Development of Multivariable PID Controller Gains in Presence of Measurement Noise

Samer S. Saab

Abstract — This paper examines the ability of a multivariable PID controller rejecting measurement noise without the use of any external filter. The work first provides a framework for the design of the PID gains comprising of necessary and sufficient conditions for boundedness of trajectories and zero-error convergence in presence of measurement noise. It turns out that such convergence requires time-varying gains. Subsequently, novel recursive algorithms providing optimal and sub-optimal time-varying PID gains are proposed for discrete-time varying linear multiple-input multiple-output (MIMO) systems. The development of the proposed optimal algorithm is based on minimizing a stochastic performance index in presence of erroneous initial conditions, white measurement noise, and white process noise. The proposed algorithms are shown to reject measurement noise provided that the system is asymptotically stable and the product of the input-output coupling matrices is full-column rank. In addition, convergence results are presented for discretized continuous-time plants. Simulation results are included to illustrate the performance capabilities of the proposed algorithms.

Keywords— Multivariable control, proportional-integral-derivative (PID) controller, discrete-time varying systems, stochastic optimization, measurement noise, sampling rate

I. Introduction

From its birth for about eight decades PID control is considered to be one of the earlier control strategies and remains to be the most common control scheme used in industry. Furthermore, PID controllers are shown to be sufficiently robust to be considered for medical applications (e.g., see [1]). The consent of PID controllers is attributed to its effective performance, simple structure and implementation, and the comprehension of the contribution produced by each of its three terms. However, PID tuning remains poorly understood in many applications.

Traditionally, PID tuning is basically obtained by analysing the step response of a plant by examining the steady-state gain and transient response. Several methods have been developed, which are applicable to single-input single-output stable systems that help in systematically tuning the gains [1]. A survey of different adaptive techniques presented in [2], and a working knowledge of PID design is provided in [3]. In recent years, a good amount of work has been studied for tuning of PID controllers for systems with time delay (see, e.g., [4]-[7]). Several PID tuning methods are based on dominant pole placement (see, e.g., [4]) and other designs also use specific gain and phase margins (see, e.g., [5]) for robustness considerations. Other recent techniques use linear matrix inequalities [6] and system transformation using Rekasius substitution [7].

Considerable amount of work pertaining to analytically determining or tuning PID gains for various MIMO systems has been proposed. One approach is based on optimal control (see, e.g., [8]). A flexible multi-objective optimization tuning design for a class of nonlinear systems has been proposed in [9]. A PID design is successfully implemented to an industrial plant [10], where the approach does not require any knowledge of the plant model including the order of the system. However, the system under consideration is assumed to be asymptotically stable linear time-invariant (LTI) subject to constant reference and disturbance signals. Other techniques based on Lyapunov stability analysis are proposed for PID tuning (see, e.g., [11]-[12]). Another recent technique for selecting appropriate discrete-time integer-order PID gains for a class of MIMO systems is achieved by stabilizing an augmented system where a solution is obtained by treating the augmented system as a static output feedback problem [13]. Another approach that requires mathematical models, which is based on frequency response measurement, is proposed in [14] for two-input two output LTI systems taking into consideration an integrity problem. The solution is obtained by intersecting over stabilizing sets of stabilizing controllers. In [15] the existence of stabilizing PID-controllers for different classes of LTI MIMO plants is presented, and synthesis procedures that guarantee robust closed-loop stability is proposed. Another approach for parameter tuning is the data-driven control, such as virtual reference feedback tuning (VRFT) (see, e.g., [16]) and correlation based tuning (see, e.g., [17], which is only based on input and output data collection. However, when restricting this data-driven technique to PID controller, VRFT optimality realization may not be possible [18]. The use of a filter is recently proposed [18] for MIMO LTI system to approximate the minima of the cost function leading to unbiased estimates.

Although all of the proposed tuning methods above and many others; e.g., based on various intelligent control schemes, come with a number of unique features, they do not consider effectively attenuating the effects of measurement noise.

A problem with PID controller is in its derivative term. This term can lead to undesired effects due to saturation [19] and amplifies higher frequency measurement noise that can cause significant degradation in the tracking performance. Many proposed PID controllers embed different robust designs limiting the effects of higher frequency noise. Even if the approach were based on a holistic multi-objective optimization design, difficulties in assigning weights that relate different performance criteria and robustness would arise [20]. A common approach used in industry to limit noise sensitivity is to filter either the measured output or the derivative part. During the past few years, considerable efforts have been oriented towards the remedy of measurement noise (see, e.g., [20]-[32]). In order to reduce the effects of noise, a method for tuning the gains is proposed [21]. Another approach is the implementation of a low-pass filter with some specific time constant [22]. The selections of PID gains and filter time constant are optimally determined based on a performance index [20], [23]-[25]. A significant reduction in noise is achieved when employing a second order Butterworth filter at a cost of a moderate decrease of performance [20].

Constrained optimization is used to design PID controllers and noise filters of different orders [26], [29]-[30]. The effectiveness of the constraint optimization method for parallel PID controllers presented in [30] is supported experimentally and numerically. A study presents insight into the trade-offs between performance and robustness with respect to process variations and model uncertainty, and effects of measurement noise [31]. A proposed PID design procedure is presented, which employs a software-based method to find controllers with different noise filters possessing optimal or near optimal load disturbance response subject to noise sensitivity constraints [32]. However, most of the methods employ low-pass filters while targeting SISO systems yet the adverse filtering effects of a filter on the system seem to be inevitable.

The aim of this paper is to examine PID controllers with the ability of rejecting measurement noise and erroneous initial conditions without using any external filter. It is important to note that rejection of measurement noise is also made possible; however, using stochastic iterative learning control [33]-[35]. This paper first provides a framework for the design of the PID gains by presenting necessary and sufficient conditions for boundedness of trajectories and zero-error convergence in a statistical sense. It is shown that the zero-error convergence in presence of measurement noise requires that the PID gains converge to zero; that is, the PID gains must be time-varying. Subsequently, a novel recursive algorithm providing optimal time-varying PID gains is proposed for the case where the number of inputs is less than or equal to the number of outputs. The development of the proposed algorithm is based on minimizing a stochastic performance index in presence of erroneous initial conditions, measurement noise, and process noise. In addition, a similar sub-optimal algorithm is proposed for the case of erroneous plant model and erroneous model of random disturbances. Both algorithms are shown to satisfy the presented conditions by the developed framework for the design of PID gains provided that the plant is asymptotically stable and the product of the output-input coupling matrices is full-column rank. In addition, convergence results for both algorithms are presented for discretized continuous-time plants. The effectiveness of the proposed method is confirmed numerically and performance is compared with other controllers. A recursive algorithm with noise rejection capability is also proposed for asymptotically stable linear discrete time-varying systems with no more inputs than outputs [36]. Although the algorithm is developed based on minimizing the same stochastic performance index, the proposed algorithm cannot be considered optimal since its development neglects the correlation between the input error and state error. It is important to note that the implementation of the proposed algorithms is much easier than the one presented in [36].

The rest of the paper is organized as follows. Section II defines the system and formulation of the three problems under consideration. The proposed recursive algorithms and characteristics are given in Section III, followed by simulation examples in Section IV. Finally, Section V summarizes the results and draw conclusions.

II. System Descriptions and Problem Formulation

The system under consideration is a discrete time-varying system described by the following state-space equation

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + w(k) \\ y(k) &= C(k)x(k) + v(k) \end{aligned} \tag{1}$$

where the argument $k \in \aleph$ is the discrete time index, $x(k) \in \Re^n$ is the state matrix, $w(k) \in \Re^n$ is the state disturbance, $y(k) \in \Re^p$ is the measured system output and $C(k)x(k)$ is the system output, $u(k) \in \Re^q$ is the system input, and $v(k) \in \Re^p$ is the output measurement error. It is assumed that either the plant is inherently discrete time with time-interval $T_s > 0$, or that the model (1) is described by an underlying linear continuous-time varying model, which is sampled with $T_s > 0$ sampling period and with a zero-order hold. The values of the parameters and variables in (1) depend on the magnitude of the sampling period whenever the system in (1) is the result of a discretized continuous-time system.

Notations: We denote by $E[.]$ to be the expectation operator, $\prod_{i=0}^{k} M_i \equiv M_0 M_1 \cdots M_k$, $\prod_{i=k+1}^{k} M_i \equiv I$, $I_m \in \Re^{m\times m}$ the identity matrix, $0_{n\times m} \in \Re^{n\times m}$ is the zero matrix, $\lambda(M)$ the eigenvalues of M and $tr(.)$ is the trace operator.

Assumptions:

(A1) The system matrices $[A(k) \quad B(k) \quad C(k)]$ are assumed bounded for all k and as $k \to \infty$.

(A2) For any realizable output trajectory or reference signal, $y_d(k) = [y_{1,d}(k) \ldots y_{q,d}(k)]^T$, and an appropriate initial condition $x_d(0)$, there exists a control input $u_d(k) \in \Re^q$ generating the desired output trajectory, $y_d(k)$, for the nominal plant. That is, the following difference equation is satisfied

$$\begin{aligned} x_d(k+1) &= A(k)x_d(k) + B(k)u_d(k) \\ y_d(k) &= C(k)x_d(k) \end{aligned} \tag{2}$$

where $x_d(k)$ is the state response due to $u_d(k)$ with a given $x_d(0)$ such that $y_d(0) = C(0)x_d(0)$.

(A3) $v(.)$, and $w(.)$ are zero-mean white noise processes mutually uncorrelated with each other and with $x(0)$. $E[w(k)w(k)^T] = Q(k) \geq 0$, and $E[v(k)v(k)^T] = R(k) > 0$.

It is important to note that (A2) is an expected assumption for an output tracking problem, that is, the reference trajectory $y_d(k)$ has a solution. However, (A2) may not be considered restrictive provided that the number of inputs equals to the number of outputs where $C(k+1)B(k)$ is nonsingular. Since the *existence* of $u_d(k)$ is guaranteed by setting, $\forall k$,

$$u_d(k) = \left(C(k+1)B(k)\right)^{-1}\left(y_d(k+1) - C(k+1)A(k)x_d(k)\right).$$

The above construction of $u_d(k)$, which is presented only to show its existence, assumes full and certain knowledge of the plant and state initial condition, and absence of disturbances. However, if the number of outputs is greater than number of inputs, then this assumption becomes necessary. In addition, (A2) may require $C(k+1)B(k)$ being full-column rank. Although such a relative degree assumption may be considered limiting in general it is not the case for discretized continuous plants. For example, consider a LTI system described by $\dot{x}(t)=A_c x(t)+B_c u(t)$, and $y(t)=Cx(t)$. If the input is discretized using a zero-order hold then $B=\int_{\tau=0}^{T_s} e^{A_c\tau} d\tau B_c$ where $e^{A_c\tau}=I+A_c\tau+\frac{1}{2!}(A_c\tau)^2+...$ If the plant is controllable, then CB becomes full-column rank provided that C is full-row rank. However, for small sampling period $\int_{\tau=0}^{T_s}\frac{1}{m!}(A_c\tau)^m d\tau B_c \approx 0$, for $m>2$, then it would be practical to assume that the relative degree of the continuous-time plant is no larger than 2 in order to have $C(k+1)B(k)$ full-column rank.

(A1)-(A3) are assumed to hold throughout this paper.

The control law under consideration[1] is given by

$$u(k+1)=u(k)+\sum_{i=1}^{3} K_i(k)e(k+2-i) \tag{3}$$

where the matrices $K_i(k), i\in\{1,2,3\}$, represent the $(p\times q)$ learning gains, and $e(k)\equiv y_d(k)-y(k)$ is the output measurement error due to the control action $u(k)$.

It is worthwhile noting that the work in [37] shows that while applying the backward rectangular rule to an ideal PID controller, the *z*-domain transfer function of the controller can be written as

$$\frac{U(z)}{E(z)}=\frac{(K_P+K_I+K_D)z^2-(2K_P+K_D)z+K_D}{z^2-z}$$

where K_P, K_I and K_D are the proportional, integration and differentiation gains, respectively. The transfer function above is based on setting the time intervals over which integration and differentiation are done to the sampling period. By contrasting the relevant parameters of the transfer function and (3), we find that $K_1=K_P+K_I+K_D$, $K_2=2K_P+K_D$ and $K_3=K_D$.

Define the state and the input errors as:
$\delta x(k)\equiv x_d(k)-x(k)$ and $\delta u(k)\equiv u_d(k)-u(k)$, respectively.

For compactness of presentation, we denote:

$$A\equiv A(k), B\equiv B(k), C\equiv C(k), C^+\equiv C(k+1)$$
$$A^-\equiv A(k-1), B^-\equiv B(k-1), C^-\equiv C(k-1)$$
$$K_i\equiv K_i(k),\ i\in\{1,2,3\}$$

The input error model corresponding to the update law can be derived from (3) as follows:

$$\begin{aligned}\delta u(k+1)=\delta u(k)-K_1e(k+1)-K_2e(k)\\ -K_3e(k-1)+\Delta u_d(k)\end{aligned} \tag{4}$$

where $\Delta u_d(k)\equiv u_d(k+1)-u_d(k)$. Equations (1) and (2) yield

$$\delta x(k+1)=A\delta x(k)+B\delta u(k)-w(k) \tag{5}$$

which further lead to

$$\begin{aligned}e(k+1)=C^+A\left(A^-\delta x(k-1)+B^-\delta u(k-1)-w(k-1)\right)\\ +C^+B\delta u(k)-C^+w(k)-v(k+1)\end{aligned} \tag{6}$$

Similarly, we can obtain

$$e(k)=CA^-\delta x(k-1)+CB^-\delta u(k-1)-Cw(k-1)-v(k) \tag{7}$$

$$e(k-1)=C^-\delta x(k-1)-v(k-1) \tag{8}$$

Inserting Equations (6)-(8) in (4) and rearranging terms yield

$$\begin{aligned}\delta u(k+1)&=(I-K_1C^+B)\delta u(k)-\left(K_1C^+AB^-+K_2CB^-\right)\delta u(k-1)\\ &-\left(K_1C^+AA^-+K_2CA^-+K_3C^-\right)\delta x(k-1)\\ &+K_1C^+w(k)+\left(K_1C^+A+K_2C\right)w(k-1)\\ &+K_1v(k+1)+K_2v(k)+K_3v(k-1)+\Delta u_d(k)\end{aligned} \tag{9}$$

In order to arrange Equations (5) and (9) into an augmented system, we define:

$$X\equiv\begin{bmatrix}\delta u(k)\\ \delta u(k-1)\\ \delta x(k-1)\end{bmatrix}, X^+\equiv\begin{bmatrix}\delta u(k+1)\\ \delta u(k)\\ \delta x(k)\end{bmatrix}, \Omega\equiv\begin{bmatrix}\Delta u_d(k)\\ 0\\ 0\end{bmatrix}$$

$$V\equiv\begin{bmatrix}v(k+1)\\ v(k)\\ v(k-1)\end{bmatrix}, W\equiv\begin{bmatrix}w(k)\\ w(k-1)\end{bmatrix}$$

We combine (5) and (9) to obtain

$$X^+=\Phi X+\bar{I}KV+(\bar{I}K\Gamma_1+\Gamma_2)W+\Omega \tag{10}$$

where $\Phi\equiv \mathrm{H}-\bar{I}K\mathrm{E}$, $K\equiv\begin{bmatrix}K_1 & K_2 & K_3\end{bmatrix}$,

$$\mathrm{H}\equiv\begin{bmatrix}I_q & 0_q & 0_{qn}\\ I_q & 0_q & 0_{qn}\\ 0_{nq} & B^- & A^-\end{bmatrix}, \bar{I}\equiv\begin{bmatrix}I_q\\ 0_q\\ 0_{nq}\end{bmatrix}, \mathrm{E}\equiv\begin{bmatrix}C^+B & C^+AB^- & C^+AA^-\\ 0_{pq} & CB^- & CA^-\\ 0_{pq} & 0_{pq} & C^-\end{bmatrix}$$

$$\Gamma_1\equiv\begin{bmatrix}C^+ & C^+A\\ 0_{pn} & C\\ 0_{pn} & 0_{pn}\end{bmatrix}, \Gamma_2\equiv\begin{bmatrix}0_{qn} & 0_{qn}\\ 0_{qn} & 0_{qn}\\ 0_n & I_n\end{bmatrix}$$

The dimensions of relevant matrices are presented for convenience:

$C, CA: p\times n$	$X,\Omega: 2q+n\times 1$	$\bar{I}: 2q+n\times q$
$B: n\times q$	$V: 3p\times 1$	$\mathrm{E}: 3p\times 2q+n$
$CB, K_i^T: p\times q$	$W: 2n\times 1$	$\Gamma_1: 3p\times 2n$
$K^-: q\times 3p$	$\mathrm{H}, P, \Phi: 2p+n\times 2p+n$	$\Gamma_2: 2q+n\times 2n$

Next, we define the covariance of the augmented system state (10) as $P^+\equiv E[X^+X^{+T}]$ and $P\equiv E[XX^T]$, which statistically represent the input error and state error.

In the following, we present three different problems under consideration for the system (1) employing the PID controller given in (3) under Assumptions (A1)-(A3).

Problem 1 Statement: Establish a framework for the design of the PID gains for boundedness of all trajectories, and zero convergence of the covariance, $P(k)$, in presence of measurement noise and initialization errors.

Problem 2 Statement: Develop optimal PID gains that satisfy the framework established in Problem 1. In addition, for discretized continuous system, show that $\lim_{k\to\infty}\|P(k)\|$ can be

[1] A more common form of a PID control law is given by $u(k)=u(k-1)+\sum_{i=1}^{3}K_i(k)e(k+1-i)$

made arbitrary small for sufficiently small sampling period in presence of measurement noise and initialization errors.

Problem 3 Statement: Design PID gains that need not use the knowledge of the state matrix, $A(k)$, as well as knowledge of all disturbances statistics, $R(.), Q(k), \text{and } P(0)$, while satisfying the framework established in Problem 1. In addition, for discretized continuous system, show that $\lim_{k\to\infty}\|P(k)\|$ can be made arbitrary small for sufficiently small sampling period in presence of measurement noise and initialization errors.

III. Main Results

This section addresses Problem 1-3. In particular, Theorems 1 and 2 provide a solution to Problem 1 by establishing relevant necessary and sufficient conditions. Corollary 1 shows that $\lim_{k\to\infty} P(k)=0 \Rightarrow \lim_{k\to\infty} K(k)=0$. Theorems 3-5 resolve Problem 2, where Theorem 3 presents an optimal recursive algorithm, and Theorem 4 shows that the proposed optimal algorithm satisfies the conditions in Theorems 1 and 2. In addition, Theorem 4 emphasizes the boundedness of all trajectories are guaranteed in presence of process noise, which is not considered in Theorem 1. Theorem 5 shows that $\lim_{k\to\infty}\|P(k)\|$ can be made arbitrary small for sufficiently small sampling period in presence of measurement noise and initialization errors. Subsequently, a suboptimal recursive algorithm is also proposed. Theorems 6-7, which are the counterparts of Theorems 4 and 5, address Problem 3.

A. Framework for the design of the PID gains

This section addresses Problem 1; in particular, it establishes necessary and sufficient conditions pertaining to the boundedness and convergence of $P(k)$.

Since processes are uncorrelated, that is, $\delta x(0)$, $v(.)$ and $w(.)$ are assumed zero-mean white noise, then (10) leads to

$$P^{+}=\Phi P\Phi^{T}+\bar{I}K\bar{R}(\bar{I}K)^{T}+(\bar{I}K\Gamma_1+\Gamma_2)\bar{Q}(\bar{I}K\Gamma_1+\Gamma_2)^{T}+E[\Omega\Omega^{T}] \quad (11)$$

where

$$\bar{R}=\begin{bmatrix} R(k+1) & 0 & 0\\ 0 & R(k) & 0\\ 0 & 0 & R(k-1)\end{bmatrix},\bar{Q}=\begin{bmatrix} Q(k) & 0\\ 0 & Q(k-1)\end{bmatrix}$$

For $k>0$, the solution is given by

$$P(k)=\bar{P}_o+\bar{P}_{\mathrm{T}} \quad (12)$$

where

$$\bar{P}_o\equiv\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)P(0)\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)^{T}$$

$$\bar{P}_{\mathrm{T}}\equiv\sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\mathrm{T}(i)\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^{T}$$

with

$\mathrm{T}(i)\equiv\bar{I}K\bar{R}K^{T}\bar{I}^{T}+(\bar{I}K\Gamma_1+\Gamma_2)\bar{Q}(\bar{I}K\Gamma_1+\Gamma_2)^{T}+E[\Omega\Omega^{T}]$.

Theorem 1: Let system (1) satisfy Assumptions A1)–A3) with $R(k)>0,\ Q(k)=0,\ \forall k\geq 0$ and $P(0)>0$, and updating law (3) be applied with K being full-row rank. The boundedness of all trajectories is guaranteed if and only if $\exists c_{\Pi}>0, c_{\Sigma}>0$ such that $\forall k>0$

$$\left\|\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)^{T}\right\|\leq c_{\Pi} \quad (13)$$

$$\left\|\sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}K(\bar{I}K)^{T}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^{T}\right\|\leq c_{\Sigma} \quad (14)$$

Proof of Theorem 1:
Since $P(k)$ is the summation of two nonnegative definite matrices, $\bar{P}_o$ and $\bar{P}_{\mathrm{T}}$, then $P(k)$ if and only if *both* $\bar{P}_o$ and $\bar{P}_{\mathrm{T}}$ are bounded. Since $P(0)>0$, $\exists \underline{c}>0, \bar{c}>0$ such that $\underline{c}I<P(0)<\bar{c}I$ hence $\bar{P}_o$ is bounded if and only if $\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)\left(\prod_{i=0}^{k-1}\Phi(k-1-i)\right)^{T}$ is bounded.

K, being full-row rank, cannot be unbounded else condition (14), would be violated as well as $\bar{P}_{\mathrm{T}}$ would become unbounded. With $R(k)>0,\ Q(k)=0,\ E[\Delta u_d \Delta u_d^{T}]\geq 0$ is bounded $\forall k\geq 0$, $KK^{T}>0$ and $K\bar{R}K^{T}>0$. Therefore, $\exists c_{\tau}\geq 0, c_{\mathrm{T}}>0$ such that

$$c_{\tau}\bar{I}\left(KK^{T}\right)\bar{I}^{T}\leq \mathrm{T}(i)=\bar{I}\left(K\bar{R}K^{T}+E[\Delta u_d\Delta u_d^{T}]\right)\bar{I}^{T}\leq c_{\mathrm{T}}\bar{I}\left(KK^{T}\right)\bar{I}^{T}$$

Consequently, (14) is satisfied if and only if $\bar{P}_{\mathrm{T}}$ is bounded. ■

Theorem 2: Let system (1) satisfy Assumptions A1)–A3) with $R(k)>0,\ Q(k)=0,\ \forall k\geq 0$ and $P(0)>0$, and updating law (3) be applied with K being full-row rank.

If $\lim_{k\to\infty}\Delta u_d(k)=0$, then $\lim_{k\to\infty}P(k)=0$ if and only if

$$\lim_{k\to\infty}\prod_{i=0}^{k-1}\Phi(k-1-i)=0 \quad (15)$$

and

$$\lim_{k\to\infty}\sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}K(i)K^{T}(i)\bar{I}^{T}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^{T}=0 \quad (16)$$

Proof of Theorem 2:
Since $P(k)$ is the summation of two nonnegative definite matrices, $\bar{P}_o$ and $\bar{P}_{\mathrm{T}}$, then $\lim_{k\to\infty}P(k)=0$ if and only if *both* $\bar{P}_o$ and $\bar{P}_{\mathrm{T}}$ converge to zero.

Necessary and Sufficient condition of (15) *pertaining to* $\bar{P}_o$:
Since $P(0)>0$, then convergence of $\bar{P}_o$ is equivalent to (15).

Necessary and Sufficient condition of (16) *pertaining to* $\bar{P}_{\mathrm{T}}$:
It is important to note that although from the proof of Theorem 1, we have: $\exists c_{\tau}\geq 0, c_{\mathrm{T}}>0$ such that $\forall k\geq 0$

$$c_{\tau}\bar{I}\left(KK^{T}\right)\bar{I}^{T}\leq \mathrm{T}(i)=\bar{I}\left(K\bar{R}K^{T}+E[\Delta u_d\Delta u_d^{T}]\right)\bar{I}^{T}\leq c_{\mathrm{T}}\bar{I}\left(KK^{T}\right)\bar{I}^{T}$$

The above upper bound may not hold as $k\to\infty$ unless $\lim_{k\to\infty}\Delta u_d(k)=0$. Subsequently, a more thorough proof is provided.

Necessary condition of (16):

Since $E[\Delta u_d \Delta u_d{}^T] \geq 0$ or equivalently $E[\Omega\Omega^T] \geq 0$, then (16) is necessary for convergence of $\overline{P}_{\mathrm{T}}$.

Sufficient condition of (16):

(16) implies the convergence of the part of $\overline{P}_{\mathrm{T}}$ corresponding to $\bar{I}K\overline{R}K^T\bar{I}^T$. It remains to show that

$$\lim_{k\to\infty}\sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}E[\Delta u_d \Delta u_d{}^T]\bar{I}^T\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T=0$$

Consequently, we need to show that $\forall\, i \geq 0$ and as $i \to \infty$

$$\lim_{k\to\infty}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}E[\Delta u_d \Delta u_d{}^T]\bar{I}^T\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T=0$$

Since $\forall\, i \geq 0$, $\bar{I}E[\Delta u_d \Delta u_d{}^T]\bar{I}^T \geq 0$, then $\exists c_\Delta > 0$ such that $0 \leq \bar{I}E[\Delta u_d \Delta u_d{}^T]\bar{I}^T \leq c_\Delta I$, hence,

$$\lim_{k\to\infty}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}E[\Delta u_d \Delta u_d{}^T]\bar{I}^T\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T \leq$$

$$c_\Delta \lim_{k\to\infty}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T=0$$

where the zero equality above is due to (15). (15) implies that any product of $\Phi(.)$ is bounded, that is, $\exists c_\varsigma > 0$ such that $\left\|\prod_\varsigma \Phi(\varsigma)\right\| \leq c_\varsigma$. Thus,

$$\lim_{\substack{i\to\infty\\k\to\infty}}\left\|\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}E[\Delta u_d(i) \Delta u_d{}^T(i)]\bar{I}^T\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T\right\|$$

$$\leq c_\varsigma^2 \lim_{i\to\infty}\left\|\bar{I}E[\Delta u_d(i) \Delta u_d{}^T(i)]\bar{I}^T\right\|=0$$

where the zero equality above is due to $\lim_{k\to\infty}\Delta u_d(k)=0.$ ■

Corollary 1: If (15) and (16) hold, then $\lim_{k\to\infty}K(k)=0.$

Proof of Corollary 1: (16) is the series of nonnegative matrices. In order for (16) to hold, then, we need to have all the terms to be zero including those at infinity; in particular,

$$\lim_{\substack{i\to\infty\\k\to\infty}}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\bar{I}K(i)K^T(i)\bar{I}^T\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)^T=0$$

$\Phi(.) \neq 0$, and indefinite products of $\Phi(.)$ are different than zero, in particular, $\lim_{\substack{i\to\infty\\k\to\infty}}\left(\prod_{j=1}^{k-1-i}\Phi(k-j)\right)\neq 0$, which is multiplied by $\lim_{i\to\infty}K(i)$. For example, consider one of the following terms:

$$\Phi(.)\bar{I}K(.)=\begin{bmatrix}(I-K_1C^+B)K_1 & (I-K_1C^+B)K_2 & (I-K_1C^+B)K_3\\ K_1 & K_2 & K_3\\ 0 & 0 & 0\end{bmatrix}$$

Consequently, we are left with $\lim_{i\to\infty}K(i)=0.$ ■

B. Optimal Gains

This section addresses Problem 2 by first presenting an optimal recursive algorithm for generating the PID gains. The optimality is based on the minimization of the $tr(P(k+1))$ at each time sample. Subsequently, the characteristics of the proposed algorithm according to Problem 2 are provided.

Theorem 3: Let system (1) satisfy Assumptions A1)–A3) and updating law (3) be applied. The PID gains, represented in $K(k)$, that minimize the mean-square of the input and state errors, that is minimizing the $tr(P(k+1))$, at each k^{th} instant are given in the following recursive formulas for all $k > 0$:

$$K=\bar{I}^T P\mathrm{E}^T(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1} \tag{17}$$

$$\begin{aligned}P^+ &= (\mathrm{H}-\bar{I}K\mathrm{E})P(\mathrm{H}-\bar{I}K\mathrm{E})^T+\bar{I}K\overline{R}(\bar{I}K)^T\\ &+(\bar{I}K\Gamma_1+\Gamma_2)\overline{Q}(\bar{I}K\Gamma_1+\Gamma_2)^T+E[\Omega\Omega^T]\end{aligned} \tag{18}$$

Proof of Theorem 3:

Expanding (11) and collecting terms

$$\begin{aligned}P^+ &= \mathrm{H}P\mathrm{H}^T-\mathrm{H}P(\bar{I}K\mathrm{E})^T-\bar{I}K\mathrm{E}P\mathrm{H}^T\\ &+\bar{I}K(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)(\bar{I}K)^T\\ &+\bar{I}K\Gamma_1\overline{Q}\Gamma_2^T+\Gamma_2\overline{Q}\Gamma_1^T(\bar{I}K)^T+\Gamma_2\overline{Q}\Gamma_2^T+E[\Omega\Omega^T]\end{aligned}$$

To minimize $tr(P^+)$ with respect to K we set $\dfrac{\partial tr(P^+)}{\partial K}\equiv 0$

$$\begin{aligned}\frac{\partial tr(P^+)}{\partial K} &= -2\bar{I}^T\mathrm{H}P\mathrm{E}^T+2\bar{I}^T\bar{I}K(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)\\ &+2\bar{I}^T\Gamma_2\overline{Q}\Gamma_1^T\equiv 0\end{aligned}$$

Therefore,

$$K=(\bar{I}^T\mathrm{H}P\mathrm{E}^T-\bar{I}^T\Gamma_2\overline{Q}\Gamma_1^T)(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}$$

Since $\bar{I}^T\mathrm{H}=\bar{I}^T$ and $\bar{I}^T\Gamma_2=0$, then

$$K=\bar{I}^T P\mathrm{E}^T(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}$$ ■

Claim 1. $0 \leq \lambda\left(P\mathrm{E}^T(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E}\right)<1$

In addition, let M be the q^{th} leading principal minor of $P\mathrm{E}^T(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E}$ (first $q\times q$ block of the matrix).

If C^+B is full-column rank, then

$$0<\lambda(M)<1$$

Proof of Claim 1. Woodbury identity implies that

$$P\mathrm{E}^T(\mathrm{E}P\mathrm{E}^T+\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E}=I-(I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}$$

P and $(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)$ are positive definite. Thus, $\lambda\left(P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E}\right)\geq 0$ and $0<\lambda\left((I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}\right)\leq 1$. Therefore,

$$0\leq\lambda\left(I-(I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}\right)<1$$

$M=\bar{I}^T\left[I-(I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}\right]\bar{I}$ or

$M=I_q-\bar{I}^T(I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}\bar{I}$

Following the same steps as above leads to $0\leq\lambda(M)<1$.

Since $P>0$, $\overline{R}>0$ and $\Gamma_1\overline{Q}\Gamma_1^T\geq 0$, then the zero eigenvalue(s) of $\left(I-(I+P\mathrm{E}^T(\overline{R}+\Gamma_1\overline{Q}\Gamma_1^T)^{-1}\mathrm{E})^{-1}\right)$ is only due to the likelihood of E not being full-column rank. That is, if

E were full-column rank, then all eigenvalues would be strictly positive regardless the values of P and $\bar{R}$. Consequently, without loss of generality, for simplicity we assume here that $P = I$ and $\bar{R} + \Gamma_1 \bar{Q} \Gamma_1^T = I$.

Therefore, $M \equiv I_q - \bar{I}^T (I + \mathrm{E}^T \mathrm{E})^{-1} \bar{I}$. Using one of the Searle set of identities leads to $(I + \mathrm{E}^T \mathrm{E})^{-1} = I - \mathrm{E}^T (I + \mathrm{E}\mathrm{E}^T)^{-1} \mathrm{E}$ and $M \equiv \bar{I}^T \mathrm{E}^T (I + \mathrm{E}\mathrm{E}^T)^{-1} \mathrm{E}\bar{I}$. Since $\mathrm{E}\bar{I} = \begin{bmatrix} C^+ B \\ 0 \\ 0 \end{bmatrix}$ is full column rank, then M is positive definite (nonsingular), that is $0 < \lambda(M) < 1$. ■

Theorem 4: Let $R(k) > 0,\ \forall k \geq 0$ and $P(0) > 0$. If $C^+ B$ is full-column rank, and the system in (1) is asymptotically stable, $\rho(A(k)) < 1\ \forall k \geq 0$, then the optimal recursive algorithm (17)-(18) guarantee that:

- K is full-row rank and $0 < \lambda(\Phi) < 1,\ \forall k \geq 0$
- the conditions (13)-(14) of Theorem 1 are satisfied and $P(k)$ is bounded $\forall k$ while $Q(k) \geq 0,\ \forall k \geq 0$
- the conditions (15)-(16) of Theorem 2 are satisfied, $\lim_{k\to\infty} P(k) = 0$ and $\lim_{k\to\infty} K(k) = 0$ whenever $Q(k) = 0,\ \forall k \geq 0$ and $\lim_{k\to\infty} \Delta u_d(k) = 0$

Proof of Theorem 4:

Since $\mathrm{E}\bar{I}$ is full-column rank and consequently $\bar{I}^T P \mathrm{E}^T$ is full-row rank, and K is also full-row rank.

Inserting (17) in $\mathrm{H} - \bar{I}K\mathrm{E}$, we obtain

$$\mathrm{H} - \bar{I}K\mathrm{E} = \mathrm{H} - \bar{I}\bar{I}^T \mathrm{H} P \mathrm{E}^T (\mathrm{E} P \mathrm{E}^T + \bar{R} + \Gamma_1 \bar{Q} \Gamma_1^T)^{-1} \mathrm{E}$$

$$= \mathrm{H} - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P \mathrm{E}^T (\mathrm{E} P \mathrm{E}^T + \bar{R} + \Gamma_1 \bar{Q} \Gamma_1^T)^{-1} \mathrm{E}$$

$$\Phi = \mathrm{H} - \bar{I}K\mathrm{E} = \begin{bmatrix} I - M & 0 & 0 \\ I & 0 & 0 \\ 0 & B^- & A^- \end{bmatrix}$$

where M is the q^{th} leading principal minor of $P\mathrm{E}^T (\mathrm{E} P \mathrm{E}^T + \bar{R} + \Gamma_1 \bar{Q} \Gamma_1^T)^{-1} \mathrm{E}$ (first $q \times q$ block of the matrix). Thus, $\lambda(\Phi) = \underbrace{\{0, 0, \cdots, 0\}}_{q} \cup \lambda(I - M) \cup \lambda(A^-)$. Since system is assumed asymptotically stable, Claim 1 yields $0 < \lambda(I - M) < 1$, therefore, $0 < \lambda(\Phi) < 1 \forall k$, which leads to (13) and (15). Since $\Phi = \mathrm{H} - \bar{I}K\mathrm{E}$ is bounded $\forall k$, then $\bar{I}K\mathrm{E}$ is bounded $\forall k$. Consider

$$\bar{I}K\mathrm{E} = \begin{bmatrix} K_1 C^+ B & K_1 C^+ A B^- + K_2 C B^- & K_1 C^+ A A^- + K_2 C A^- + K_3 C^- \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $C^+ B$ and CB^- are full-column rank and C^- must be full-row rank, then K is bounded $\forall k$.

$$P^+ = (\mathrm{H} - \bar{I}K\mathrm{E}) P (\mathrm{H} - \bar{I}K\mathrm{E})^T + \bar{I}K\bar{R}(\bar{I}K)^T + (\bar{I}K\Gamma_1 + \Gamma_2)\bar{Q}(\bar{I}K\Gamma_1 + \Gamma_2)^T + E[\Omega\Omega^T]$$

Since $\bar{I}K\bar{R}(\bar{I}K)^T + (\bar{I}K\Gamma_1 + \Gamma_2)\bar{Q}(\bar{I}K\Gamma_1 + \Gamma_2)^T + E[\Omega\Omega^T]$ is bounded and $0 < \lambda(\Phi) < 1 \forall k$, imply that $P(k)$ is bounded $\forall k$, hence (14) is satisfied.

Expanding and rearranging terms in (11) with $Q = 0$, yield

$$P^+ = (\mathrm{H} - \bar{I}K\mathrm{E}) P \mathrm{H}^T - \mathrm{H} P (\bar{I}K\mathrm{E})^T + \bar{I}K(\mathrm{E}P\mathrm{E}^T + \bar{R})(\bar{I}K)^T + E[\Omega\Omega^T]$$

Inserting (17) with $Q = 0$ and rearranging terms, lead to

$$P^+ = \Phi P \mathrm{H}^T + (\bar{I}\bar{I}^T - \mathrm{H}) P \mathrm{E}^T (\bar{I}K)^T + E[\Omega\Omega^T]$$

We have

$$\Phi = \mathrm{H} - \bar{I}K\mathrm{E} = \begin{bmatrix} I - M & 0 & 0 \\ I & 0 & 0 \\ 0 & B^- & A^- \end{bmatrix}$$

$$\bar{I}\bar{I}^T - \mathrm{H} = \begin{bmatrix} 0 & 0 & 0 \\ -I & 0 & 0 \\ 0 & -B^- & -A^- \end{bmatrix}, (\bar{I}K\mathrm{E})^T = \begin{bmatrix} M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\bar{I}^T \Phi = (I - M)\bar{I}^T$, $\mathrm{H}^T \bar{I} = \bar{I}$ and $\bar{I}^T(\bar{I}\bar{I}^T - \mathrm{H}) = 0$

$P_u(k) \equiv E[\delta u(k) \delta u^T(k)] = \bar{I}^T P(k) \bar{I}$. Therefore,

$$P_u(k+1) = (I - M) P_u(k) + \Delta_d(k)$$

where $\Delta_d(k) \equiv E\left[\Delta u_d(k) \Delta u_d(k)^T\right]$.

From the proof of Claim 1, we observe that if $P(k)$ is bounded and whenever $\lim_{k\to\infty} P(k) \neq 0$, then $\exists 0 < c_M < 1$ such that $0 < \lambda\left(\lim_{k\to\infty}(I - M)\right) \leq c_M < 1$. If $\lim_{k\to\infty} P(k) = 0$, then this ends the proof, else, we have $0 < \lambda\left(\lim_{k\to\infty}(I - M)\right) \leq c_M < 1$. In addition, since $\lim_{k\to\infty} \Delta u_d(k) = 0$, then $\lim_{k\to\infty} \Delta_d(k) = 0$. Consequently, $\lim_{k\to\infty} P_u(k) = 0$. Furthermore,

$$P_x(k) \equiv E[\delta x(k) \delta x^T(k)] = A^- P_x(k-1) A^{-^T} + B^- P_u(k-1) B^{-^T}$$

Since system is asymptotically stable and $\lim_{k\to\infty} P_u(k) = 0$, then $\lim_{k\to\infty} P_x(k) = 0$ and thus $\lim_{k\to\infty} P(k) = 0$ since $P(k)$ is a covariance matrix with diagonal block elements: $P_u(k)$, $P_u(k-1)$ and $P_x(k)$. The latter implies (16).

Whereas $\lim_{k\to\infty} K(k) = 0$ is readily clear from (17). ■

Remark 1. Assuming that $C(k+1)B(k)$ is full-column rank requires that the number of outputs to be greater than or equal to the number of inputs.

Remark 2. Although the assumption that $\lim_{k\to\infty} \Delta u_d(k) = 0$, may be considered restrictive and especially for time-varying systems, Theorem 4 indicates the capability of the controller rejecting random measurement noise, and random initial conditions. However, $\lim_{k\to\infty} P(k) = 0$ requires absence of random state disturbances. The capability of rejecting random measurement noise, whenever $\lim_{k\to\infty} \Delta u_d(k) \neq 0$, is studied in Theorem 5, discussed in Remark 3, and further justified with numerical examples in Section IV.

Definition 1. A trajectory $u_d(k)$ is said to be smooth if for any given sampling period T_s and any consistent norm $\|.\|$, $\exists c_u > 0$ such that $\forall\ k \geq 0$,

$$\|u_d(k+1) - u_d(k)\| \leq c_u T_s$$

Theorem 5. Consider the optimal recursive algorithm presented in (17) and (18). If C^+B is full-column rank, the system in (1) is asymptotically stable, the trajectory of $u_d(k)$ is smooth, $R(k) > 0$, and $Q(k) = 0$, $\forall\, k \geq 0$, then $\exists c_P(c_d T_s) > 0$ such that:

$$\lim_{k\to\infty} \|P(k)\| \leq c_P(c_d T_s) \tag{19}$$

where $c_P(c_d T_s)$ is a decreasing function of $c_d T_s$ and $\lim_{c_d T_s \to 0} c_P(c_d T_s) = 0$.

Proof of Theorem 5.

From the proof of Theorem 4, we have $P_u(k+1) = (I - M)P_u(k) + \Delta_d(k)$. Solving for $P_u(k)$,

$$P_u(k) = \left(\prod_{i=0}^{k-1}(I-M)\right)P_u(0) + \sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}(I-M)\right)\Delta_d(i)$$

If $\lim_{k\to\infty} P(k) = 0$, then this ends the proof, else, we have from the proof of Claim 1, we observe that if $P(k)$ is bounded and whenever $\lim_{k\to\infty} P(k) \neq 0$, then $\exists 0 < c_M < 1$ such that $0 < \lambda\left(\lim_{k\to\infty}(I-M)\right) \leq c_M < 1$. Consequently, $\lim_{k\to\infty}\sum_{i=0}^{k-1}\left(\prod_{j=1}^{k-1-i}(I-M)\right)$ is bounded. Since the trajectory is smooth, $\exists c_d > 0$ such that $\forall\, i \geq 0$, $\|\Delta_d(i)\| \leq c_d T_s$ and $\lim_{k\to\infty}\left(\prod_{i=0}^{k-1}(I-M)\right)P_u(0) = 0$. Therefore, $\exists c_{P_u}(c_d T_s) > 0$ such that $\lim_{k\to\infty}\|P_u(k)\| \leq c_{P_u}(c_d T_s)$, where $c_{P_u}(c_d T_s)$ is a decreasing function of $c_d T_s$ and $\lim_{c_d T_s\to 0} c_{P_u}(c_d T_s) = 0$.

In addition, $P_x(k) = A^- P_x(k-1) A^{-T} + B^- P_u(k-1) B^{-T}$

Since system is asymptotically stable and $\lim_{k\to\infty}\|P_u(k)\| \leq c_{P_u}(c_d T_s)$, then $\exists c_{P_x}(c_d T_s) > 0$ such that $\lim_{k\to\infty}\|P_x(k)\| \leq c_{P_x}(c_d T_s)$, where $c_{P_x}(c_d T_s)$ is a decreasing function of $c_d T_s$ and $\lim_{c_d T_s\to 0} c_{P_x}(c_d T_s) = 0$.

Since $P(k)$ is a covariance matrix with diagonal block elements: $P_u(k)$, $P_u(k-1)$ and $P_x(k)$ where $\lim_{k\to\infty}\|P_u(k)\| \leq c_{P_u}(c_d T_s)$ and $\lim_{k\to\infty}\|P_x(k)\| \leq c_{P_x}(c_d T_s)$ then $\exists c_P(c_d T_s) > 0$ such that: $\lim_{k\to\infty}\|P(k)\| \leq c_P(c_d T_s)$, where $c_P(c_d T_s)$ is a decreasing function of $c_d T_s$ and $\lim_{c_d T_s\to 0} c_P(c_d T_s) = 0$. ■

Remark 3. It is important to note that Theorem 5 does *not* assume that $\lim_{k\to\infty} \Delta u_d(k) = 0$. Equation (19) implies that the steady-state output error decreases as $c_d T_s$ decreases in presence of measurement noise and initial errors, and in absence of process noise. That is, regardless the size of measurement noise, arbitrary small errors can be achieved at steady state for sufficiently small sampling period. Therefore, the proposed algorithm is capable of rejecting measurement noise. This remark also applies to Theorem 7.

Remark 4. Theorem 5 does not indicate what sampling rate to choose in order to achieve a desired error bound. However, since the system under consideration is linear, then one would expect a smaller value of c_d for "smoother" desired output trajectories or for desired output trajectories with smaller bandwidth. Consequently, higher sampling rates should be used for desired output trajectories with higher bandwidth.

C. *Sub-Optimal PID Gains*

This section addresses Problem 3 by first presenting a non-optimal recursive algorithm that updates the PID gains. The algorithm mimics the proposed optimal algorithm, however, without using the "exact" knowledge: 1) of the state matrix, $A(k)$; 2) the covariance of measurement noise, process noise, initialization error, $R(k), Q(k)$, and $P(0)$; and 3) $E[\Omega\Omega^T]$. Subsequently, characteristics of the proposed non-optimal algorithm are provided according to Problem 3.

Notations:

Let $\hat{A}(k) \in \Re^{n\times n}$ be any stable matrix, and

$$\hat{\mathrm{H}} \equiv \begin{bmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & B^- & \hat{A} \end{bmatrix}, \hat{\mathrm{E}} \equiv \begin{bmatrix} C^+B & C^+\hat{A}B^- & C^+\hat{A}^2 \\ 0 & CB^- & C\hat{A} \\ 0 & 0 & C^- \end{bmatrix}$$

$\hat{\Phi} = \hat{\mathrm{H}} - \bar{I}K\hat{\mathrm{E}}$ and for compactness, we also set $\hat{A} \equiv \hat{A}(k)$.

Proposed recursive algorithm:

$$K = \bar{I}^T \hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1} \tag{20}$$

$$\hat{P}(k+1) = \hat{\Phi}\hat{P}(k)\hat{\Phi}^T + \bar{I}KS(k)(\bar{I}K)^T + \beta(k)\bar{I}\bar{I}^T \tag{21}$$

where $\hat{P}(0) > 0$, $S(k) > 0, \forall k$, and $\beta(k) \geq 0, \forall k$.

Remark 5: In presence of a stable reference (erroneous) model of the plant and estimates of $R(k), Q(k)$, $P(0)$ and $E[\Omega\Omega^T]$, it is suggested to employ (17) and (18). However, (21) may not well estimate the actual $P(k)$.

Claim 2: $0 \leq \lambda\left(\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}\hat{\mathrm{E}}\right) < 1$

In addition, let $\hat{M}$ be the q^{th} leading principal minor of $\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}\hat{\mathrm{E}}$ (first $q \times q$ block of the matrix).

If C^+B is full-column rank, then $0 < \lambda(\hat{M}) < 1$.

Proof of Claim 2:

The first two parts of proof follow exactly the same steps as in the proof of Claim 1, thus omitted.

Since the first block column of E and $\hat{\mathrm{E}}$ are equal, then $\hat{M}$ is also equal to the q^{th} leading principal minor of $\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}\mathrm{E}$. The rest follows the same steps as in the proof of Claim 1, thus omitted ■

Theorem 6: Let $R(k) > 0,\ \forall k \geq 0$ and $P(0) > 0$. If C^+B is full-column rank, and the system in (1) is asymptotically stable, then the sub-optimal recursive algorithm (20)-(21) guarantee:

- K is full-row rank and $0 < \lambda(\Phi) < 1,\ \forall k \geq 0$
- the conditions (13)-(14) of Theorem 1 are satisfied and $P(k)$ is bounded $\forall k$ while $Q(k) \geq 0,\ \forall k \geq 0$
- the conditions (15)-(16) of Theorem 2 are satisfied, $\lim_{k\to\infty} P(k) = 0$ and $\lim_{k\to\infty} K(k) = 0$ whenever $Q(k) = 0,\ \forall k \geq 0$ and $\lim_{k\to\infty} \Delta u_d(k) = 0$

Proof of Theorem 6:

Since $\hat{\mathrm{E}}\bar{I}$ is full-column rank and consequently $\bar{I}^T\hat{P}\hat{\mathrm{E}}^T$ is full-row rank, then K is also full-row rank.

Inserting (20) in $\mathrm{H} - \bar{I}K\mathrm{E}$, we obtain

$$\mathrm{H} - \bar{I}K\mathrm{E} = \mathrm{H} - \bar{I}\bar{I}^T\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}\mathrm{E}$$

$$= \mathrm{H} - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}\mathrm{E}$$

$$\Phi = \mathrm{H} - \bar{I}K\mathrm{E} = \begin{bmatrix} I - \hat{M} & 0 & 0 \\ I & 0 & 0 \\ 0 & B^- & A^- \end{bmatrix}$$

From Claim 2, we have $0 < \lambda(\hat{M}) < 1$. The rest of proof follow exactly the same steps as in the proof of Theorem 4, thus omitted. ■

Theorem 7: Consider the recursive algorithm presented in (20) and (21). If C^+B is full-column rank, the system in (1) is asymptotically stable, the trajectory of $u_d(k)$ is smooth and $Q(k) = 0,\ \forall k \geq 0$, then $\exists c_{P_N}(c_d T_s) > 0$ such that:

$$\lim_{k\to\infty} \|P(k)\| \leq c_{P_N}(c_d T_s)$$

where $c_{P_N}(c_d T_s)$ is a decreasing function of $c_d T_s$ and $\lim_{c_d T_s \to 0} c_{P_N}(c_d T_s) = 0$.

Proof of Theorem 7: Following similar steps as in the proof of Theorem 5, we conclude that $\exists c_{\hat{P}} > 0$ such that $\lim_{k\to\infty}\|\hat{P}(k)\| \leq c_{\hat{P}} T_s$. Since $K = \bar{I}^T\hat{\mathrm{H}}\hat{P}(k)\hat{\mathrm{E}}^T(\hat{\mathrm{E}}\hat{P}(k)\hat{\mathrm{E}}^T + S(k))^{-1}$ and $S(k) > 0, \forall k$, then $\exists c_K > 0$ such that $\lim_{k\to\infty}\|K(k)\| \leq c_K T_s$.

Similarly, we can write

$$P_u(k+1) = (I - \hat{M})P_u(k) - \bar{I}^T P(\bar{I}K\mathrm{E})^T\bar{I} + K(\mathrm{E}P\mathrm{E}^T + \bar{R})K^T + \Delta_d(k)$$

Again $(\bar{I}K\mathrm{E})^T\bar{I} = \mathrm{E}^T K^T \bar{I}^T\bar{I} = \mathrm{E}^T K^T$. Thus,

$$P_u(k+1) = (I - \hat{M})P_u(k) - \bar{I}^T P\mathrm{E}^T K^T + K(\mathrm{E}P\mathrm{E}^T + \bar{R})K^T + \Delta_d(k)$$

Similar to the proof of Theorem 5, since $P(k)$ is bounded, $0 < \lambda\left(\lim_{k\to\infty}(I - \hat{M})\right) \leq c_M < 1$, $\lim_{k\to\infty}\|K(k)\| \leq c_K T_s$, $\|\Delta_d(k)\| \leq c_d T_s, \forall k \geq 0$, then $\exists c_1 > 0$ such that $\lim_{k\to\infty}\|P_u(k)\| \leq c_1(c_d T_s)$. The rest of the proof follows similar relevant steps as in the proof of Theorem 5. ■

Remark 6: Unlike Theorems 5 and 7, Theorems 4 and 6 do not necessitate a discretized system, where $\lim_{k\to\infty} P(k) = 0$ is guaranteed in presence of measurement noise provided that $\lim_{k\to\infty} \Delta u_d(k) = 0$. On the other hand, Theorems 5 and 7 do not require $\lim_{k\to\infty} \Delta u_d(k) = 0$, where arbitrary small values of $\lim_{k\to\infty} P(k)$ can be achieved in presence of measurement noise for sufficiently small sampling period provided trajectories are smooth as described in Definition 1. In all cases, rejection of measurement noise is attained at steady state.

IV. NUMERICAL EXAMPLES

In this section we illustrate the performance capabilities of the proposed sub-optimal algorithm (20) and (21). All simulations are carried out using MATLAB, and all random errors are assumed zero-mean white Gaussian noise. The norm, $\|.\|$, employed throughout this example is the 2-norm. In addition, we set $u(0) \equiv 0$ and $u(1) \equiv K_1(0)e(1) + K_2(0)e(0)$.

Example 1 illustrates the boundedness of trajectories in presence of random disturbances and tracking performance. Example 1 considers a two-input three-output fourth-order linear time varying system. Example 2 compares the performance of an adaptive fuzzy PID controller coupled with a Kalman filter with the proposed algorithm. Example 3 considers a plant of a large transport airplane.

A. *Example* 1: Two-input three-output time varying system

We consider a two-input three-output fourth-order linear time-varying system with plant nominal parameters $\begin{bmatrix} A(k) & B(k) & C \end{bmatrix}$ used to update the output of the plant. The role of the nominal model is basically used to depict the actual plant, which is not available to the controller. An erroneous model of the plant is considered as a reference model in order to only update the recursive formulas (20)-(21). The parameters of the erroneous model are denoted by the triplet $\begin{bmatrix} \hat{A}(k) & \hat{B}(k) & C \end{bmatrix}$.

$\begin{bmatrix} A(k) & B(k) & C \end{bmatrix}$ and $\begin{bmatrix} \hat{A}(k) & \hat{B}(k) & C \end{bmatrix}$ are obtained by discretizing their associated continuous-time models with sample period T_s, $\begin{bmatrix} A_c(t) & B_c(t) & C \end{bmatrix}$ and $\begin{bmatrix} \hat{A}_c(t) & \hat{B}_c(t) & C \end{bmatrix}$, respectively. In the following, we present the employed nominal and erroneous continuous-time parameters employed.

$$A_c(t) = \begin{bmatrix} -2.2 & 0.33 & 0 & -0.11 \\ -0.68 & -2.95 + 0.55\cos(1.05t) & 0.75 & 0.48 \\ 0.54 & 0.37 & -2.5 + 0.55\sin(1.9t) & -0.22 \\ -0.11 & -0.09 & 0 & -2.7 \end{bmatrix}$$

$$\hat{A}_c(t)=\begin{bmatrix}-2 & 0.3 & 0 & -0.2\\ -1 & -3+0.5\cos(t) & 1 & 1\\ 0.7 & 0.5 & -3+0.5\sin(2t) & -0.3\\ -0.2 & -0.1 & 0 & -3\end{bmatrix}$$

$$B_c(t)=\begin{bmatrix}0 & 0\\ 3.3 & 0.45\sin(1.05t)\\ 0.45\cos(1.9t) & 4.4\\ 0 & 5.5\end{bmatrix}$$

$$\hat{B}_c(t)=\begin{bmatrix}0 & 0\\ 3 & 0.5\sin(t)\\ 0.5\cos(2t) & 4\\ 0 & 5\end{bmatrix},\ C=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

It turns out that $\max_t \text{Real}\{\lambda(A_c(t))\}=-1.585$ and $\min_t \text{Real}\{\lambda(A_c(t))\}=-3.6155$.

The proposed method is expected to track all the state variables, which should also result in output tracking. Although the choice of output matrix is independent of time, the product $CB(k)$ is time variant. Consequently, the output matrix is first selected to be in the simplest form. The choices of the state and input coupling matrices are almost randomly generated while making sure that state matrix is stable and $CB(k)$ is full-column rank.

It is important to note that the number of outputs is greater than the number of inputs in which not any set of desired output trajectories may be feasible. Consequently, one simple and effective way to generate feasible desired trajectories, $x_d(k)$ and $y_d(k)$, is by constructing a profile for $u_d(k)$ with some $x_d(0)$. These values are applied to (2) while using the nominal plant. This exercise yields the desired trajectories, $x_d(k)$ and $y_d(k)$. In particular, we consider in this example

$u_{d_1}(k)=10e^{-0.5T_s k}-20$ and $u_{d_2}(k)=20(1-e^{-T_s k})+10$

with $x_d(0)=\begin{bmatrix}15 & 15 & 15 & 15\end{bmatrix}^T$. Fig. 1 shows the desired trajectories of the state variables, $x_{d_i}(k), i\in\{1,2,3,4\}$, in blue. The trajectories, $x_{h_i}(k), i\in\{1,2,3,4\}$ of Fig. 1, shown in black, are the responses of erroneous model, $\begin{bmatrix}\hat{A}(k) & \hat{B}(k) & C\end{bmatrix}$, with input $u_d(k)$ and initial condition $x_d(0)$. The difference between $x_{d_i}(k)$ and $x_{h_i}(k)\, i\in\{1,2,3,4\}$, being quite significant, illustrates the sensitivity of the model and the inadequate performance of the erroneous model when used in an open-loop fashion. The trajectories $x_i(k), i\in\{1,2,3,4\}$ of Fig. 1, in red, are the results of the proposed PID algorithm with $T_s=0.1$ sec while considering the following setup:

- Inaccurate initial conditions are used, in particular, $x(0)\equiv 0$ and $u(0)\equiv 0$.
- Measurement errors are included; in particular, zero-mean white Gaussian noise, with variance equals to one, is added to all three outputs.
- Erroneous statistical information is provided to the controller, in particular, the measurement covariance matrix, $\hat{\bar{R}}=2\bar{R}=2I$ while $P(0)=500I$.
- The erroneous model, $\begin{bmatrix}\hat{A}(k) & \hat{B}(k) & C\end{bmatrix}$, is provided to the controller.

Fig. 2 shows the corresponding state variables errors for two different scenarios: a) the above setup (in red), and in particular with $\bar{R}=I$, and b) the above setup (in black) but with $\bar{R}=100I$, that is, the variance of the measurement noise, corresponding to each output, is equal to 100.

While examining Fig. 1 and Fig. 2, the following can be concluded, for a fixed and relatively large sampling period:

1. Tracking capability of the sub-optimal controller in presence of erroneous initial conditions and measurement noise. It can be noted from Fig. 1 that, as time increases, $x_i(k)\to x_{d_i}(k), i\in\{1,2,3,4\}$. This phenomena is due to that fact that $\lim_{k\to\infty}\Delta u_d(k)=0$, which illustrates part of the results in Theorem 6.
2. Capability of rejecting measurement noise. Similarly, it can be also noticed that the state variables errors converge to zero as time increases regardless the size of measurement noise.

In order to illustrate a) boundedness of trajectories, and b) demonstrating the ability of the proposed algorithm estimating all errors by extracting the variances of each error from the diagonal entries of $\hat{P}(k)$ in (21), $\sqrt{\hat{P}_{u_i}(k)}$ and $\sqrt{\hat{P}_{x_j}(k)}$ with $i\in\{1,2\}$ and $j\in\{1,2,3,4\}$, we consider the setup above while adding zero-mean white Gaussian process noise with variance equals to 10 to each state variable and also zero-mean measurement noise with variance equals to 1000 to each output. Again the proposed algorithm is based on the erroneous model, and erroneous noise statistics, in particular, we set $\hat{\bar{R}}=2\bar{R}=2000I$ and $\hat{\bar{Q}}=1.2\bar{Q}=12I$.

The corresponding results, which are shown in Fig. 3, illustrate the above two points. It is important to note that $\lim_{k\to\infty}P(k)\neq 0$ since in presence of process noise the errors never converge to zero.

The norms of the associated PID gains are shown in Fig. 4. As time increases, the gains decrease as the errors due to initial condition and measurement noise are rejected. Simulations show that the PID gains, $K_i(k), i\in\{1,2,3\}$, are full-row rank.

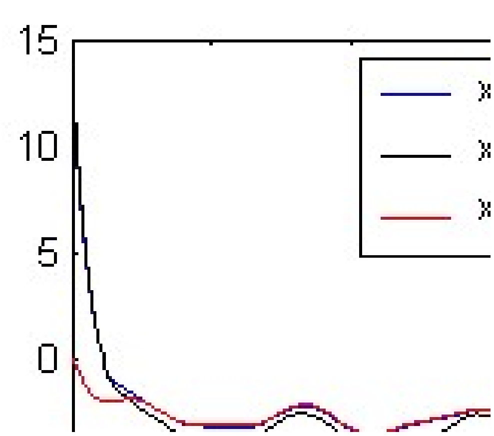

Fig. 1. Desired trajectories (blue), sensitivity of modelling errors (black), and tracking performance (red) with $T_s = 0.1$ sec, $\overline{R} = I$ and $\overline{Q} = 0$.

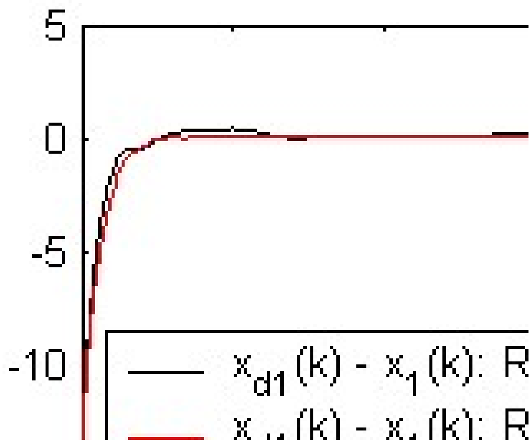

Fig. 2. Tracking state errors with $T_s = 0.1$ sec, $\overline{Q} = 0$, $\overline{R} = I$ (red), and $\overline{R} = 100I$ (black).

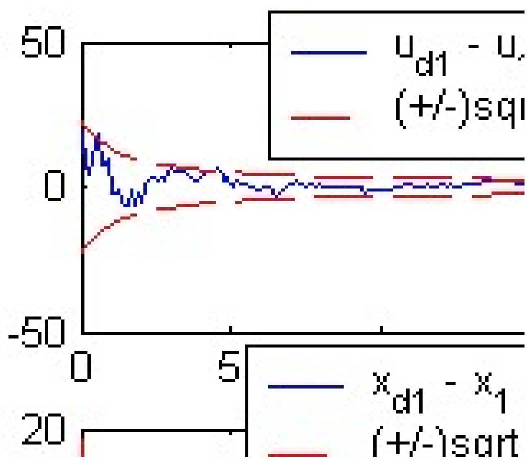

Fig. 3. Input and state errors (blue), $\sqrt{\hat{P}_{u_i}(k)}, i \in \{1,2\}$ (red) and $\sqrt{\hat{P}_{x_i}(k)}, i \in \{1,2,3,4\}$ (red). $T_s = 0.1$ sec, $\overline{R} = 1000I$ & $\overline{Q} = 10I$.

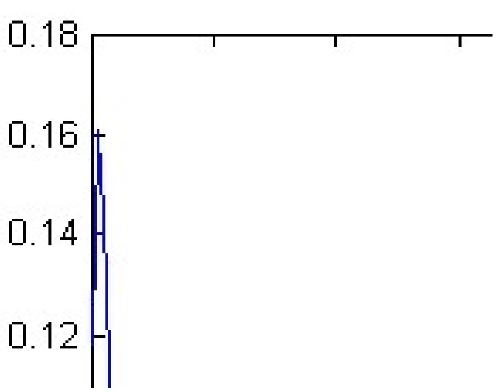

Fig. 4. Norm of PID gains with $T_s = 0.1$ sec, $\overline{R} = 1000I$ and $\overline{Q} = 10I$.

B. *Example 2: Line-of-sight control of an inertially stabilized platform*

This example compares the performance between the method presented in [38] and the proposed method based on the example and reference trajectories considered in [38]. The system under consideration reflects a line-of-sight control of an inertially stabilized platform consisting of a DC motor and a gyroscope. The plant is described by the following state-space system

$$\dot{x}(t) = A_c x(t) + B_c(u(t) + w(t)), \quad y(t) = Cx(t) + v(t)$$

where

$$A_c = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_a}{L_a} \\ \frac{K_b}{J} & -\frac{B_m}{J} \end{bmatrix}, \quad B_c = \begin{bmatrix} \frac{1}{L_a} \\ 0 \end{bmatrix}, \quad C = [0 \quad 1]$$

and $x_1(t)$ is the armature current, $x_2(t)$ is the angular speed, $u(t)$ is the armature voltage, R_a is the armature resistance, B_m is the viscous-friction coefficient, L_a is the armature inductance, J is the equivalent moment of inertia, K_b is the torque constant, and K_a is the speed constant. The values of parameters are listed in Table 1.

TABLE 1: DC MOTOR PARAMETERS [38]

R_a Ω	L_a mH	B_m N×m×sec	K_a rad/sec	K_b N×m/A	J kg×m^2
43.1	0.921	0	0.307	0.2963	2.655×10^{-5}

The process noise and measurement noise are assumed Gaussian with $w \sim (0, 0.001)$ and $v \sim (0, 0.1)$ and all initial conditions are to zero as in [38].

The controller presented in [38] is a fuzzy adaptive PID controller coupled with a discrete-time Kalman filter. The Kalman filter is employed in order to reduce the effects of random disturbances. Two scenarios are considered in [38] in presence of random disturbances:

Scenario A: Stabilization where the reference output trajectory is zero.

Scenario B: Tracking where the reference trajectory $y_d(t) = 10\sin\frac{2\pi}{3}t$.

The plant eigenvalues $s_{1,2} = \lambda(A_c) \in \{-79.628, -4.6717\times10^4\}$. We choose a sampling period $T_s = 10^{-4}$ sec and we run the simulations for 5 seconds. We set $\overline{Q} \equiv 10^{-6}$, $\overline{R} = 10^{-2}I$, $\forall k$, and $P(0) = I$ for both scenarios and $E[\Omega\Omega^T] = 0$, for Scenario A. The variance of errors of the proposed method and the ones in reported in [38] are listed in Table 2. It is important to note that the performance results reported in [38] show that their proposed method outperforms a PID controller. However, the variance of residuals corresponding to our proposed method is 188 times less than the best results reported in [38].

TABLE 2. PERFORMANCE COMPARISON SCENARIO A: EXAMPLE 2

	Fuzzy PID without Kalman filter	Fuzzy PID with Kalman filter	Proposed
$\mathrm{var}(0-y)$	3.53×10^{-2}	3.4×10^{-3}	1.8×10^{-5}

Scenario B: We set $E[\Omega\Omega^T]=0$, except for the first diagonal entry where we set $E[\Omega\Omega^T]_{1,1}=100T_s^2$. Fig. 5 shows the tracking performance of the proposed method. The resulting residual variance $\mathrm{var}(y_d(k)-y(k))=9.4\times10^{-4}$. However, the residual errors presented in Fig. 10 of [38] shows a noisy sinusoidal trend, approximated by $0.1\sin\frac{2\pi}{3}t+\bar{v}(t)$, where $\bar{v}(t)$ is a zero mean noise with variance no less than 3.4×10^{-3}.

It is important to note that since $\overline{Q}\neq 0$ then $\lim_{k\to\infty}K(k)\neq 0$. In fact, in about 0.25sec, $K(k)\to10^{-3}\times[5.94\quad 5.60\quad 5.26]$.

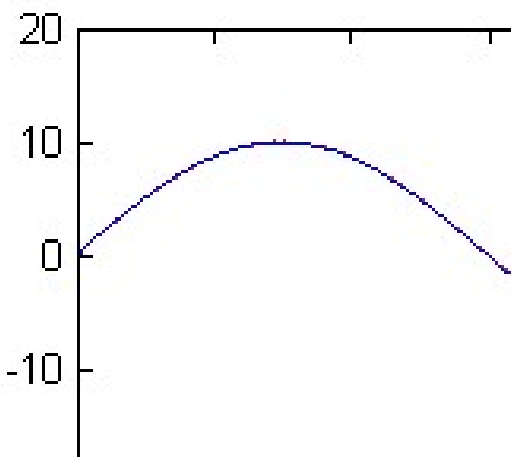

Fig. 5. Angular velocity tracking: Example 2, Scenario B.

C. *Example* 3*: Large transport airplane*

In this example we consider the linearized lateral motion dynamic model of a large transport airplane [39] described by

$$\dot{x}=A_c x+B_c u,\quad x=[v_b\quad p_b\quad r_b\quad \phi\quad \psi]^T,\quad u=[d_r\quad d_a]^T$$

where v_b is the lateral velocity, p_b is the roll rate, r_b is the yaw rate, ϕ is the roll angle, ψ is the yaw angle, d_r is the rudder position, and d_a is the aileron position. The state matrix, A and the output coupling matrix B are [39]

$$A_c=\begin{bmatrix}-0.13858 & 14.326 & -219.04 & 323.167 & 0\\ -0.02073 & -2.1692 & 0.91315 & 0.000256 & 0\\ 0.00289 & -0.16444 & -0.15768 & -0.00489 & 0\\ 0 & 1 & 0.00618 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}$$

$$B_c=\begin{bmatrix}0.15935 & 0.00211\\ 0.01264 & 0.021326\\ -0.012879 & 0.00171\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

It is important to note that one eigenvalue of the continuous-time state matrix corresponding the yaw angle is zero. The main purpose of this example is to consider the performance of the proposed algorithm in presence of colored noise. In particular, we consider a nonstationary process reflected by the presence of random walk due to gyro bias drift.

The objective is to have the lateral velocity and the yaw angle track the following reference trajectories [39]

$$v_b^{ref}=\sin(0.01t)\text{ ft/sec and }\psi^{ref}=5-5\cos(0.01t)\text{ deg.}$$

An aircraft mechanism incorporates gyros to measure the angular rates, including the yaw rate, and a multiple-beam Doppler radar to provide the three velocity components including the lateral velocity. Consequently, we use only two output measurements, the lateral velocity and the yaw rate. Thus, $C=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}$.

Consequently, we set our reference trajectory for the yaw rate

$$r_b^{ref}\equiv\dot{\psi}^{ref}=0.05\sin(0.01t)$$

The measurement noise corresponding to the lateral velocity is modelled as zero-mean Gaussian noise with standard deviation $\sigma_1=5\times10^{-2}$ ft/sec. However, for gyro bias drift, we assume a random walk described by the following difference equation

$$v(k+1)=v(k)+\eta(k)$$

where $\eta(k)$ is zero-mean Gaussian noise. For an aircraft gyro with adequate data acquisition, we assume that the standard deviation of $\eta(k)$ is $\sigma_2=5\times10^{-5}$ deg/sec, which results in a typical angular random walk of 0.003 deg in our hour (see the top lot of Fig. 7). Consequently, we model the measurement error covariance matrix as follows

$$R(k)=\begin{bmatrix}\sigma_1^2 & 0\\ 0 & r_v(k)\end{bmatrix},\quad r_v(k+1)=r_v(k)+\sigma_2^2$$

We set all the elements of $P(0)$ to zero except for the first four diagonal elements to one. We simulate the system for 800 sec with sapling period $T_s=8\times10^{-2}$.The relevant aircraft outputs, lateral velocity, yaw rate and yaw angle, are shown along with their corresponding reference signals in Fig. 6. The outputs uniformly track their reference trajectories. It can be noted from the two top plots of Fig. 6, which are associated with the measured outputs, show the ability of the proposed controller in rejecting nonstationary measurement noise. For example, towards the end of the run, the yaw rate error is less than 5×10^{-5} deg/sec, where the gyro drift at that time (see top plot of Fig. 7) is around 2×10^{-3} deg/sec. The corresponding control signals, rudder position and aileron position shown in the bottom plot of Fig. 7, are relatively smooth.

In [39] multivariable model reference adaptive control (MRAC) is applied to the plant in absence of measurement noise using measurements of of lateral velocity and yaw angle. Based on Fig. 2 in [39], long transient (≈ 600 sec) is observed in the tracking of the yaw angle. In addition, the yaw error at around 110 sec is approximately 0.82 deg, whereas the maximum absolute yaw error achieved over the entire time period using the proposed controller is 0.8 deg including the measurement noise under consideration, and 0.1 deg in absence of measurement noise. It is important to note that MRAC [39] implements full state feedback whereas the proposed system only uses measurements of lateral velocity and yaw rate. It is very important to note that whenever we use measurements of lateral velocity and yaw *angle*, instead of the yaw rate, significantly better performance is achieved. For example, the maximum absolute yaw error achieved over the entire time period using the proposed controller is 0.025 deg in absence of measurement noise. The latter corresponds to a situation where the product of the output-input coupling matrices CB_c associated with the continuous-time plant is not full rank. However, due to discretization the corresponding product, CB , becomes full rank.

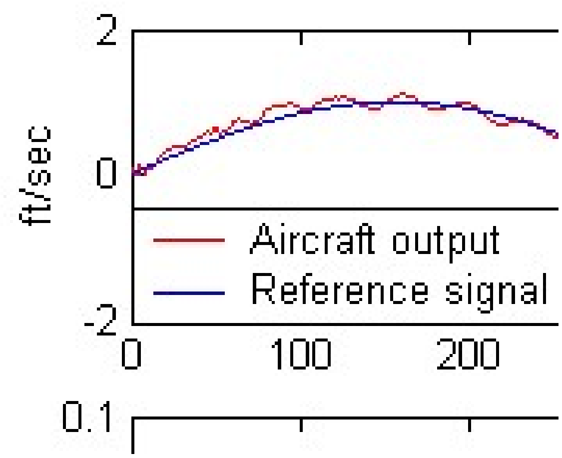

Fig. 6. Lateral velocity (top), yaw rate (center), and yaw angle (bottom).

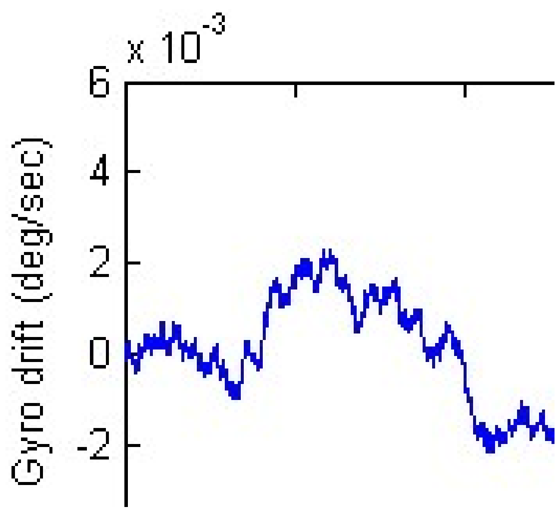

Fig. 7. Gyro drift (top) and control signals (bottom).

V. Conclusion

This paper provided a framework for selecting PID gains targeting complete rejection of measurement noise and initial errors while considering boundedness of all trajectories. The system under consideration is a discrete-time varying MIMO system. The study in this paper did not consider integrating any external filter to the system. The associated necessary and sufficient conditions indicate that such PID gains should be time varying. Optimal and sub-optimal recursive algorithms were proposed, where the sub-optimal algorithm is based on erroneous plant and erroneous statistical models. Provided that the plant is stable and the product of the input and output coupling matrices is full-column rank the proposed recursive algorithms were shown to possess the following characteristics:
(a) withstanding process noise and measurement noise of any size; (b) rejecting measurement noise and errors due initial conditions; (c) automatically producing the PID gains in a discrete-time fashion; (d) providing indicative statistical estimates of errors at every time sample; and (e) skillfully tracking any realizable reference trajectories in presence of measurement noise. Simulation results illustrated all presented characteristics. We have also numerically implemented the proposed controller to different plants; in particular, line-of-sight control of an inertially stabilized platform, and large transport airplane with nonstationary measurement noise. Based on the obtained results, it can be concluded that the proposed controller can effectively perform under a broader class of applications beyond the analytical setup considered in this manuscript.

References

[1] K. Van Heusden, G. A. Dumont, G.A., K. Soltesz, C.L. Petersen, A. Umedaly, N. West, J.M. Ansermino, Design and Clinical Evaluation of Robust PID Control of Propofol Anesthesia in Children, *IEEE Trans. on Contr. Sys. Tech.* 22 (2) (2014) 491-501.

[2] P. Cominos, N. Munro, PID controllers: recent tuning methods and design to specification, *IEE Proc. Contr. Theory and Applications* 149 (1) (2002) 56-53.

[3] K. J. Åström, T. Hägglund, C. C. Hang, and W. K. Ho, Automatic Tuning and Adaptation for PID Controllers – a Survey, *Contr. Engineering Practice* 1 (4) (1993) 699–714.

[4] P. Zitek, J. Fiser, T. Vyhlidal, Dimensional analysis approach to dominant three-pole placement in delayed PID control loops, *J. Process Control* 23 (2013)1063–1074.

[5] S. Srivastava, V. S. Pandit, A PI/PID controller for time delay systems with desired closed loop time response and guaranteed gain and phase margins, *J. Process Control* 37 (2016) 70-77.

[6] F.O. Souza, et al., LMI design method for networked-based PID control, *Int. J. of Contr*. Accepted author version posted online: 29 Jan 2016.

[7] E. Almodaresi, M. Bozorg, H. D. Taghirad, Stability domains of the delay and PID coefficients for general time-delay systems, *Int. J. of Contr*. 89 (4) (2016) 783-792.

[8] Y. Cao, X.B. Chen, An output-tracking-based discrete PID-sliding mode control for MIMO systems, *IEEE/ASME Trans. on Mechatronics* 99 (2013) 1–12.

[9] G. Reynoso-Meza, S. García-Nieto, J. Sanchis, and F.X. Blasco, Controller Tuning by Means of Multi-Objective Optimization Algorithms: A Global Tuning Framework, *IEEE Trans. on Contr. Sys. Tech.* 21 (2) (2013) 445-458.

[10] W.S.W. Wang, D.E. Davison, and E.J. Davison, Controller Design for Multivariable Linear Time-Invariant Unknown Systems, *IEEE Trans. Autom. Control* 58 (9) (2013) 2292-2306.
[11] J. Orrante-Sakanassi, V. Santibánez, V. M. Hernández-Guzmán, New tuning conditions for a class of nonlinear PID global regulators of robot manipulators, *Int. J. of Contr*. 87 (4) (2014) 728-741.
[12] M. Mendoza, A. Zavala-Río, V. Santibáñez, F. Reyes, A generalised PID-type control scheme with simple tuning for the global regulation of robot manipulators with constrained inputs, *Int. J. of Contr*. 88 (10) (2015) 1995-2012.
[13] S. Saab, R. Toukhtarian, "A MIMO Sampling Rate Dependent Controller," *IEEE Trans. Ind. Elect*., 62 (6) (2015) 3662-3671.
[14] P. Kallakuri, L.H. Keel, S.P. Bhattacharyya, "Multivariable controller design with integrity," *in the proc. Amer. Contr. Conf.* (2013) 5159-5164.
[15] A.N. Gundes, A.B. Ozguler, "PID Stabilization of MIMO Plants," *IEEE Trans. Autom. Contr.* 52 (8) (2007) 1502-1508.
[16] S. Formentin, S. Savaresi, L. del Re, Non-iterative direct data-driven controller tuning for multivariable systems: theory and application, IET Control Theory Appl. 6 (2012)1250–1257.
[17] L. Miskovic, A. Karimi, D. Bonvin, and M. Gevers, Correlation-based tuning of decoupling multivariable controllers, *Automatica* 43 (2007) 1481–1494.
[18] L. Campestrini, D. Eckhard, L. A. Chía, E. Boeira, Unbiased MIMO VRFT with application to process control. *J. of Process Control* 39 (2016) 35-49.
[19] A.Theorin, T. Hägglund, Derivative backoff: The other saturation problem for PID controllers, *J. of Process Control* 33 (2015) 155-160.
[20] V. Romero Segovia, T. Hägglund and K.J. Åström, "Measurement Noise Filtering for PID Controllers," *J. of Process Control* 24 (4) (2014) 299–313.
[21] S. Skogestad, Tuning for Smooth PID Control with Acceptable Disturbance Rejection, *Ind. and Engineer. Chem. Research* 45 (2006) 7817–7822.
[22] J. Zhong, L. Li, Tuning Fractional-Order $PI^{\lambda}D^{\mu}$ Controllers for a Solid-Core Magnetic Bearing System, *IEEE Trans. on Contr. Sys. Tech.* 23 (4) (2015) 1648 - 1656.
[23] B. Kristiansson and B. Lennartson, Evaluation and Simple Tuning of PID Controllers with High-Frequency Robustness, *J. Process Contr.* 16 (2006) 91–103, 2006.
[24] A.J. Isaksson and S.F. Graebe, Derivative Filter is an Integral Part of PID Design, *IEE Proc., Control Theory and Appl.* 149 (2002) 41–45.
[25] P.H. Chang and J.H. Jung, A Systematic Method for Gain Selection of Robust PID Control for Nonlinear Plants of Second-Order Controller Canonical Form, *IEEE Trans. on Contr. Sys. Tech.* 17 (2) (2009) 473-483.
[26] P.-O. Larsson, T. Hägglund, Control Signal Constraints and Filter Order Selection for PI and PID Controllers, *in the proc. of the Amer. Contr. Conf.* (2011) 4994–4999.
[27] Y. Lee, D. Shin and C.C. Chung, PID Controller with Feedforward Low Pass Filters for Permanent Magnet Stepper Motors Control, *in the proc. of the 12th Int. Conf. Auto. and Sys*. (2012) 1597-1600.
[28] K.K. Tan, S. Huang, W. Liang, A.A. Mamun, E.K. Koh and H. Zhou, Development of a Spherical Air Bearing Positioning System, *IEEE Trans. Ind. Elect.* 59 (9) (2012) 3501-3509.
[29] T.B. Sekara, M. Matausek, Optimization of PID Controller based on Maximization of the Proportional Gain under Constraints on Robustness and Sensitivity to Measurement Noise, *IEEE Trans. Autom. Contr.* 54 (1) (2009) 184–189.
[30] A.D. Micic, M.R. Matausek, Optimization of PID controller with higher-order noise filter, *J. Process Control* 24 (5) (2014) 694–700.
[31] O. Garpinger, T. Hägglund, K.J. Åström, Performance and robustness trade-offs in PID control, *J. Process Contr.* 24 (5) (2014) 568–577.
[32] O. Garpinger, T. Hägglund, Software-based optimal PID design with robustness and noise sensitivity constraints, *J. Process Contr*. 25 (9) (2015) 90-101.
[33] S. S. Saab, A Discrete-Time Stochastic Learning Control Algorithm, *IEEE Trans. Autom. Contr.* 46 (6) (2001) 877-887.
[34] S. S. Saab, Selection of the Learning Gain Matrix of an Iterative Learning Control Algorithm in Presence of Measurement Noise, *IEEE Trans. Autom. Contr.* 50 (11) (2005) 1761-1774.
[35] D. Shen, Y. Wang, Survey on stochastic iterative learning control, *J. of Process Control* 24 (2014) 64–77.
[36] S. S. Saab, A Stochastic PID Controller for a Class of MIMO Systems, *Int. J. of Control*. Accepted author version posted online: 02 May 2016.
[37] D. Y. Abramovitch, H. Storrs, R. Workman, Semi-automatic tuning of PID gains for atomic force microscopes, *Asian J. of Contr*. 11 (2) (2009) 188-195.
[38] Q. Tang, X. Wang, and Q. Yang, Line-of-sight stabilization control based on a fuzzy PID controller and Kalman filter, *Contr. Engineer. and Electronics Engineer.* 95 (2014) 33-40.
[39] J. Guo, Y. Liu, G. Tao, Multivariable MRAC with state feedback for output tracking, *in the proc. of the Amer. Contr. Conf.* (2009) 592-597.