a slope and focused on the downstream region of the flow known as the “head” of the gravity current. They carried out experiments over a wide range of slope angles ($5^\circ \leq \theta \leq 90^\circ$) and observed the front Froude number to be nearly independent of the slope. They also found the size of the head to increase due to entrainment of ambient fluid as well as entrainment of heavy fluid from the trailing body of the current. Two-dimensional buoyant clouds moving along inclined boundaries under a gravitational force were investigated theoretically and experimentally by [25]. By considering the gravity current as an inclined thermal, they developed the classic thermal theory to describe the front velocity and formed the basis for many subsequent studies. The theory was extend to gravity currents on a slope with decreasing buoyancy due to particle settling [26] and increasing buoyancy as a result of the presence of denser sediments [27]. [28] and [29] conducted a series of experiments to re-examine the gravity currents instantaneously released with finite buoyancy on a sloping open channel. The results demonstrated an acceleration phase that extends well beyond the downstream distance given by [25]. The discrepancy was explained by the observation that head of the current is being fed by the following flow that increased its buoyancy as it propagates downstream. In agreement with [28], [30] also found a feeding current from the following fluid that maintains the head velocity. Modified version of the thermal theory of [25] was suggested in [28] and [29] using the measured increase in buoyancy instead of the original assumption of constant buoyancy and found to give results that closely agreed with the experimental measurements. More recently, [31] assessed the validity of classic thermal theory with direct numerical simulations. The prediction based on the theory are shown to be appropriate only for the acceleration phase but not for the entire gravity current motion. More details on the dynamics of gravity currents on different bottom slopes ($0^\circ \leq \theta \leq 9^\circ$) has also been experimentally studied in [32, 33] for both Boussinesq and non-Boussinesq gravity currents. [34, 35] further investigated those currents in the acceleration phase down a slope in a range of $0^\circ \leq \theta < 90^\circ$ using high-resolution two-dimensional numerical simulations.
In comparison, investigations of an instantaneous release on an unconfined slope has been relatively limited. Unlike planar currents on a horizontal or sloping boundary, or axisymmetric currents on a horizontal boundary, a current released on a slope with no lateral boundaries is non-planar and non-axisymmetric, which makes the flow fully three-dimensional. Such a flow configuration has many practical relevance such as snow avalanches and pyroclastic flows spreading downhill. [36] presented a model of the motion of a circular heavy gas cloud down a uniform slope without entrainment using the solutions of shallow water equations. They close the problem by applying a constant Froude number condition at the front of the current with zero height at the rear of the current. The model predicts that the gravity current will assume a self-similar circular wedge shape. The predictions from this wedge integral model are shown to agree adequately well with experimental data in [37] under calm conditions. [38] expanded the circular wedge model of [36] to include the effect of entrainment through a simple model based on the current advection velocity. The entrainment leads to an increase in the size of the current, decrease in the density and deceleration of the cloud frontal speed with distance. However, neither model can accurately predict the shape of the gravity current. Based on the off-axis predictions, [38] suggested that the cloud front may be more curved than predicted by the circular wedge model and that the trailing edge material moves at a lower velocity. [39] performed Boussinesq saline experiments and found that the gravity current takes on a shape that is more akin to a triangular shaped or crescent shaped wedge. The entrainment coefficient is found to be almost independent of the slope. They extended the original circular wedge integral model of [36] to a new triangular wedge integral model with entrainment. The model does not include an explicit constant Froude number condition. Instead the drag of the slope and the ambient fluid is accounted using the drag coefficient and added mass terms in the momentum equation. Interestingly, the wedge integral model also predicts a constant Froude number, which implies a constant Richardson number, thus, a constant entrainment coefficient. Recently, [40] and [41] reported data on the dynamics of circular finite release Boussinesq gravity currents on a uniform slope using fully-resolved three-dimensional direct numerical simulations. Their data showed that the gravity current evolves to a shape that is similar to a triangular wedge shape as shown by [39]. The front velocity of the gravity current is found to go through two acceleration phases, which is not observed in any of the previous studies [see 36, 39]. They attributed the second acceleration phase to the
rearrangement and redistribution of the heavy fluid to the front of the current down the slope. Similar experiments on the non-Boussinesq type of flow on a steep open slope was conducted by [42] and on a larger scale by [43].

Furthermore, studies of gravity currents beyond the classical canonical configurations are very scarce despite the fact that the majority of currents in real situations have an arbitrary, non-circular or non-axisymmetric, initial source. Dynamics of non-circular finite release gravity currents on a horizontal boundary have been recently studied in [44, 45] experimentally and numerically. They demonstrated that the effects of the initial shape influence the evolution of current well into the long-time phase, which would corresponds to the inertial self-similar phase in the case of planar or axisymmetric configurations. It was observed that the local velocity of the current can vary significantly, thus leading to long-lasting local speed variations, which may result in dramatically different front locations that depend on the shape of the initial release [44]. An extended box model based on partitioning of the initial release using geometric rays that are perpendicular to the front was proposed and shown capable of capturing the dynamics of such releases. Following the earlier work of [44], [45] showed that such non-axisymmetric currents eventually reach a self-similar regime during which the local front propagation scales as $t^{1/2}$ as in circular releases and the non-axisymmetric front has a self-similar shape that primarily depends on the aspect ratio of the initial release. Further complementary experiments of non-Boussinesq currents and top-spread currents conducted by [45] suggested that this self-similar dynamics is independent of the density ratio, vertical aspect ratio, wall friction and Reynolds number, provided $Re \geq O(10^4)$.

In the present investigation, we present results from fully-resolved direct numerical simulations of three-dimensional elliptical, finite release, Boussinesq gravity currents propagating down an unconfined uniform slope. The study comprises a series of simulations of elliptical releases on a range of slope angles. The shape parameters are varied to study the effects of the initial cross-sectional aspect ratio and mean height to lock radius ratio on the dynamics of the gravity current. Our data show that the long-time development of the spatial mass distribution of the gravity current is significantly influenced by its initial shape at the time of release for relative shallow slopes ($\theta = 5^\circ$ and $10^\circ$) while it is less influenced when released on steeper slopes ($\theta = 15^\circ$ and $20^\circ$). The switching of axes for non-circular releases are observed using both a subjective and a objective measures of the aspect ratio of the gravity
current. An increase in the slope angle results an enhanced longitudinal mass spreading in the streamwise direction due to the increased force along the sloping boundary. Multiple acceleration phases are observed for the current centre of mass in the case of the current with a small or moderate initial cross-sectional aspect ratio whereas one single acceleration phase exists for the current with a large initial cross-sectional aspect ratio. The Froude numbers for the currents released with the same slope angle but different initial shapes are observed to attain a similar constant value after the second acceleration phase. The mean Froude number is seen to increase with increasing slope angles. The mean height to lock radius ratio is found to be important during the early development of the current but does not have a strong influence on the long-time motion of the current.

II. NUMERICAL FORMULATION

We consider the case of a slanted elliptical cylinder containing heavy fluid surrounded by an infinite extent of light fluid on a sloping boundary. The density difference between the two fluids is assumed to be small enough so that the Boussinesq approximation is valid. With this approximation, density variations are only retained in the buoyancy term. The dimensionless system of equations governing the motion of the flow reads

\[ \nabla \cdot \mathbf{u} = 0, \]  
\[ \frac{D\mathbf{u}}{Dt} = \rho \mathbf{e}^g - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \]  
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{1}{Sc Re} \nabla^2 \rho. \]

Here, \( \mathbf{u}, p \) and \( \rho \) are the divergence-free dimensionless velocity vector, pressure and density in the flow, respectively, and \( \mathbf{e}^g \) is a unit vector pointing in the direction of gravity. The dimensionless density and pressure are given by

\[ \rho = \frac{\rho^* - \rho_0}{\rho_1^* - \rho_0}, \quad p = \frac{p^*}{\rho_0^*(U^*)^2}. \]

Any variable with an asterisk is to be understood as dimensional. The variables \( \rho^* \), \( \rho_0^* \) and \( \rho_1^* \) represent the local, ambient and initial heavy fluid densities, respectively. Hence, the value of \( \rho \) is bounded between 0 and 1. The variables \( p^* \) and \( U^* \) denote the local pressure and velocity scale. The two dimensionless numbers in Eq. (3) are the Reynolds number...
(Re) and Schmidt number (Sc), defined as
\[
Re = \frac{U^* L^*}{\nu^*}, \quad Sc = \frac{\nu^*}{\kappa^*},
\]
where \(\nu^*\) is the kinematic viscosity and \(\kappa^*\) is the molecular diffusivity of the fluid. The length scale \(L^*\), the velocity scale \(U^*\) and the time scale \(T^*\), following the definitions from [39], are given by
\[
L^* = (V_0^*)^{1/3}, \quad U^* = \sqrt{\frac{g^* (P_1^* - P_0^*)}{\rho_0^*}} L^*, \quad T^* = L^*/U^*,
\]
where \(V_0^*\) is the initial volume of heavy fluid in the slanted elliptical cylinder and \(g^*\) denotes the gravitational acceleration.

All the simulations considered in the present investigation are run in a rectangular box of size \(L_x \times L_y \times L_z\), as shown in Fig. 1. The system of governing equations is solved using a de-aliased pseudo-spectral code [46]. Fourier expansions are employed for the flow variables along the streamwise (\(x\)) and spanwise (\(y\)) directions. In the wall normal direction (\(z\)), a Chebyshev expansion with Gauss-Lobatto quadrature points is used, which provides higher resolution near the wall boundary. The flow field is time advanced implicitly using the Crank-Nicolson scheme for the diffusion term. The advection terms and the buoyancy term are treated using the third-order Runge-Kutta scheme. More details on the implementation of the numerical scheme can be found in [47]. Periodic boundary conditions are employed along the streamwise and spanwise directions. This implies that a periodic array of gravity currents is being simulated. In the wall normal direction, no slip boundary conditions is used for the velocity field at the bottom wall and a free slip boundary condition is used at the top wall.

Owing to the periodic boundary conditions, the computational box is typically taken to be quite long along the streamwise and spanwise directions in order to allow unhindered development of the current on a sloping boundary for sufficiently long time. [19] and [21] indicated that the interaction of the advancing front with the boundary becomes important when the front is within one dimensionless unit from the boundaries. Careful examinations have showed that a large domain size of \(L_x = 18\) and \(L_y = 15\) is sufficient to ensure there is no influence from the periodic boundary conditions used. Additionally, a wall normal length of \(L_z = 2.5\) ensures that the top boundary is sufficiently far from the flow to have any significant effect on it [41]. Simulations were carried out at \(Re = 5000\) with a grid resolution.
of $700 \times 600 \times 201$ ($N_x \times N_y \times N_z$) corresponding to approximately 85 million grid points. This is consistent with the requirement of a grid size of the order of $O(ReSc)^{-1/2}$ [16, 48]. The heavy fluid was initially confined inside a truncated elliptical segment of mean height $h_0$ as shown in Fig. 1, whose initial cross-sectional aspect ratio is $\Lambda_0 = a/b$. When $\Lambda_0 > 1$, $a$ and $b$ represent the major and minor axes of the elliptical cross section, respectively, and vice versa when $\Lambda_0 < 1$. The lock radius along the slope is defined as $r_0 = a/\cos \theta$ (see Fig. 1), where $\theta$ represents the inclination of the bottom boundary with respect to the horizontal along the $x$-axis. The initial density was smoothly varied from $0$ to $1$ over a thin region at the interface between the ambient fluid and the heavy fluid. The flow was started from rest with a small random disturbance of 5% amplitude superposed on the density field to accelerate the three-dimensional development [see 18]. Since it has been shown that the dynamics of gravity currents is weakly dependent on Schmidt number provided that Reynolds number of the flow is large [19, 49], the Schmidt number of unity was employed for all the simulations. The time step was chosen to ensure a Courant number less than 0.5.

III. RESULTS AND DISCUSSION

Two sets of simulations were conducted to study the front dynamics of elliptical gravity currents on a uniform slope. The first series of gravity current simulations performed were designed to investigate the influence of slope angle and initial cross-sectional aspect ratio of the elliptical segment, including combinations of 28 simulations with four slope angles ($\theta = 5^\circ$, $10^\circ$, $15^\circ$ and $20^\circ$) and seven initial cross-sectional aspect ratios ($\Lambda_0 = 0.1$, $0.2$, $0.5$, $1$, $2$, $5$ and $10$). The mean height $h_0$ and the initial volume $V_0$ of the elliptical cylinder were both held at unity. The second series of simulations investigated the effect of mean height to lock radius ratio ($\Gamma = h_0/r_0$), in which both $h_0$ and $r_0$ were adjusted so that the ratio $\Gamma$ amounts to half and twice the value employed in the case with $\Lambda_0 = 1$ and $\theta = 15^\circ$, respectively. The rest of the configuration parameters remain unchanged, i.e. the initial volume is kept at unity so that the Reynolds number remains unchanged for all simulations.
A. Spatial flow structure and mass distribution

In order to better understand spatial development of the gravity current, we study the equivalent height of the current (also known as mass hold-up),

\[ h_z(x, y, t) = \int_0^{L_z} \rho(x, y, z, t) dz. \] (7)

The interface between the current and the ambient fluid can be defined as the position where the wall normal integrated height of the current \( h_z \) exceeds a small threshold value \( \epsilon = 10^{-3} \). The current-ambient interface is however sensitive to the chosen value of \( \epsilon \) in the range \([10^{-4}, 10^{-3}]\). The evolution of \( h_z \) for the currents with \( \Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10 \) released on a slope angle of \( \theta = 10^\circ \) at \( Re = 5000 \) is shown in Fig. 2. The coloured regions correspond to values of \( h_z \geq 10^{-2} \) and indicate the position of the current and the mass distribution within. High value of \( h_z \) is shown in red, the low value (\( h_z = 10^{-2} \)) is coloured in blue. The light grey colour corresponds to bare regions of heavy fluid (\( 10^{-3} \leq h_z < 10^{-2} \)). The discussion to follow will focus on the influence of \( \Lambda_0 \) on the shape and mass distribution within the current at various times. At \( t = 1 \), the influence of the slope is not yet perceived, and the current evolves as if it were on a horizontal boundary. The effect of the slope is important when it becomes equivalent to the vertical aspect ratio of the current along the streamwise direction [39]. Shortly after release (\( t = 3 \)), the currents with a small initial cross-sectional aspect ratio (\( \Lambda_0 = 0.1 \) and 0.2) appear to split from the centre along the major axis in the spanwise direction and partitions themselves into a downstream and an upstream advancing current. The majority of heavy fluid, however accumulates on the downstream end. The front along the initial minor axis is observed to travel faster than that along the initial major axis. As a result, the time-dependent, horizontal aspect ratio of the current (\( \Lambda \)) increases with the spreading. The current with a moderate initial cross-sectional aspect ratio (\( \Lambda_0 = 0.5, 1 \) and 2) is found to spread almost in an axisymmetric manner, forming a nearly circular shape. The heavy fluid forms an outer ring with the majority of mass accumulating at the downstream end, except for a thin layer of fluid residing in the interior of the ring. Currents with a large initial cross-sectional aspect ratio (\( \Lambda_0 = 5 \) and 10) are observed to split in half from the centre along the major axis in the streamwise direction. As expected, the majority of the heavy fluid accumulates at the downstream end within each of the two portions of the current. As a consequence, the time-dependent aspect ratio
of the current decreases with the spreading.

At $t = 6$, the effect of the inclined boundary becomes apparent. For currents with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2), the downstream portion of the current has developed into a chrysalis shaped structure in which the majority of mass converges towards the centre near the front, which is known as the head of the gravity current. On the other hand, the upstream portion develops into a shell like structure with a thin layer of fluid residing at the downstream end, near the centre of the release. The heavy fluid on the upstream end is also observed to converge towards the centre but not as quickly as the heavy fluid at the downstream end. The currents with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2) remain circular. The mass in the outer ring is found to move from the rear of the ring and converge towards the front of the current, forming a crescent shaped structure. A thin patch of heavy fluid resides in the rear of the current. The mass in the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) continues spreading outwards in the spanwise direction while propagating downstream with the majority of mass accumulating near the downstream shoulder, forming a horseshoe shaped structure. As a result, the aspect ratio of the current further decreases. Multiple undulations at the front of the current are observed for all the currents examined. These are due to the lobe and cleft instability [3, 19, 22]. The lobes and clefts continuously merge and split resulting in a complex pattern that evolves dynamically at the front of the current.

As the current propagates further downstream ($t = 15$), the upstream and downstream portions for the cases with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2) become fully detached. The downstream portion grows in size but remains in a chrysalis shaped structure. On the upstream end, the current has slowed down significantly after travelling about twice the current’s mean height and begins to reverse directions and spread slowly in the direction of the slope. The outer ring for the current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2) becomes fully detached at its rear and has developed into a boomerang like structure, which wraps around a thin patch of heavy fluid residing in the centre. For the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10), the mass converges to the front of the current. Due to the strong spanwise partitioning and spreading of the current at the beginning, the current has developed into a bow shaped structure. For the current with an initial cross-sectional aspect ratio less than one ($\Lambda_0 = 0.1, 0.2$ and 0.5), the minor axis of the initial release along the streamwise
direction becomes the new major axis whereas the major axis of the initial release along the spanwise direction becomes the new minor axis. Vice versa for the current with an initial cross-sectional aspect ratio greater than one ($\Lambda_0 = 2, 5$ and 10). Hence, the switching of the major and minor axes as compared to the initial shape is always observed for the current with non-circular shape ($\Lambda_0 \neq 1$) at all the slope angles studied. Note that this switching of axes for non-circular releases has been previously reported for saline and particle-laden currents on a horizontal boundary [45, 50]. This switching of axes is a consequence of the initial local volume partitioning of the current and the azimuthally varying current height, which lead to local fast and slow fronts along the circumference.

Fig. 3 depicts the spatial distribution of $h_z$ at the end of each simulation ($t = 30$). At $\theta = 5^\circ$, the final spatial mass distribution of the current is dependent on its initial shape at the time of release. For the current with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2), the local volume at the front remains as a chrysalis shaped structure. The current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2) has developed into a boomerang like structure. The structure of the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) remains in a bow shape with distinctive dual front locations. For all cases at $\theta = 5^\circ$, a thin layer of heavy fluid resides near the initial centre of the release. At $\theta = 10^\circ$, the local volume at the front of the current with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2) becomes more triangular with less fluid left at the rear. For the current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2), the majority of the mass is observed to aggregate near the front and forms a V-shaped structure. With increased slope angle, dual mass peaks located at the dual front locations for the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) converge to the centre. At $\theta = 15^\circ$, all the currents are observed to become more turbulent. For the current with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2), the local volume at the front begins to lose its chrysalis shape whereas the local volume at the rear becomes less calm and begins to re-attach to the fluid at the front. The current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2) starts to lose its V-shape with less fluid residing between the two tails. The current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) is seen to develop into a triangular shaped structure with the majority of mass centred at the front. At $\theta = 20^\circ$, all the currents lose their pre-developed shape and break up into small-scale structures. The mass of the current is found to converge faster
and more quickly to form the head of the current with a smaller $\Lambda_0$ at a larger slope angle. Therefore, long-time development of the spatial mass distribution of the gravity current is influenced by its initial shape when released at smaller slope angles ($\theta = 5^\circ$ and $10^\circ$) while it is insensitive to the initial shape of the current when released at larger slope angles ($\theta = 15^\circ$ and $20^\circ$).

**B. Temporal evolution of the aspect ratio**

In order to further understand longitudinal and lateral spreading of the gravity current, we study the temporal evolution of the aspect ratio of the current $\Lambda$,

$$\Lambda(t) = \frac{x(t)_{\text{max}} - x(t)_{\text{min}}}{y(t)_{\text{max}} - y(t)_{\text{min}}} \bigg|_{\epsilon = 10^{-3}},$$

where $x(t)_{\text{max}}$, $x(t)_{\text{min}}$, $y(t)_{\text{max}}$ and $y(t)_{\text{min}}$ are the maximum and minimum longitudinal and lateral locations of the detected interface of $\bar{h}_z$ at a small threshold value $\epsilon = 10^{-3}$ in the streamwise direction and spanwise direction, respectively. The evolution of $\Lambda$ for the present simulations is shown in Fig. 4 as solid lines. At $\theta = 5^\circ$, the aspect ratio $\Lambda$ for the current with a circular initial shape ($\Lambda_0 = 1$) stays almost at a constant value of unity and the current remains in a circular shape. $\Lambda$ for the current with an initial cross-sectional aspect ratio less than one ($\Lambda_0 = 0.1$, $0.2$ and $0.5$) is observed to increases after the release of heavy fluid. At approximately $t = 15$, $\Lambda$ reaches a constant value that is greater than one. At the end of the simulations ($t = 30$), the minor axis of the initial release along the streamwise direction becomes the new major axis whereas the major axis of the initial release along the spanwise direction becomes the new minor axis. Vice versa for the current with an initial cross-sectional aspect ratio greater than one ($\Lambda_0 = 2$, $5$ and $10$). The switching of axes for non-circular releases is due to a relatively slower spreading along the original major axis and a relatively faster spreading along the original minor axis. This switching of axes is a consequence of the initial local volume partitioning of the current that leads to local fast and slow fronts, as described in [44, 45, 50] for both saline currents and particle-laden current. Such long-lasting speed variations among the different sections of the front may result in dramatically different front locations that depend on the shape of the initial release. At $\theta = 10^\circ$, the longitudinal mass spreading in the streamwise direction is enhanced due to the increased gravitational force along the sloping boundary, resulting an increasing in $\Lambda$ from
its previous constant value after the switching of axes at $\theta = 5^\circ$. As slope angle further increases ($\theta = 15^\circ$ and $20^\circ$), the final aspect ratio for the all currents becomes greater than unity.

The aforementioned criterion (Eq. 8) used to determine the current aspect ratio can be deemed to be a subjective measure as this method incorporates a threshold value of $\epsilon$. It might be interesting to consider a more objective measure without using any threshold. Here, we introduce an integral measure of the current longitudinal and lateral spreads using the moment of inertia tensor $I_{xx}$ and $I_{yy}$, which can be written as

\begin{align}
I_{xx} &= \frac{1}{M} \int_{0}^{L_x} \int_{0}^{L_y} \int_{0}^{L_z} (x - x_{CM})(x - x_{CM})\rho(x, y, z, t) \, dz \, dy \, dx, \\
I_{yy} &= \frac{1}{M} \int_{0}^{L_x} \int_{0}^{L_y} \int_{0}^{L_z} (y - y_{CM})(y - y_{CM})\rho(x, y, z, t) \, dz \, dy \, dx,
\end{align}

where $M$ is the total mass of the heavy fluid, $x_{CM}$ and $y_{CM}$ are the streamwise and spanwise locations of the current centre of mass, respectively. Hence, a good quantitative description of the current aspect ratio can be expressed as $\sqrt{I_{xx}/I_{yy}}$. The temporal evolution of $\sqrt{I_{xx}/I_{yy}}$ for the present simulations is also shown in Fig. 4 as dotted lines in the same colour as the corresponding $\Lambda$ for the same case. Overall, it can be seen that $\sqrt{I_{xx}/I_{yy}}$ agrees quite well with the corresponding $\Lambda$. However, $\sqrt{I_{xx}/I_{yy}}$ overestimates the current aspect ratio compared to $\Lambda$ for the currents with an initial cross-sectional aspect ratio no greater than one ($\Lambda_0 = 0.1, 0.2, 0.5$ and 1) whereas $\sqrt{I_{xx}/I_{yy}}$ underestimates the current aspect ratio compared to $\Lambda$ for the currents with an initial cross-sectional aspect ratio greater than one ($\Lambda_0 = 2, 5$ and 10). The discrepancy of $\sqrt{I_{xx}/I_{yy}}$ from $\Lambda$ is likely due to the inclusion of distribution of the thin layer of heavy fluids that is not considered by $\Lambda$ using a subjective threshold. It is clear that both measures give a good description of the evolution of the current aspect ratio.

C. Front location and front velocity

The front location of the current is defined as the maximum streamwise location of the detected interface between the current and the ambient fluid with the small threshold value $\epsilon = 10^{-3}$. Consequently, the distance travelled by the current $x_F$ along the streamwise direction can be calculated as the difference between the front location of the current at
$t > 0$ and the front location of the current at $t = 0$. The temporal evolution of the front location from the present simulations is shown in Fig. 5, where the square symbols represent the experimental data from the study by [39]. Their experiments were conducted in a tank of dimensions $2.0m \times 2.5m \times 0.85m$ (width $\times$ length $\times$ depth). The bottom boundary of the tank can be adjusted to give a slope in the range $5^\circ$ to $20^\circ$. A slanted circular cylinder that contains food dye coloured saline water of density no more than a few percent of fresh water was used to release the heavy fluid by rapidly lifting the cylinder from the water tank. The small density difference in the experiments justifies the use of the Boussinesq approximation.

The present numerical data is found to agree well with the published experimental data. In general, the distance travelled by the current increases as the slope angle increases. This is due to the increased gravitational force in the direction of the sloping boundary. At $\theta = 5^\circ$, it can be observed that the current with a smaller $\Lambda_0$ travels farther downstream than the current with a larger $\Lambda_0$. As the slope angle increases ($\theta = 10^\circ$), the difference in the distance travelled by the current among the currents with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and $0.2$) and a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5$, $1$ and $2$) is reduced. At $\theta = 15^\circ$ and $20^\circ$, the current with a small or moderate $\Lambda_0$ travels almost the same distance. However, the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and $10$) still falls behind, in which the current with an $\Lambda_0 = 10$ travels the slowest.

The difference in the distance travelled among the currents with different $\Lambda_0$ values can be explained by the initial local volume partitioning of the current and its switching of axes. The current with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and $0.2$) partitions along its major axis in the spanwise direction with the majority of mass accumulating in the local volume at the front. The mass rapidly converges towards the centre, creating a relatively fast travelling current head. The current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5$, $1$ and $2$) is found to spread radially outwards in a nearly axisymmetric manner forming a circular mass ring. The majority of mass accumulates in the front half of the ring, which travels along the circumference of the ring towards the front of the current and quickly develops into the head of the current. As the slope angle increases, this process of mass redistribution becomes more efficient. As a result, the distance travelled by the current with a moderate $\Lambda_0$ converges towards those travelled by the current with a small initial aspect ratio. At $\theta = 15^\circ$ and $20^\circ$, there is almost no difference between the two groups of currents. On the other hand, the relatively slower velocities of the currents with
a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) is due to the initial spanwise partitioning and the preferential spreading, which delays the process of mass redistribution. Consequently, the current either fails to form a single aggregated current head ($\theta = 5^\circ$ and $10^\circ$) or spends a significant amount of time to form a single aggregated current head ($\theta = 15^\circ$ and $20^\circ$), resulting in a slow moving current.

Fig. 5 also shows the temporal evolution of the front velocity $u_F$ for the currents released at four slope angles ($\theta = 5^\circ$, $10^\circ$, $15^\circ$ and $20^\circ$) with seven initial cross-sectional aspect ratios ($\Lambda_0 = 0.1$, $0.2$, $0.5$, $1$, $2$, $5$ and $10$). The front velocity is obtained using a second order central finite difference scheme on the front location with respect to time. The symbols represent the front velocity from the present simulations and each solid line with the same colour is the corresponding $12^{th}$ order least square polynomial fit to guide the trend of the front velocity. The results reveal some interesting aspects of elliptical finite release gravity currents on a uniform sloping boundary. Initially, the front velocity is seen to increase at the same rate for all the currents simulated. This clearly indicates that the current is not significantly influenced by either its initial shape or the slope during the acceleration phase. At the end of the acceleration phase, the current attains a local maximum velocity. This maximum velocity is observed to increase monotonically with larger slope angles whereas it is found to initially decrease then increase as the current $\Lambda_0$ increases. The current subsequently decelerates to a local minimum velocity. Interestingly, the current undergoes a second acceleration phase after the first acceleration phase and arrives at a second local maximum velocity, whose value ranges between approximately 70% to 140% of the local maximum velocity at the end of the first acceleration phase. The presence of a second acceleration phase significantly raises the front velocity and results in a higher front velocity at later times. It can be seen that the front velocity increases with steeper slopes after the first acceleration phase (approximately $t > 1.5$). However, the front velocities of the currents released at the same slope angle converge to the same level for $t > 20$, which suggests that the velocity of the gravity current eventually “forgets” its initial shape during the later self-similar phase. The observation of a second acceleration phase immediately following the first acceleration phase was also reported for circular density currents on a uniform slope by [40] and [41]. Those studies showed that the second acceleration is the result of the redistribution of the heavy fluids towards the downstream end of the current near the mid-plane ($y = 0$ plane).
D. Acceleration and deceleration phases

In order to understand the dynamics of the current, durations of the first acceleration phase ($A_1$), the first deceleration phase ($D_1$) and the second acceleration phase ($A_2$) for the currents released at four slope angles ($\theta = 5^\circ$, $10^\circ$, $15^\circ$ and $20^\circ$) with seven initial cross-sectional aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5$ and $10$) are shown in Table I. It can be observed that the duration of $A_1$ increases with steeper slopes while it decreases with larger $\Lambda_0$ values. As the slope angle increases, the component of the gravitational force acting in the direction of the slope grows, which results in a longer duration of $A_1$. The preferential spreading of the mass along the initial minor axes of the release leads to a longer duration of $A_1$ for the current with a small $\Lambda_0$ (faster spreading in the streamwise direction) but a shorter duration of $A_1$ for the current with a larger $\Lambda_0$ (faster spreading in the spanwise direction).

The duration of $D_1$ is seen to decrease with both larger slope angles and larger initial cross-sectional aspect ratios. The convergence of mass towards the current head near the mid-plane is deemed to be the primary cause for the second acceleration phase. The shallower the slope the weaker the flux of mass towards the centre of the current at the downstream end, thus the longer the duration of the converging phase. Consequently, the duration of $D_1$ is shortened with steeper slopes. However, the influence of $\Lambda_0$ is less intuitive. For currents with small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and $0.2$), the current splits along its major axis of the initial release in the spanwise direction with the majority of mass accumulating in the local volume at the front. Since $\Lambda_0$ for the current at the initial release is small, the local volumes at the front retains a similar shape with an even smaller $\Lambda_0$ after the split. The convergence of mass towards the mid-plane takes more time due to the elongated width of the current in the spanwise direction. As a result, the duration of $D_1$ is lengthened for the current with a smaller $\Lambda_0$. On the other hand, the current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and $2$) appears to form a circular shape with the mass accumulating at the downstream end of the outer ring. In such a configuration, the mass is found to travel more efficiently and converges more quickly towards the current head near the mid-plane at the downstream end. Consequently, the duration of $D_1$ is shortened for the current with a moderate $\Lambda_0$.

The mechanism behind the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$...
and 10) is found to be different from those aforementioned. The current initially splits along
the initial major axis of the release in the streamwise direction. Subsequently, the mass is
quickly shifted to the downstream end within each of the two portions of the current as a
result of the relatively large $\Lambda_0$ and the influence of the sloping boundary. Consequently,
the duration of $D_1$ is shortened for the current with a large $\Lambda_0$.

The second acceleration phase ($A_2$) takes place at the end of $D_1$ after the current re-
arranges and redistributes the mass towards its head, whose duration increases with both
steeper slopes and larger initial cross-sectional aspect ratios. Similarly, the duration of $A_2$ is
lengthened with larger slope angles due to increased gravitational force acting in the direc-
tion of the sloping boundary. Since the mass is more centralised in the current head for the
current with a smaller $\Lambda_0$, there is almost no extra buoyancy fed into the current head during
the second acceleration phase, which results in a short duration of $A_2$. On the contrary, the
buoyancy is continuously fed into the current head for the current with a larger $\Lambda_0$, which
leads to a longer lasting second acceleration phase. Therefore, it can be concluded that
the initial shape of the release of a gravity current has some subtle and unique long-term
influence on its acceleration and deceleration phases compared to the slope angle.

Table II presents the maximum velocity during the first acceleration phase ($A_{1_{\text{max}}}$),
the maximum velocity during the second acceleration phase ($A_{2_{\text{max}}}$) and the ratio of the
second local maximum velocity over the first local maximum velocity ($A_{2_{\text{max}}}/A_{1_{\text{max}}}$) for
the currents at four slope angles ($\theta = 5^\circ$, $10^\circ$, $15^\circ$ and $20^\circ$) with seven initial cross-sectional
aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5$ and $10$). It can be observed that both $A_{1_{\text{max}}}$ and
$A_{2_{\text{max}}}$ increase with steeper slopes. The initial accelerations are the same for all the currents
up to approximately $t = 0.5$. The duration of $A_1$ for the current released on a steeper slope
lasts longer than the one released on a shallower slope, which results in a monotonically
increasing $A_{1_{\text{max}}}$ with larger slope angles. On the other hand, both the duration of $D_1$ and
the associated rate of deceleration decreases with the current released on a steeper slope
and the duration of $A_2$ together with its rate of acceleration generally increases with the
current released on a larger slope angle. Consequently, $A_{2_{\text{max}}}$ increases as the slope becomes
steeper.

For the currents released on the same sloping boundary, it can be seen that both $A_{1_{\text{max}}}$
and $A_{2_{\text{max}}}$ increase initially and then decrease as $\Lambda_0$ increases. The maximum $A_{1_{\text{max}}}$ and
maximum $A_{2_{\text{max}}}$ occur with a current having a more circular shape ($\Lambda_0 = 0.5$ or 1) at the
time of release. The ratio of $A_{2\text{ max}}$ over $A_{1\text{ max}}$ for the currents considered is calculated and presented in Table II as well. The ratio is coloured in red if $A_{1\text{ max}} > A_{2\text{ max}}$ while it is coloured in blue if $A_{1\text{ max}} < A_{2\text{ max}}$. It can be observed that the lowest $A_{2\text{ max}}/A_{1\text{ max}}$ occurs with $\theta = 5^\circ$ and $\Lambda_0 = 0.1$ whereas the highest $A_{2\text{ max}}/A_{1\text{ max}}$ occurs with $\theta = 20^\circ$ and $\Lambda_0 = 10$. $A_{2\text{ max}}/A_{1\text{ max}}$ increases with steeper slopes and larger $\Lambda_0$ values. It can also be seen that $\Lambda_0$ seems to have a stronger influence on $A_{2\text{ max}}/A_{1\text{ max}}$ at a larger slope angle ($\theta = 15^\circ$ and $20^\circ$) compared to a smaller slope angle ($\theta = 5^\circ$ and $10^\circ$). By colouring $A_{2\text{ max}}/A_{1\text{ max}}$ into either red or blue, it clearly indicates how both the slope angle and the initial cross-sectional aspect ratio influence the level of $A_{2\text{ max}}$ with respect to $A_{1\text{ max}}$. The dynamical behaviour of both $A_{1\text{ max}}$ and $A_{2\text{ max}}$ is in itself interesting from a fluid mechanics point of view. The maximum difference in $A_{1\text{ max}}$ and $A_{2\text{ max}}$ is found be 200% ($\Lambda_0 = 0.5, \theta = 20^\circ$ vs $\Lambda_0 = 10, \theta = 5^\circ$) and 164% ($\Lambda_0 = 1, \theta = 20^\circ$ vs $\Lambda_0 = 10, \theta = 5^\circ$) respectively with respect to its lowest value. The values for $A_{2\text{ max}}/A_{1\text{ max}}$ can vary between 73% and 138%. These results show that the level of $A_{1\text{ max}}$ and $A_{2\text{ max}}$ are quite sensitive to the slope angle and the initial cross-sectional aspect ratio of the current.

E. Temporal evolution of front Froude number

The Froude number quantifies the local ratio of inertia and gravitational force at the front of the current. It is a key quantity of interest for gravity currents, which is a relationship consisting three natural characteristics of a gravity current, namely a characteristic velocity $u^*_c$ (the usual choice being the front velocity of the current), a characteristic depth $h^*_c$ and a characteristic reduced gravity $g^*_c$. In the present study, the Froude number can be calculated using the maximum height $\bar{h}_{z, \text{Max}}$, which is defined as the maximum value of $\bar{h}_z(x, y, t)$. Hence, the Froude number can be expressed as

$$F_r = \frac{u_F}{(\bar{h}_{z, \text{Max}})^{1/2}}. \quad (11)$$

Fig. 6 shows the temporal evolution of the Froude number from the present simulations. The symbols represent the Froude number as calculated from Eq. 11 and each solid line with the same colour is the corresponding 12th order least square polynomial fit to guide the trend of the Froude number. Numerical investigation of the temporal evolution of front Froude number for finite circular gravity currents on an unbounded uniform slope at $\theta = 5^\circ, 10^\circ, 15^\circ$.
and $20^\circ$ was carried out using the same method in an earlier study by [41]. Their simulation results are in reasonable agreement with the experimental data of [39]. The Froude number is observed to attain a nearly constant value beyond the second acceleration phase. In the present study, it can be seen that currents released with the same slope angle but different initial shapes attain a very similar constant $Fr$ value after the second acceleration phase. The dependence of interdecile mean Froude number $Fr$ on the slope angle $\theta$ beyond the second acceleration phase is shown in Fig. 7, where the interdecile mean Froude number is given as

$$Fr = \frac{1}{t_f - t_s} \int_{t_s}^{t_f} Fr(t) dt,$$

(12)

where $t_s$ and $t_f$ represent the non-dimensional times that mark the first decile and ninth decile for the duration beyond the second acceleration phase till the end of the simulations, respectively. For the currents released on the same slope angle, the interdecile mean Froude numbers are almost the same. However, $Fr$ is seen to increase with increasing slope angles from a value of approximately 0.7 at $\theta = 5^\circ$ to 1.1 at $\theta = 20^\circ$, respectively.

F. Centre of mass

In order to further understand how the centre of mass of the current evolves at the various stages of spreading, we investigate the streamwise temporal evolution of the current centre of mass $x_{CM}$ and the corresponding velocity $u_{CM}$. The temporal evolution of the current centre of mass from the present simulations is shown in Fig. 8. The position of the centre of mass is also indicated in Fig.s 2 and 3 as a red dot. The maximum streamwise displacement of $x_{CM}$ increases as the slope angle increases. This is due to the increased gravitational force in the direction of the sloping boundary. Interestingly, $x_{CM}$ is found to travel farther for the current with a larger $\Lambda_0$ whereas the front location of the current is observed to travel farther for the current with a smaller $\Lambda_0$ from those aforementioned. For the cases with a smaller $\Lambda_0$ is found to travel at slower rate compared to currents with a larger $\Lambda_0$. However, the difference for $x_{CM}$ among the currents with different $\Lambda_0$ values reaches a constant after approximately $t = 15$. The initial difference can be explained with the initial local volume partitioning of the current. The current with a smaller initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2) is observed to partition in the spanwise direction during the initial acceleration phase, forming a downstream and an upstream advancing current.
upstream advancing current initially travels backwards in the upstream directional, then slows down and eventually travels in the downstream direction. As a result, the centre of mass for currents with a smaller $\Lambda_0$ initially travels at a slower rate.

The temporal evolution of $u_{CM}$ is presented in Fig. 8. The results again reveal some interesting aspects of elliptical finite release gravity currents on a uniform slope. The velocity is initially seen to increase at the same rate for all the currents up to $t = 1.5$, which indicates that the current is not significantly influenced by either its initial shape or the sloping boundary during the initial acceleration phase. Gravity currents with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1, 0.2$ and $0.5$) experience two more acceleration phases at the end of the first acceleration phase, the current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 1$ and $2$) experiences another acceleration phase phase at the end of the first acceleration phase while there is no more acceleration phase for the current with a larger initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and $10$). The multiple acceleration phases of the centre of mass for the current with a small or moderate $\Lambda_0$ is due to initial local volume partitioning and the dynamics of each of the partitioned local volumes. However, the current centre of mass velocity is observed to collapse for $t > 15$, which suggests that eventually the initial shape of the current has no influence on $u_{CM}$.

G. Effect of mean height to lock radius ratio

The second series of simulations were conducted to investigate the influence of mean height to lock radius ratio ($\Gamma$). Both $h_0$ and $r_0$ were adjusted so that $\Gamma$ amounts to half and twice the value employed in the case with $\Lambda_0 = 1$ and $\theta = 15^\circ$, which is $\Gamma_0$, respectively. The rest of the configuration parameters remain unchanged, i.e. the initial volume is kept at unity so that the Reynolds number remains unchanged for all simulations.

Fig. 9 presents the evolution of spanwise integrated density of the current $\tilde{w}_y$ in the cases of circular finite release of gravity currents with mean height to lock radius ratios of $0.5 \times \Gamma_0$, $\Gamma_0$ and $2 \times \Gamma_0$ at $Re = 5000$ on a sloping boundary with $\theta = 15^\circ$, respectively. The interface between the current and the ambient fluid can be defined as the position where $\tilde{w}_y$ exceeds a small threshold value $\epsilon = 10^{-3}$. The current-ambient interface is not however sensitive to the chosen value of $\epsilon$ in the range $[10^{-4}, 10^{-2}]$. High value of $\tilde{w}_y$ is shown in red, the threshold value ($\tilde{h}_z = 10^{-3}$) is coloured in blue. The red dot with a black outline indicates
the location of the current centre of mass. The black arrow corresponds to the direction of the gravitational force. \( \vec{w}_y \) is defined as

\[
\vec{w}_y(x, z, t) = \int_0^{L_y} \rho(x, y, z, t) dy.
\] (13)

The side view of the initial shapes of the slanted elliptical heavy fluid can be observed at \( t = 0 \). At \( t = 1 \), the height of the heavy fluid decreases due to the collapse caused by the gravitational force. As the collapse of heavy fluid continues (\( t = 3 \)), more heavy fluid is pushed into the head at the downstream end. At \( t = 6 \), a semi-elliptical current head forms followed by a tail. Note that the size of the current head increases as the current spreads down the slope because of both entrainment of the ambient fluid and convergence of the mass to the current head. As the current further propagates (\( t = 15 \)), some heavy fluid is left behind the current head and forms a separated tail current. At the end of the simulation (\( t = 30 \)), the amount of buoyancy contained in the current head is observed to decrease for all the currents.

The spatial evolution of the current centre of mass from its original position for the currents released with three mean height to lock radius ratios (0.5 \( \times \Gamma_0 \), \( \Gamma_0 \) and 2 \( \times \Gamma_0 \)) and the corresponding wall normal temporal evolution of the current centre of mass (\( z_{CM} \)) are also presented in Fig. 9. It can be seen that \( z_{CM} \) initially decreases up to \( t = 2 \), which is equivalent to the duration of the first acceleration phase of the current front velocity (see Fig. 10). Subsequently, \( z_{CM} \) is observed to rise to a local maximum before it drops again at approximately \( t = 4 - 5 \), which is at the end of the first deceleration phase and prior to the second acceleration phase. This shows the front dynamics are interconnected with the potential to kinetic energy transfer of the current as a whole. Interestingly, the wall normal centre of mass of the current (for all three simulations) collapses to the same value at approximately \( t = 15 \), which is coincident with the time when the current front velocity collapses. The steady rise of \( z_{CM} \) beyond \( t \approx 15 \) is likely related to the entrainment rate of ambient fluid as the currents reach a self-similar shape.

The Kulikovskiy-Sveshnikova-Beghin (KSB) model is a simple integral model for powder snow avalanches on an incline that has conservation equations for mass, momentum and volume. [51] extended the model further with an alternative assumption, according to which the volume growth rate is controlled solely by the overall Richardson number (Ri). The Richardson number dependent entrainment function could reproduce the velocities and
volumes well compared to the experimental and field data [51]. More recently, modifications to the model were proposed by [52], which eliminate the problem of predicting physically impossible densities when there is significant particle entrainment by including the entrained snow volume. With the modified model, physically realistic mean densities are predicted, which have a significant impact on the Richardson number-dependent ambient entrainment. In this study, we compare the present numerical data to the model in [51] to examine whether $z_{CM}$ satisfies a Richardson number entrainment criterion. The volume growth rate $\alpha_v$ was found empirically as a function of Richardson number in the experiments by [51], in which $\alpha_v$ is fitted by a function (see Fig. 9)

$$\alpha_v = \begin{cases} e^{-\lambda R_i^2}, & R_i \leq 1, \\ \frac{e^{-\lambda}}{R_i}, & R_i > 1, \end{cases} \tag{14}$$

where $\lambda = 1.6$ and the overall Richardson number in the present study can be defined as

$$R_i = \frac{\cos \theta}{u_{CM}^2}. \tag{15}$$

The volume growth rate $\alpha_v$ satisfies the following relationship

$$\frac{dz_{CM}}{dt} = \alpha_v u_{CM} = \alpha_v \frac{dx_{CM}}{dt}, \tag{16}$$

this would imply

$$z_{CM} = \alpha_v x_{CM} + c, \tag{17}$$

where $c$ is the constant as a result of the integration. Hence, $\alpha_v$ in the cases of gravity currents with mean height to lock radius ratios of $0.5 \times \Gamma_0$, $\Gamma_0$ and $2 \times \Gamma_0$ can be estimated using Eq. 17 for $t \geq 10$. As shown in Fig. 9, the volume growth rates from the present simulations compare well with the prediction by the empirical function from the study by [51].

The temporal evolution of the front location $x_F$ is shown in Fig 10. It is clear that the front displacements are the same for all the currents up to approximately $t = 10$, after which the displacement of the current with $2 \times \Gamma_0$ falls behind slightly. The difference reaches a maximum at approximately $t = 15$ and remains almost constant till the end of the simulation. After the inspection of $h_z$, it is found that the growing difference of the front location can be due to the earlier emergence of turbulent structure for the current with $2 \times \Gamma_0$ at $t = 9$. 

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Once the currents turn into a self-similar shape at $t = 15$, they all travel at a comparable velocity, which results in a constant difference afterwards.

Fig. 10 also presents the temporal evolution of the front velocity $u_F$ of the currents. The symbols represent the front velocity from each of the simulations and individual solid line with the same colour as the symbol is the corresponding 12th order least square polynomial fit. It can be seen that the mean height to lock radius ratio clearly affects the development of the current, including the duration of both acceleration phases ($A_1$ and $A_2$), deceleration phase ($D_1$) and local maximum velocities ($A_{1_{\text{max}}}$ and $A_{2_{\text{max}}}$). With the limited data sets from the present simulations, it is difficult to draw a definite conclusion about the influence of mean height to aspect ratio. However, the front velocity collapses onto each other when the current develops into a self-similar shape at $t = 15$. It can be concluded that the shape parameter, mean height to lock radius ratio, has influence on the short-time dynamics of the gravity current but doesn’t seem to have a strong long-time influence on the motion of the gravity current.

IV. CONCLUSIONS

We present results from fully-resolved numerical simulations to investigate the dynamics of elliptical finite release gravity currents on a uniform slope. Simulations were performed at $Re = 5000$. In the first series of calculations, gravity currents with seven different initial cross-sectional aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5$ and 10) were released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ$ and $20^\circ$). Both the initial shape of the release and the slope angle were shown to significantly influence the spatial mass distribution and propagation of the gravity currents. The time evolution of the circular releases ($\Lambda_0 = 1$) compare favourably, both qualitatively and quantitatively, with previous experiments of [39]. The non-circular current ($\Lambda_0 \neq 1$) is observed to switch its initial major and minor axes at the release. This switching of axes is a consequence of the initial local volume partitioning of the current and the azimuthally varying current height, which leads to local fast and slow fronts along the circumference as described in [45, 50]. The mass is seen to aggregate near the front of the current to form a current head.

On a shallow sloping boundary ($\theta = 5^\circ$), the initial shape of the release has greater influence on the final spatial mass distribution of the current. For the current with a small
initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2), the local volume at the front remains as a chrysalis shaped structure with a detached thin film of fluid at the rear. The current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2) develops into a boomerang like structure with a thin layer of fluids residing in the centre. The current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) evolves into a bow shape with a distinctive dual front locations. At $\theta = 10^\circ$, the front volume for the current with a small initial cross-sectional aspect ratio ($\Lambda_0 = 0.1$ and 0.2) becomes more triangular shaped with less fluid left at the rear. For the current with a moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.5, 1$ and 2), the majority of mass is observed to aggregate near the front and forms a V-shaped structure. The dual mass peaks in the front for the current with an initial large cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) are seen to converge to a single blunt current head. As the slope becomes steeper ($\theta = 15^\circ$), all the currents become more turbulent. The current with a small or moderate initial cross-sectional aspect ratio ($\Lambda_0 = 0.1, 0.2, 0.5, 1$ and 2) begins to lose its shape. However, the current with a large initial cross-sectional aspect ratio ($\Lambda_0 = 5$ and 10) develops into a triangular shaped structure. At $\theta = 20^\circ$, all the currents lose their pre-developed shaped and develop into a self-similar shape with lots of small-scale turbulence. The long-time development of the spatial mass distribution of the gravity current is significantly influenced by its initial shape at the release on a shallower slope ($\theta = 5^\circ$ and 10$^\circ$) while it is less influenced when released with larger slope angles ($\theta = 15^\circ$ and 20$^\circ$).

In order to further understand longitudinal and lateral spreading of the gravity current, both a subjective and an objective measures of the current aspect ratio are applied to the present numerical data. The switching of is found for all the non-circular releases, which is a consequence of the initial local volume partitioning of the current that leads to local fast and slow fronts. Both measures provide a good quantitative description of mass spreading of the gravity current. The discrepancy of $\sqrt{I_{xx}/I_{yy}}$ from $\Lambda$ is likely due to the inclusion of distribution of the thin layer of heavy fluids that is not counted by $\Lambda$ using a subjective threshold.

The temporal evolution of the current front location and front velocity are studied to explore the dynamical development of the gravity current. The displacement of the front location compares well with the experimental data from [39]. The current with a smaller $\Lambda_0$ is observed to travel the farthest. The temporal evolution of the front velocity reveals that
the current undergoes a second acceleration phase immediately after the first acceleration phase. The presence of the second acceleration phase indicates a redistribution of mass within the current to increase the buoyancy at the downstream end of the current. The preferential accumulation and convergence of mass increase the buoyancy at the current head and subsequently result in a second surge of the front velocity. The velocities of the currents at late times \( t > 20 \) are observed to attain the same value if released at the same slope angle. This indicates that the initial shape of the release weakly influences the long-time development of the gravity current.

The acceleration and deceleration phases \( (A_1, A_2, \text{and } D_1) \) of the current are further explored to investigate the influence of the initial shape and the slope angle on the early development of the current. The duration of \( A_1 \) is found to increase with steeper slopes while it decreases with larger \( \Lambda_0 \) values. The duration of \( D_1 \) is seen to decrease with both larger slope angles and larger \( \Lambda_0 \) values. However, the duration of \( A_2 \) increases with both steeper slope and larger \( \Lambda_0 \) values. The underlying mechanism is associated with the mass redistribution within the current of different initial shapes, which indicates that the initial shape has a subtle and unique long-term influence on its dynamics during the acceleration and deceleration phases.

The maximum velocity during the first acceleration phase \( (A_{1\text{max}}) \), the maximum velocity during the second acceleration phase \( (A_{2\text{max}}) \) and the ratio of the two local maximum velocities \( (A_{1\text{max}}/A_{2\text{max}}) \) are also compared. Both \( A_{1\text{max}} \) and \( A_{2\text{max}} \) are observed to increase with steeper slopes. The maxima of \( A_{1\text{max}} \) and \( A_{2\text{max}} \) occur for currents having a more circular shape at the time of release \( (\Lambda_0 = 0.5 \text{ or } 1) \) on a fixed sloping boundary. The extent of the second acceleration of the maximum velocity at the end of the first acceleration phase \( (A_{1\text{max}}/A_{2\text{max}}) \) varies from 73\% to 138\% for \( \theta = 5^\circ \) and \( 20^\circ \), respectively. \( A_{1\text{max}}/A_{2\text{max}} \) is found to increase with \( \Lambda_0 \) and \( \theta \).

The font condition of a gravity current is studied through the temporal evolution of Froude number. In the present simulations, the currents released with the same slope angle but different initial shapes attain a very similar constant \( Fr \) value after the second acceleration phase. The mean Froude number \( \overline{Fr} \) is seen to increase with increasing slope angles from a value of approximately 0.7 at \( \theta = 5^\circ \) to 1.1 at \( \theta = 20^\circ \), respectively.

The streamwise temporal evolution of the current centre of mass \( x_{CM} \) and its velocity \( u_{CM} \) are also explored. The displacement of the current centre of mass increases as the slope
angle increases. The mass centre for the current with a smaller \( \Lambda_0 \) is observed to travel at a slower rate, which can be explained with the initial local volume partitioning of the current in the spanwise direction, forming a backward moving upstream advancing current and forward moving downstream advancing current. The current centre of mass velocity is initially seen to increase at the same rate for all the currents up to \( t = 1.5 \). Interestingly, three acceleration phases are observed for the current with a small initial cross-sectional aspect ratio (\( \Gamma = 0.1, 0.2 \) and 0.5) and two acceleration phases are found for the current with a moderate initial cross-sectional aspect ratio (\( \Gamma = 1 \) and 2) while only one acceleration phase exists for the current with a large initial cross-sectional aspect ratio (\( \Gamma = 5 \) and 10).

The second series of simulations were carried out to explore the influence of mean height to lock radius ratio (\( \Gamma \)). From the spatial distribution of \( \bar{w}_y \), it can be seen that the currents initially collapse to form a semi-elliptical current head at the downstream end with a long tail. The wall normal position of the current centre of mass is seen to decrease (up to \( t = 2 \)) and subsequently increases to a local maximum (at \( t = 4 - 5 \)), which coincide with the initial acceleration phase and the deceleration phase prior to the second acceleration phase of the front velocity. It shows how the front dynamics are interconnected with the potential to kinetic energy transfer of the current as a whole. Interestingly, the centre of mass of the current collapses to the same value at approximately \( t = 15 \). By comparing the present numerical data to the model in [51], it is found that wall normal current centre of mass \( z_{CM} \) satisfies a Richardson number entrainment criterion and the volume growth \( \alpha_v \) from the present simulations compared well with the prediction by the empirical function from the study by [51]. The temporal evolution of the front velocity clearly indicates that the mean height to lock radius ratio affects the early development of the current. The front velocity eventually collapses to the same value as the current centre of mass after the current develops into a self-similar shape (\( t = 15 \)). It can be concluded that the shape parameter, mean height to lock radius ratio, has influence on the short-time dynamics of the gravity current but doesn’t seem to have a strong long-time influence on the motion of the gravity current.

[17] F. Necker, C. Härtel, L. Kleiser, and E. Meiburg, “High-resolution simulations of particle-


FIG. 1. Schematic of the rectangular computational domain with the temporal evolution of the a circular gravity current released at $\theta = 10^\circ$ (left). The current is visualised with isosurfaces of density with a value of $\rho = 0.05$ at $t = 0$ (red), $t = 3$ (green), $t = 6$ (brandeis blue), $t = 15$ (sky blue) and $t = 30$ (teal blue), respectively. Top and side view of the initial shape of the slanted elliptical cylinder containing the heavy fluid (right).
FIG. 2. Evolution of the equivalent height of the current $h_z$ at $Re = 5000$, $\theta = 10^\circ$ and $\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$ (top to bottom) at $t = 1, 3, 6, 15$ (left to right). The red circle corresponds to the position of the centre of mass in the $x$-$y$ plane.
FIG. 3. Spatial distribution of the equivalent height of the current $\tilde{h}_z$ at $Re = 5000$, $\theta = 5^\circ$, $10^\circ$, $15^\circ$, $20^\circ$ (left to right) and $\Lambda_0 = 0.1$, $0.2$, $0.5$, $1$, $2$, $5$, $10$ (top to bottom) at $t = 30$. The red circle corresponds to the position of the centre of mass in the $x$-$y$ plane.
FIG. 4. Temporal evolution of the aspect ratio $\Lambda$ (shown in solid lines) and the equivalent integral measure of the aspect ratio using the moment of inertia tensor $\sqrt{I_{xx}/I_{yy}}$ (shown in dashed lines) for the currents with seven initial aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$) released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$).
FIG. 5. Temporal evolution of the front location of the current $x_F$ and the front velocity $u_F$ for the currents with seven initial aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$) released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$).
FIG. 6. Temporal evolution of the Froude number $Fr$ for the currents with seven initial aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$) released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$).
FIG. 7. Interdecile mean Froude number $\overline{Fr}$ as a function of slope angle $\theta$ for the currents with seven initial aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$) released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$).
FIG. 8. Temporal evolution of the streamwise centre of mass of the current $x_{CM}$ and the corresponding velocity $u_{CM}$ for the currents with seven initial aspect ratios ($\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$) released at four slope angles ($\theta = 5^\circ, 10^\circ, 15^\circ, 20^\circ$).
FIG. 9. Evolution of spanwise integrated density of the current $\bar{w}_y$ at $Re = 5000$, $\theta = 15^\circ$ and $\Lambda_0 = 1$ at $t = 0, 1, 3, 6, 15, 30$ for the cases with $0.5 \times \Gamma_0$ (top left), $\Gamma_0$ (middle left) and $2 \times \Gamma_0$ (bottom left) with the temporal evolution of the wall normal position of the centre of mass (top right), spatial evolution of the centre of mass from its original release position in the $x$-$z$ plane (middle right) and the comparison of the volume growth rate $\alpha_v$ to the empirical function from the study by [51] (bottom right), respectively. The black arrow in the contours of $\bar{w}_y$ corresponds to the direction of the gravitational force.
FIG. 10. Temporal evolution of the front location $x_F$ and the front velocity $u_F$ for the current with an initial cross-sectional aspect ratio $\Lambda_0 = 1$ release at $\theta = 15^\circ$ at $Re = 5000$ for the cases with $0.5 \times \Gamma_0$, $\Gamma_0$ and $2 \times \Gamma_0$. 
TABLE I. Durations of the first acceleration phase ($A_1$), the first deceleration phase ($D_1$) and the second acceleration phase ($A_2$) at $Re = 5000$ for the four slope angles ($\theta = 5^\circ$, $10^\circ$, $15^\circ$, $20^\circ$) and seven initial aspect ratios ($A_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$)

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TABLE II. The maximum velocity during the first acceleration phase ($A_{1\text{max}}$), the maximum velocity during the second acceleration phase ($A_{2\text{max}}$) and the ratio of the second local maximum velocity over the first local maximum velocity ($A_{2\text{max}}/A_{1\text{max}}$) at $Re = 5000$ for the four slope angles ($\theta = 5^\circ$, 10$^\circ$, 15$^\circ$, 20$^\circ$) and seven initial aspect ratios ($A_0 = 0.1, 0.2, 0.5, 1, 2, 5, 10$).

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