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Why is CVaR Superior to VaR?

A Unified Framework for Mean-Risk Portfolio Optimization

by

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Abstract

Until recently, value-at-risk (VaR) has been a widely used risk measure in portfolio optimization. The large number of recent bank failures shows that VaR failed to account for the expected losses which resulted from the outburst of a rare event such as the global financial crisis, thereby questioning its reliability and credibility as a measure of risk. Alternatively, previous work concurs that conditional value-at-risk (CVaR) is a coherent tail risk measure, and has established the superiority of CVaR over traditional measures of risk (variance and VaR) from a theoretical standpoint. This study aims at investigating the reasons that render CVaR superior to other traditional risk measures from an empirical perspective. We develop a theoretical model that solves the mean-risk portfolio optimization problem within a unified framework for all three different measures of risk (variance, VaR, and CVaR). We test our model empirically using financial data on return indices over a period covering the financial crisis. Our results support the theoretical predictions regarding the superiority of CVaR. We find that the mean-CVaR framework can be applied to multi-model returns, unlike mean-variance (where variance is a dispersion measure) and mean-VaR (where VaR is a non-coherent risk measure) which are only valid when returns are normal. The mean-CVaR framework respects diversification, and we find that CVaR is the most conservative measure of risk.

Keywords: *portfolio optimization, risk, variance, value-at-risk, conditional value-at-risk, linear programming.*

Chapter 1: Introduction

The term 'risk' cannot be easily defined because the definition of risk varies based on application and context. From an economic point of view, risk is a quantitative notion of quantifiable (measurable) uncertainty of an economic activity. Economic agents engage in different activities with outcomes that are uncertain. Losses and gains that result from these activities determine whether the economic agent or investor's wealth remains the same, deteriorates or improve. However, there is a consensus that risk invokes notions of dispersion, uncertainty, randomness and probability. This probabilistic and Brownian nature of risk makes risk measurement and management problematic.

Several measures have been used to measure risk, including variance, Value-at Risk and Conditional Value-at-Risk, and the latter is recognized as a superior proxy for risk. Risk needs not only be defined and measure, it also needs to be managed. Portfolio optimization is a technique for managing risk. It provides a tool which addresses the nature of risk and identifies optimal tradeoff between risk and reward.

The theory and practice of portfolio optimization was pioneered by Harry Markowitz in the 1950s. Since then, it has developed substantially to the point where portfolio optimization is now regarded as a distinct area in the theory of finance. Portfolio optimization has attracted a considerable amount of intellectual energy, not just from finance specialists, but also from specialists in other disciplines such as operations research, computer science, mathematics, and statistics due to the challenging intellectual problems that the subfield poses. In this light, portfolio optimization is a challenging topic of rising importance and increasing concern in the

financial industry. However, optimization depends upon a set of inputs which if biased, will result in imprecise or poor assessment of risk management.

1.1 Motivation of the study

The emergence of portfolio optimization and financial risk management as a discipline is due to a number of factors. One factor is an environment with increased volatility. A volatile environment exposes firms to greater levels of financial risks and provides incentives for firms to find new and better ways of managing this risk. Another factor that triggered the need for the development of portfolio optimization is financial market integration. The globalization of market economies is followed by a significantly larger and more liquid and risky equity market.

A different factor is the phenomenal growth in trading activity since the late 1960s, illustrated by the fact that the average number of shares traded per day on the New York Stock Exchange has grown phenomenally from around \$4 million in 1961 to around \$3.6 trillion in early 2007 (BIS, 2007). There have also been massive increases in the range of instruments traded over the past couple of decades, and trading volumes in these new instruments have also grown very rapidly.

The rapid advance of information technology has also contributed to the development of portfolio optimization and risk management. Improvements in the field of information technology have made possible huge increases in both computational power and in the speed with which calculations can be carried out as well as huge reductions in computing costs. These improvements have expanded the range of risk management options available to managers. They can now use sophisticated algorithms programmed into computers to carry out real-time calculations that were not possible before.

On another front, the development of risk management was spurred by a series of risk management disasters since the early 1990s when financially prudent institutions went bankrupt or announced massive losses, often through unauthorized trades, and always because of inadequate risk management processes. Such examples include Barings Bank (1995), Long Term Capital Management (1998) and Bayou Hedge Fund Group (2006). Most of these disasters involved losses in billions of \$US, which have made the financial community keenly aware of the importance of good risk management practices.

In parallel, research in the financial risk community all preached pretty much the same message – the need for good management controls including the need to use value-at-risk (VaR) models to estimate firmwide risks, the need for adequate systems to manage risk, and the importance of adequate disclosure.

The bank of international settlements (BIS) recommends the use of VaR as the ‘dominant’ measure of financial risk. However, the recent financial meltdown and chain of financial outbursts undermine the credibility of VaR as a proper risk measure.

1.2 Objective and implication of the study

In lack of a proper risk measure and a practical mean-risk portfolio optimization model, this study investigates the reasons that render CVaR (conditional value-at-risk) superior to VaR (value-at-risk).

Prior literature has established superiority of CVaR compared to VaR from a theoretical stand point. While there has been several public statements criticizing VaR, no study effectively investigates these claims using empirical data. In light of the recent financial crisis and the failure of the VaR measure to account for the risk exposure of financial institutions (Taleb,

2007), there is an opportunity to assess ex-post the usefulness of VaR and empirically corroborate the superiority of CVaR using most recent data before and during the financial meltdown.

In order to establish CVaR's superiority over the conventional risk measures, this thesis provides a unified theoretical framework of mean-risk analysis, and we test it empirically using financial data on return indices over a period covering the financial crisis. In line with previous work, the thesis builds upon the Markowitz Mean-Variance model to develop a more comprehensive Mean-Risk model of portfolio optimization. We propose to use VaR and CVaR as measures of risk instead of the variance. Both a Mean-VaR and Mean-CVaR portfolio optimization models are developed and implemented. Prior research compared mean-variance and mean-VaR or mean-VaR and mean-CVaR separately as part of solving the portfolio optimization problem. In this work, we take a more global stand by examining all possible mean-risk models using different measures of risk (Variance, VaR and CVaR) both theoretically and empirically. The empirical results of the three frameworks (mean-variance, mean-VaR and mean-CVaR) are compared.

We find that CVaR is superior to VaR both from empirical and theoretical standpoints. We find that VaR is a misleading risk measure when it comes to detecting rare unexpected extreme events. CVaR is superior to the traditional measures of risk on the basis that mean-CVaR portfolio optimization problem results in a more diversified portfolio and a more conservative risk measure. We recommend that financial institutions upgrade their risk management systems from VaR to CVaR.

The thesis is divided into six chapters. Chapter two overviews the validity of the most recently used measure of risk, VaR, in light of the financial crisis. Chapter three reviews the literature starting from the desired properties of risk measures, the different mean-risk portfolio optimization models, to portfolio optimization using CVaR versus other risk measures, and the empirical applications using mean-CVaR models. In Chapter four, we describe the data used and the methodology formulated and implemented for the three frameworks – mean-variance, mean-VaR and mean-CVaR. Empirical findings are presented in chapter five and the conclusion is offered in chapter six.

Chapter 2: VaR in Light of the Financial Crisis

Measuring risk has been an issue of major concern throughout the years. The seminal work of Harry Markowitz on modern portfolio theory addresses the tradeoff between standard deviation as the measure of risk and return as the measure of reward. Nevertheless, as discussed in the next chapter, critics argue that the mean-variance modern portfolio theory is unreliable. It conveys poor parameter estimates because variance does not capture all the features of risk. In addition, it considers a restrictive assumption which is risk normality. Considering the shortcomings of using variance as a measure of risk, in this chapter, we introduce risk measures that go beyond Harry Markowitz's modern portfolio theory. Namely, we highlight the advantages and disadvantages of the most widely used measure of risk, VaR, in light of the financial crisis.

2.1 The quest for an adequate measure of risk

Several attempts have been contended beyond Harry Markowitz to come up with a proper measure of risk. Maracine (2002) and Alderweireld, Garcia and Léonard (2006) suggest going back to traditional measures such as duration gap analysis and scenario or what-if analysis to analyze risk exposure. However, despite their simple practical implementation, such approaches have certain drawbacks. Maracine (2002) and Alderweireld, Garcia and Léonard (2006) criticize that such approaches ignore risks other than interest rate risk, tell us nothing about the likelihood of extreme risks, and depend to a certain extent on the skill of the risk analyst. Eventually, the authors point out that the results of duration gap analysis and scenario analysis are inaccurate and subjective.

In view of the lack of a proper measure of risk exposure, the Basel Capital Accord recommended that internationally active banks evaluate the riskiness of their assets using internally generated risk models instead of relying on outside credit ratings provided by ratings agencies. A number of major financial institutions started working on internal models to measure and aggregate risks across the institution as a whole. This practice was deemed even more important as financial firms became more complex and the market more volatile. The implementation of such models allowed financial institutions to calculate the VaR (Value of Risk) which is the maximum portfolio loss over a specified period of time within a certain confidence level. The Bank for International Settlements (BIS) even recommended the use of VaR as a 'dominant' measure of financial risk. VaR then rapidly became the most commonly used measure of risk and every financial institution was keen in developing and strengthening its own VaR system.

2.2 Advantages of VaR

VaR presents a significant number of advantages over traditional risk measures. It is expressed in the simplest and most easily understood unit for measuring risk which is the possible amount of money loss.

Marshall & Michael (1997) explain that VaR plays an important role in unifying the comparison of risk exposure among financial institutions. It can also be employed to measure the risk of any type of portfolio return distributions, hence facilitating the comparison of risks of different portfolios. For instance, VaR allows measuring risk associated with a fixed-income position in a way that is comparable to risk associated with an equity position.

Another feature of VaR is that it enables the aggregation of risks into an overall measure of portfolio risk. According to Duffie and Pan (1997), VaR takes into account the ways in which different risk factors interact (or correlate) with each other. Similarly, Billio and Pelizzon (2000) argue that VaR is holistic because its focus is on a complete portfolio, often at the firmwide level, and not just on individual positions.

Ammann and Reich (2001) show that, unlike previous measures of risk, VaR does not provide answers to 'what-if' questions. Instead, VaR gives an indication of the likelihood of loss by associating probabilities with specified loss amounts.

Shirletov (2007) argues that VaR can have several desirable applications. Specifically, (a) senior management can use VaR to set their overall risk target. For example, if they want the firm to increase its risks, they would increase the overall VaR target, and vice versa. (b) VaR can be used to determine capital requirements, both at the firmwide and business unit level; the riskier the activity, the higher the VaR and the greater the capital requirement. (c) VaR applications extend to marked-to-market reporting and disclosing purposes, even reporting VaR information in annual reports. (d) VaR-based decision rules can guide investment, hedging, trading and portfolio management decisions by taking into account the implications of alternative choices for portfolio risk as a whole. VaR can also be used to carry out portfolio-wide (or macro) hedging strategies that are otherwise difficult to implement. (e) Finally, VaR information can provide the basis for remuneration packages for managers, discouraging excessive risk-taking that would otherwise reward employees on the basis of profits alone without any regards to risk. In short, VaR is not only an attractive risk measure for portfolio optimization and resource allocation, it also helps establish a more consistent and integrated approach for a better overall risk management.

In light of the widespread support for VaR as a measure of risk, it has had an overarching and mass-scale application across industries.

2.3 Criticisms of VaR

Although most risk practitioners embrace VaR, there are those who warn that VaR has deeper problems and can be misleading as a measure of risk. One key argument is the validity of the statistical assumptions of VaR. Both Nassim Taleb (1997) and Richard Hoppe (1998) are critical of the naïve transfer of mathematical and statistical models from sciences where they are well suited to social systems where they are often invalid. The mathematical and statistical models often ignore important features such as the ways in which risk managers learn and react to their environment and the dynamic non-stationary interdependence of the market processes. VaR estimates do not take such features into consideration and hence are wide open to major errors.

Ibragimov (2004) argues that VaR estimates are imprecise to be of much use. The VaR framework liberates us from the confines of normality in the sense that it provides a risk measure that can be estimated for any distribution. However, the usefulness of VaR as a measure of risk is highly questionable outside the confines of normal distributions. For instance, VaR does not account for the subadditivity property if the distribution is not normal. In this case, VaR is going to misinform portfolio managers about the diversification effect in the portfolio, and they may make the irrational decision about portfolio allocation. As a consequence, the portfolio risk may actually increase.

Even before the outburst of the financial crisis, Danielsson (2002) and Taleb (2007) suggest that the use of VaR may destabilize the financial system. Inaccurate regulatory VaR

constraints can weaken the financial system by inducing banks to increase their risk taking. A misleading VaR constraint may give risk managers an incentive to protect themselves against mild losses, but not against larger ones in excess of VaR. Also, VaR regulatory constraints can exacerbate cyclical effects, and so aggravate financial crises. Taleb's concerns proved well grounded. He calls the financial events of the early 1990's and the recurring financial crisis in 2007 "Black Swans"¹. Such rare events and abnormal markets are excluded from the VaR estimates. Taleb argues that VaR is 'charlatanism' because it claims to estimate the risks triggered by rare events, and this is impossible. As a result, VaR gives false confidence and can be easily exploited by traders.

In the Global Association of Risk Professionals Review (2008), Einhorn, a hedge fund manager, describes VaR as "an airbag that works all the time, except when you have a car accident." He charges that VaR, instead of mitigating risk, may lead to excessive risk-taking and leverage at financial institutions. It ignores risks at the tails and focuses instead on the risks near the center that is easily manageable. VaR creates an incentive to take more risks under the illusion of getting higher return, and it is potentially catastrophic when it provides a false sense of security to risk executives and managers who may lose much more than they had bargained for. For instance, using VaR to build reserves in order to cover losses in times of crises, then underestimation may be fatal and overestimation may lead to inefficient use of capital. As Taleb puts it: "You're worse off relying on misleading information than on not having any information at all. If you give a pilot an altimeter that is sometimes defective he will crash the plane. Give him nothing and he will look out the window."

¹ Black swans refer to hard-to-predict rare events with a large negative impact.

More recently, Joe Nocera, a New York Times reporter, wrote a comprehensive article on Risk Mismanagement on January 4, 2009 whereby he discusses the role that VaR played in the financial crisis of 2007-2008. After interviewing risk managers, the article suggests that VaR was very useful to risk experts, but nevertheless exacerbated the crisis by giving false confidence and security to bank executives and regulators. VaR is a powerful tool for risk management, but it is depicted as “both easy to misunderstand, and dangerous when misunderstood.”

Proponents of VaR may respond that many of these criticisms are not specific to VaR, but would apply to other conventional risk measures as well. They may also argue that some of these crises and problems are due to the misuse of VaR rather than the VaR itself. VaR is not a useless concept to be abandoned altogether. For example, it can be used in risk-reporting only as a characteristic of the portfolio return distribution since it has a straightforward interpretation. However, the financial crisis that stunned the world not only undermines the VaR's position as the dominant measure of financial risk, but also destroys the VaR's claim to be regarded as a ‘proper’ measure of financial risk in the true sense of the term. This takes us to the next major development in financial risk management, the theory of coherent financial risk measures and CVaR.

Chapter 3: Literature Review

In portfolio theory, several measures of risk have been modeled; each with different properties and serving different purposes. Some of these risk measures like the standard deviation of returns, the interquartile range and the mean absolute deviation are descriptive values. In contrast, other risk measures like implied volatility, stochastic volatility and generalized autoregressive conditions are empirical in nature. These measures describe the second moment of the random returns. Other risk measures estimate risk as a process, including analytical measures such as VaR and CVaR which feature the likelihood of extreme risks or tail probabilities. The value-of-risk (VaR) is widely used by financial firms to get a better sense of their overall risk exposure and to achieve a more optimal allocation of capital. Alternatively, the conditional value at risk (CVaR) has been recently established as being a ‘coherent’, ‘true’ and ‘practical’ measure of risk. Mean-Risk portfolio optimization problems recommend using CVaR as a superior measure of risk compared to other commonly used risk measures and proxies such as VaR, variance or standard deviation. This chapter outlines the progress that has been made in developing the concept of a ‘coherent’ risk measure from both a theoretical and a practical viewpoint. It highlights the new trend that has recently emerged to examine the implications of mean-CVaR models using finance and economic applications.

The chapter is organized as follows. Section 1 overviews the nature of risk and the desirable properties to of a ‘coherent’ risk measure and which are satisfied by CVaR. Section 2 presents different mean-risk portfolio optimization problems modeled using CVaR as a measure of risk. Section 3 reviews research that compares mean-CVaR to other mean-risk models,

including VaR and other dispersion measures. Section 4 discusses some of the recent empirical applications using mean-CVaR optimization models.

3.1 Properties of Risk Measures with emphasis on CVaR

There has been a major debate on the differences and common features of risk measures. Several studies have attempted to come up with a 'true' measure of risk, but none has actually explicitly deduced an ideal and comprehensive measure. Nevertheless, a set of desirable properties that a risk measure should satisfy has been developed throughout the course of the literature.

Holton (2004) explores the nature of risk and argues that risk is a subjective phenomenon involving exposure and uncertainty. Exposure is defined as susceptibility to risk, and uncertainty is the lack of complete certainty, that is, the existence of more than one possibility. Financial institutions are becoming increasingly complex and they should manage and control for the sources of risk in. If risk can be measured, then the assets that contribute to portfolio risk may be identified, leading to a reallocation of assets that minimizes potential loss and portfolio risk. However, modeling for exposure and uncertainty is limited to the range of risks that are perceived. A true functional definition of investment risk is out of reach, and no perfect risk measure exists. Financial institutions are making a lot of effort to model risk, but they are only able to capture some of the characteristics of risk, rendering common risk metrics used such as the variance as incomplete.

Artzner, Delbaen, Eber and Heath (ADEH, 1998) develop the desirable features of a "coherent" measure of risk. They develop a framework for measuring risk and demonstrate that the characteristics of a risk measure ought to satisfy four axiomatic properties. ADEH denote the

risk measure by $\rho(X)$ and assign a real-valued number to a random variable. If investment A (B) has a random payoff equal to Y (X), then the four properties of a coherent risk measure are:

1) The Monotonicity Property

$$\rho(Y) \leq \rho(X), \text{ if } Y \geq X \quad \forall \text{ random variable } X \text{ and } Y$$

Monotonicity states that if investment A has random return (payoff) Y that is at least equal to the return (payoff) X of investment B, then the risk of A is less than the risk of B.

2) The Positive Homogeneity Property

$$\rho(0) = 0, \rho(\lambda X) = \lambda \rho(X), \text{ for all random variable } X \text{ and all } \lambda > 0.$$

According to the positive homogeneity property, scaling the return (payoff) of the portfolio by a positive factor scales the risk by the same factor. In other words, if the investment doubles, so does the risk.

3) The Subadditivity Property

$$\rho(X + Y) \leq \rho(X) + \rho(Y), \text{ for all } X \text{ and } Y.$$

If X and Y describe random payoffs, then the subadditivity property implies that the risk of the portfolio is not greater than the sum of the risks of the two random payoffs.

The positive homogeneity and the subadditivity properties imply that:

$$\rho(\lambda X + (1-\lambda)Y) \leq \rho(\lambda X) + \rho((1-\lambda)Y) = \lambda \rho(X) + (1-\lambda) \rho(Y)$$

Where $\lambda \in [0,1]$. If X and Y describe random returns, then the random quantity $\lambda X + (1-\lambda)Y$ stands for the return of a portfolio composed of two financial instruments with returns X and Y having weights λ and $(1 - \lambda)$ respectively. Differently stated, the risk of a portfolio is not greater than the sum of the risks of its constituents. This property lies at the heart of the diversification effect.

4) The Translation Invariance Property

$$\rho(X + C) = \rho(X) - C, \text{ for all } X \text{ and } C \in \mathbb{R}$$

If X describes a random payoff, then the translation invariance property suggests that adding cash to the investment reduces risk by the amount of cash added. As a result, the risk measure can be used as a guideline to set capital requirements for a certain portfolio. The risk measure $\rho(X)$ can be interpreted as the minimal amount of cash or reserve necessary to serve as a cushion against risk,

$$\rho(X + \rho(X)) = 0$$

Szego (2002) highlights the importance of the ADEH (1998) properties in identifying coherent measures of risk. The author discusses the conditions under which the classical risk measures (variance, linear correlation coefficient and VaR) can function. The variance is best suitable when the distribution function of the random returns of a portfolio is normal. The correlation coefficient describes a multivariate distribution. It is strictly related to the slope parameter of the linear regression of the random variable Y on the random variable X where X and Y are returns of assets. In other words, the linear correlation coefficient measures only the co-dependence between the linear components of X and Y . Hence, it is best suitable in case of symmetric and elliptic distributions. Recent research on new risk measures has been enhanced by setting new regulations for the financial institutions which require the use of sophisticated risk control models. In 1994, the concept of Value-at Risk, VaR came into the picture as reliable measure of risk because it sets no restrictions on the distribution function i.e. VaR functions properly with multimodal distribution functions. VaR, however, fails to account for portfolio diversification which represents a critical normalization rule used while controlling for risk. As a result, VaR is not an adequate measure of risk because acceptable risk measures should satisfy

ADEH (1998) four properties which VaR does not. Szego finds that the problem of identifying a risk measure can be easily solved by using coherent risk measures satisfying ADEH (1998) conditions.

Before the publication of the paper by ADEH (1998) on coherent risk measures, there were no standards of how to identify a valid risk measure. Ortobelli, Rachev, Stoyanov, Fabozzi and Biglova (ORSFB 2005) evaluate the properties of risk measures in order to understand how the latter has to be used in the context of portfolio optimization. They differentiate between two “admissible” classes of risk measures; safety risk measures and dispersion measures². The authors explain that the main characteristic of risk is its subjectivity which arises because two managers may define the same investment as having different risk. To illustrate the difference between safety risk measures and dispersion measures, ORSFB propose an empirical comparison among three different portfolio choice models that depend on measures of increasing moments. In particular, they use the mean, risk measures (MiniMax, mean absolute deviation (MAD) and standard deviation) and a skewness parameter as first, second and third degree moments respectively, then they compare the impact of the choice of different measures on the optimization of wealth or return of the portfolio. MiniMax minimizes the maximum possible risk, and MAD is the average absolute deviation from the mean of the data set. All three measures are monotone, consistent with second-order stochastic dominance, positively homogeneous, convex and linearizable. However, ORSFB note that the three commonly used risk measures are, in reality, a measure of uncertainty i.e. dispersion. They find that only under certain conditions, such as at a fixed mean (return) level, may an uncertainty or dispersion

² Following the tradition of Frank Knight (1921), some financial analysts make a distinction between risk and uncertainty. Risk exists when a probability based on past experience can be attached to an event in the future, whereas uncertainty exists when there is no objective way to place a probability on an event.

measure be used to minimize risk and to obtain optimal portfolios for prudent non-satiable investors. Since, the mean level is rarely assumed to be fixed, the standard deviation, as with every dispersion measure, is not a proper risk measure.

In an extended paper, Rachev, Ortobelli, Stoyanov, Fabozzi and Biglovae (ROSFB 2006) search for an ideal measure of risk in the portfolio selection problem. The authors argue that the classical Von Neumann–Morgenstern framework for maximizing the expected utility function cannot distinguish all types of human behavior observed in financial markets. Conventional portfolio selection problems fail to take into account several investors' attitudes towards risk. ROSFB identify the intrinsic properties of risk, which is an asymmetric, relative multidimensional and heteroskedastic concept. They also demonstrate that investor choice has to account for several factors, among which, are stochastic dominance, diversification, down-side risk, and risk aversion. They also evaluate investor choice using a portfolio selection among 13 international indices.

Ortobelli, Rachev, Shalit and Fabozzi (ORSF 2006) examine ideal measures of risk financial risk and show that CVaR, conditional value at risk, is a “practical” risk measure. Previous work has proffered alternative definitions of the best proxy for risk to employ in selecting optimal portfolios, from the traditional mean-variance (MV) criterion to the axiomatic properties of ADEH's (1998) coherent measure of risk (positive homogeneity, translation invariance, subadditivity, and monotonicity). Building on the seminal work of ADEH (1998), ORSF use stochastic dominance orders and the desirable features of the measure of risk to prove that CVAR is a spectral coherent measure. CVaR is also called “expected shortfall” or ‘expected

tail losses". They demonstrate that CVAR is a Gini³ risk measure in assessing the degree of dissimilarity or discrepancy between two random variables (portfolio and market index) and that is preferred over the variance. ORSF contribute to the literature by linking probability metric theory with stochastic orderings and using CVAR as a reliable measure of risk in portfolio theory.

CVaR was first introduced, as risk measure, by Uryasev (2000) and more extensively by Rockafellar and Uryasev (RU 2001). The authors argue that the widely used value-at-risk (VaR) is not a consistent measure of risk because it does not meet the subadditivity property. It is also non convex with respect to portfolio positions and provides stable statistical estimates. Instead, a reliable measure of risk should satisfy the mathematical characteristics previously established by ADEH (1998). Uryasev shows that CVaR is superior to VaR as a risk measure because it accounts for risks beyond VaR (i.e. it is more conservative than VaR). RU further highlight the crucial role of a measure of risk under uncertainty, especially in coping with the losses that might be incurred in the financial industry. However, the authors criticize VaR as being unstable, difficult to work with numerically and that it lacks account for diversification. They prove that CVaR has the ability to quantify risk beyond VaR because it measures downside risk and it is also applicable to non-symmetric and non-normally distributed loss distributions. CVaR is consistent with the mean-variance approach since, for normal loss distributions, optimal variance and CVaR portfolios coincide. In addition, the CVaR risk measure is easy to optimize when dealing with non-normal distributions, since linear programming techniques can be used for optimizing very large problems (over 1,000,000 instruments and scenarios). Finally, CVaR is

³ Gini (1965): The Gini coefficient is a measure of statistical dispersion. It is a ratio that belongs to $[0,1]$. A low Gini coefficient indicates more equal income or wealth distribution, while a high Gini coefficient indicates more unequal distribution. Both CVaR and the Gini coefficient measure are coherent risk measures which analyze statistical dispersion in terms of wealth or return distribution.

consistent with the preference relations of risk-averse investors. RU conclude that CVaR has an intuitive interpretation to portfolio optimization. The convenient and practical ways for computing and estimating CVaR allow for its application in optimal portfolio problems like mutual funds, pension funds and insurance.

Similar to Szego (2002), Danielsson (2002) reviews the properties of VaR both from a computational and a regulatory point of view to warn financial institutions of the drawbacks of VaR. When financial institutions model risk, they assume the underlying financial data takes place during stable periods and irrespective of the fact that a crisis might take place. Since investors' actions and decisions differ during the times of stability or crisis, the VaR risk measure is not robust and volatile. It is based on historical data and does not provide an accurate statistical forecast. Its violation ratio⁴ is lower than 1, and its volatility range does not cover frequent fluctuations across the time horizon. Hence, the theoretical properties of the VaR measure may provide misleading information about a financial institution's risk level. Danielsson finds that VaR not only can make the financial institution pay considerable costs, but it may perversely increase both idiosyncratic and systemic risk.

3.2 Modeling Mean-Risk portfolio optimization problems

After introducing CVaR as a coherent measure of risk, several papers have studied mean-risk portfolio analysis using CVaR as an alternative and superior measure of risk compared to VaR. The idea is to benefit from the computational and practical advantages of CVaR to generate sound, effective and pragmatic mean-risk portfolio optimization models.

⁴ Violation ratio is the realized number of VaR violations over expected number of violations. A ratio lower than 1 indicates that the realized loss is larger than the VaR forecast.

Rockafellar and Uryasev (2000) develop a new approach for simultaneous calculation of VaR and optimization of CVaR that is applicable to a general class of problems. They implement a new algorithm which simultaneously calculates the VaR and minimizes the CVaR. Generally, this problem is a non-smooth stochastic optimization problem which can be solved using general stochastic optimization approaches. By minimizing CVaR, the authors also minimize VaR because CVaR 'majorates' VaR. They collect data on the NIKKEI portfolio of options and conduct numerical experiments to test their approach. They show that portfolio positions converge quite accurately to the true values, thus demonstrating that their framework provides valid results particularly when dealing with optimal hedging problems. The authors also find that CVaR can be efficiently minimized using Linear Programming and Non-smooth Optimization techniques.

Krokhmal, Palmquist, and Uryasev (2002) extend the work of Rockafellar and Uryasev (2000) for the simultaneous calculation of VaR and CVaR optimization. They suggest a new approach to optimize CVaR by for maximizing the reward function (e.g. expected returns) under CVaR constraints. Identifying such constraints with different confidence levels gives room to shaping the loss distribution according to the preferences of the decision maker. The authors introduce CVaR as the percentile risk measure instead of VaR assuming that return can be either continuously distributed or discrete. Modeling the new approach, the objective function maximizes expected return and the constraints are divided into transaction costs, value constraints, liquidity constraints and bounds on position. To demonstrate the efficiency and flexibility of their new approach as a risk management tool, the authors construct the efficient frontier of a portfolio consisting of stocks in the S&P100 index. Using the MV framework as a benchmark, the authors compare their model with the traditional Markowitz model. The

difference between the MV and CVaR approaches is not very significant. The authors explain that such a result is because the selected dataset has a 'close-to-normal' distribution in historical returns. This minor difference is expected since, according to Uryasev (2000), CVaR and MV methods converge to the same optimal portfolio when returns are normally distributed. The authors also find that their optimization algorithm is stable, efficient and can handle a large number of instruments.

Beliakov and Bagirov (2005) compare two modeling techniques for calculating CVaR and optimizing portfolios, including linear programming and convex non-smooth optimization method. They use the discrete gradient method to generate the efficient solutions and establish that non-smooth optimization is superior to linear programming. They also argue that CVaR possesses a number of desired mathematical properties, notably convexity and subadditivity, and is a more attractive measure of risk than VaR. Unlike VaR, which underestimates the risks associated with the individual securities, CVaR correctly embodies the reduction of risks associated with diversification. Finally, the authors conclude that solving portfolio optimization problems using the non-smooth optimization technique method and discrete gradient method is suitable for optimizing small to medium size portfolios (40 securities) and the optimal portfolio can be found using a reasonable central processing unit time.

Roman, Darby-Dowman and Mitra (RDM 2006) argue that previous modeling techniques using mean-risk analysis may produce non optimal portfolios because they are dominated with respect to second order stochastic dominance. Broadly speaking, there are three main types of portfolio optimization models – mean-risk models, expected utility maximization and stochastic dominance models. Mean-risk models are convenient from a computational point of view, but which, depending on the risk measure that is used, may lack a rational and theoretical basis for

making a choice, thus questioning the validity of the results. Expected utility maximization is usually difficult to put into practice since the choice of a suitable utility function is somewhat subjective. Stochastic dominance relations, Second-Order Stochastic Dominance (SSD) in particular, are the soundest theoretical basis for making rational choices corresponding to observed economic behavior. SSD is important in investment problems since it describes the preference of rational and risk-averse decision-makers. Unfortunately, they are difficult to apply in practice. In order to construct a portfolio which is non-dominated with respect to second order stochastic dominance, RDM define a multi-objective problem for portfolio selection which is solved using a single objective function. They use CVAR as a measure of risk and append their model by constraints based on multiple scenarios. Using a dataset drawn from the Hang Seng index, RDM employ the reference point method to solve the model and retrieve the set of feasible portfolios. They also demonstrate that their model produces portfolios that are consistent with SSD and which are interactive with the decision maker's aspirations and reservations.

Similarly, Künzi-Bay A. and Mayer J. (2006) consider optimization problems for minimizing CVaR from a mathematical and theoretical point of view. They reformulate these CVaR optimization problems as two-stage recourse problems of stochastic programming and use the L-shaped method⁵ to generate a new algorithm for minimizing CVaR. Implementing the L-shaped method for two-stage recourse problems on the original linear CVaR model produces a computationally efficient CVaR portfolio optimization problem. To illustrate the competence and efficiency of the new algorithm, the authors randomly generate test problems and use the

⁵ Birge and Louveaux (1997): The L-shaped method is a decomposition method that is useful for solving problems that have the form of a master problem and several subproblems. It is a process that solves a sequence of much smaller problems. The complete problem with both the master and subproblems may be very large and beyond the capabilities of the available Excel solver. It reduces the complexity of stochastic convex programming.

CVaRMin solver to produce the optimal portfolio. They compare the minimized algorithm and the traditional CVaR linear program and find that it takes the CVaRMin solver less iterations to find an accurate optimal solution.

Quaranta and Zaffaroni (2008) research how robust optimization techniques can be used to minimize Conditional Value-at-Risk, CVaR. The idea is to develop a robust optimization technique that deals with uncertainty instead of stochastic orders as suggested by RU (2002). CVaR is used as the major component in their optimization model because it is a coherent risk measure. They pay keen attention to the estimation of expected returns, another critical component of the optimization program. In contrast to the approach of RU, the historical returns have been weighed exponentially with the aim to reduce the presence of the ghost-effect⁶. Quaranta and Zaffaroni then follow Soyster's⁷ (1973) approach to generate the robust optimization model. The authors run some numerical examples to prove the efficiency of their model.

3.3 Portfolio optimization using CVaR versus Other Risk Measures

Another related strand of literature compares the usefulness of using dispersion measures and VaR as opposed to CVaR in mean-risk analyses.

Konno, Waki and Yuuki (2002) review the characteristics of lower partial risk or downside risk measures and compare four alternative risk measures. These include lower semi-variance, lower semi-absolute deviation, first order below target risk and conditional value-at-risk. The authors argue that these risk measures, in general, and CVaR, in particular, are useful to

⁶ The ghost-effect appears when homogeneous weights are used to compute expected returns, thus attributing the same importance to more recent and closer data.

⁷ Soyster (1973) discusses in a very specific setting, uncertain hard constraints in linear programming models.

control downside risk when the distribution of assets is non-symmetric and not normal. After reviewing the advantages of each risk measure, they criticize VaR and demonstrate that CVaR possesses theoretically and computationally favorable properties of CVaR. They also formulate portfolio optimization mean-risk linear models using each of the considered risk measures and perform simulations using historical data. The authors conclude that Mean-CVaR model is superior to other models when controlling for the downside risk of a portfolio.

Alexander and Baptista (2004) analyze portfolio optimization by imposing two different constraints on the mean-variance model, VaR constraints and CVaR constraints. The authors set propositions on the boundaries of VaR and CVaR constraints and show that, for a certain confidence level, a CVaR constraint is more effective than a VaR constraint in controlling the decisions of risk-averse agents. They conclude that CVaR constraints dominate VaR constraints as a risk management tool and that VaR may be misleading as a risk measure.

Similar to Alexander and Baptista (2004), Cox, Lin, Tian and Zuluaga (2008) also extend the traditional portfolio optimization MV Markowitz problem with CVaR-like constraints to demonstrate the superiority of the latter as compared to VaR constraints. Using the CVaR optimization technique, the authors integrate the positive and negative skewness of the asset-liability portfolios in order to better model the preference of the decision makers. In a fashion similar to that of Krokhamal P., Palmquist J., and Uryasev S. (2002) model, the authors use CVaR-like constraints to better reshape the return distribution according to the customers' preferences. These preferences are specified directly in percentile terms, as compared to the traditional approach which specifies risk preferences in terms of utility functions. In addition, the authors compare the CVaR-like constraints approach with the traditional mean-variance Markowitz method and the mean-variance-skewness (MVS) recent method described by Boyle

and Ding (2006). The MV model is a one-moment model, CVaR is a two-moment model and MVS is a third moment model. The authors explain that the third moment (or skewness) condition of MVS is difficult to handle because it results in a non-linear constraint which is too complex and iterative to linearize. Finally, the authors compute the optimal portfolios for five assets based on the Markowitz model with a CVaR-like constraint (MV + CVaR) and the Boyle-Ding (2006) MVS. They evaluate the models by comparing their efficient frontiers and provide empirical support for the superiority of CVaR-like constraints approach over the other two approaches presented.

3.4 Empirical Applications using Mean-CVaR Models

Existing literature, which focused on the computational and theoretical aspect of mean-risk analysis, has established mean-CVaR as a superior model for portfolio optimization. A new trend has recently emerged to examine the implication of mean-CVaR models on finance and economics applications.

Romero-Meza and Laengle (2005) optimize CVaR of an average Chilean pension fund. The authors explain caveats of using VaR as a risk measure and use CVaR along with stochastic programming to find the average portfolio held by Chilean Pension Fund Administrators (AFPs) between October 2, 2002 and March 31, 2005. They use linear optimization modeling and show that the CVaR optimization model can be implemented on a concrete example with “intuitive” economic and financial implications. The authors find that the minimum return guarantee restriction set by AFP is seriously affecting the efficiency of the portfolio assignment.

Mulvey and Erkan (2006) apply a decentralized risk management strategy using the CVaR risk measure to reduce enterprise risks and increase profits. The authors particularly show

the advantages of CVaR over traditional risk measures such as value-at-risk and provide a concrete example that illustrates how mean-CVaR optimization model can be implemented within the context of an insurance company. In the insurance industry, portfolio managers and underwriters require sophisticated analytical tools to achieve enterprise goals such as evaluating the effects of adding a new account(s) or subtracting an existing account(s) from the current portfolio. Decision Support System tools previously used like SmartWriter are centralized and complicated to implement (Lowe and Stanard, 1996). Instead, the authors introduce decentralized dynamic financial analysis (DFA) in order to design a practical system for analyzing and managing a large-scale global financial organization. Formulating the optimization problem with CVaR constraints for the insurance company, the authors implement the decentralized DFA model for a global earthquake insurance firm with two-divisions located in USA and Europe. They find that a global insurance company has greater opportunities for risk diversification across many regions, thus reducing its overall capital requirements. In addition, a global insurance company is difficult to manage in a centralized fashion due to complex local regulations/policies.

Sakalauskas and Kriksciuniene (2006) use VaR and CVaR optimization methodologies to study short-term investment risk in the currency market. The authors explain that risk estimations in the stock markets are usually performed on a daily basis or even over longer periods. However, due to substantial market volatility, daily risk evaluations may not be sufficient. The authors suggest risk evaluations according to hourly profit alterations of the financial instrument. They implement mean-VaR and mean-CVaR econometric models to examine the risk of hourly time intervals using long position EUR/USD exchange hourly rate. They find that, with

increased volatility, stop-loss values have to be increased. Thus their work confirms the efficiency and practicality of mean-CVaR optimization model in recent financial applications.

These reviewed applications of mean-CVaR optimization models (pension funds, insurance control and short-term investments in the currency market) open up new applications horizons for CVaR. Possible extensions include investment companies, brokerage firms, hedge funds, pension funds, mutual funds, insurance industry, etc... Compared to the voluminous body of literature on mean-risk portfolio optimization, however, empirical studies using CVaR with real world applications lag behind.

The perspective of this paper is that there should be more interaction between two trends of the literature, operations research and finance. On the one hand, most of the financial studies focus on the implications of the underlying data, often ignoring the econometric techniques of operations research from which they can greatly benefit. On the other hand, much of the operations research literature considers as given the estimates used in the empirical applications, without considering economic and financial forces and inferences.

The following chapter provides a concise and straightforward model to estimate CVaR using a diversified portfolio of market indices. The traditional Markowitz Mean-Variance (M-V) analysis is generalized into a Mean-Risk framework using a true measure of risk other than the variance. We contribute to the literature by converting a complex optimization problem into a simple and linear program which can be easily solved. The final output is a structured linear programming portfolio optimization mean-CVaR model.

Chapter 4: Data and Methodology

Modern portfolio theory has developed considerably after the pioneering work of Harry Markowitz in the 1950s. Since the 1990s, significant progress has been made to further understand the concept of a risk measure from both a theoretical and a practical viewpoint. A distinction between risk and uncertainty, which translates into a distinction between a risk measure and a dispersion measure, offers a new way of looking at the problem of optimal portfolio selection.

As concepts develop, other tools become appropriate to exploring evolved ideas than existing techniques. In applied finance, these tools are being imported from mathematics. Optimization, which is a field in applied mathematics, is well-positioned for the purpose of this study.

This chapter is organized in the following way. The first section statistically describes the data employed in the analysis. The second section overviews modern portfolio theory introduced by Harry Markowitz (1952) or mean-variance analysis. Markowitz suggested that the portfolio choice be made with respect to two criteria: the expected portfolio return and the variance of the portfolio return, the latter being used a proxy of risk. The third section extends mean-variance analysis by adopting a risk measure other than the variance. This generalization framework leads to mean-risk analysis developed in section three where VaR is used as a measure of risk. The fourth section substitutes CVaR for VaR, as a coherent risk measure, and we demonstrate how the underlying optimization problem can be simplified where CVaR is selected as the risk measure. The chapter concludes with a simple linear optimization problem which can be empirically implemented to identify the optimal portfolio selection.

4.1 Data Description

Daily data on eight market indices is retrieved from Thomson 4.0 Datastream. The historical data set contains 1,680 daily values starting on January, 1st, 2003 and ending on June, 8th, 2009. It consists of 5 equity indices and 3 bond indices which cover different geographic areas. We collected data on MSCI WORLD (*EW*), MSCI USA (*EUS*), MSCI EUROPE (*EE*), MSCI FAR EAST (*EFE*), MSCI EMERGING MARKETS (*EEM*), the DataStream US Bond Index (*BUS*), Europe Bond Index (*BEU*) and Emerging Market Bond Index (*BEM*).

Figure 1 presents the trend in the variation of the price indices over our sample's period.

Figure 1: The original data set consisting of 6 financial indices

Figure 1 shows the sharp drop in equity prices during the financial crisis which started towards the beginning of 2007.

The returns of the price indices are computed geometrically using the formula, $r_t = 100 \times \ln \left(\frac{P_t}{P_{t-n}} \right)$, where n stands for the holding period in days. RiskMetrics suggests using a holding period of 10 days, so returns are also calculated over a 10-day period as $r_t = 100 \times \ln \left(\frac{P_t}{P_{t-10}} \right)$. Figure 2 shows the box-plots for the returns of the different indices employed.

Figure 2: Box Plot of 10-day returns

The box-plot, or box-and-whiskers diagram, is a convenient way of depicting several statistical characteristics of a sample. The size of the box equals the difference between the 3rd and the 1st quantile (75% quantile – 25% quantile), or the interquartile range. The line in the box corresponds to the median of the data (50% quantile). The lines extending out of the box are

called whiskers and each of them is long up to 1.5 times the interquantile range. All observations outside the whiskers are outliers and are depicted by a plus sign.

The box-plot generated from our sample shows a large interquantile range for equity indices, but this range is narrow for bond indices, pointing to a greater variability in equity indices. The plot also shows that the sample data contains many outliers. Within the advent of the financial crisis towards the beginning of 2007, equity prices exhibited abrupt changes, thereby resulting in extreme values as represented by outliers. These outliers are not excluded from our sample data because we need to shed the light on the implications of such acute values and tail probabilities.

Table 1 provides descriptive statistics on the different indices.

Table 1: Summary Statistics of Returns

(%)	EW	EE	EUS	EFE	EEM	BUS	BEU	BEM
Mean	-0.1054	0.1689	0.0446	0.1944	0.4726	-0.0013	0.1698	0.1662
Median	0.5258	0.7560	0.5667	0.5160	1.3731	-0.0129	0.2040	0.1847
Variance	28.2128	19.0837	12.8714	17.1092	28.5266	0.8825	5.0539	5.2723
Std. dev.	5.3116	4.3685	3.5877	4.1363	5.3410	0.9394	2.2481	2.2962

Table 1 indicates that the equity indices have a return higher than those of the bond indices. Consistent with the relationship between risk and returns, equity indices are riskier than bond indices, and their standard deviation of returns exceeds that of bond indices.

A central element in mean-variance portfolio optimization is the normality assumption which, if violated, would give misleading results. In order to assess normality, we first present a normal probability plot.

Figure 3: Normal Probability Plots

The normality plot of each return index has three graphical elements. The plus signs show the empirical distribution of data as represented by each point in the sample. The solid line connects the 25th and 75th percentiles or the majority of the data and represents a robust linear fit (i.e., insensitive to the extreme values in the sample). The dashed line extends the solid line to both ends of the sample. The y-axis values are probabilities and, hence, vary from 0 to 1. The distance between the tick marks on the y-axis matches the distance between the quantiles of a normal distribution for each index. If all data points fall very close to the solid line, the assumption of normality is reasonable, else if the data is nonnormal, the plus signs follow a curve. Figure 3 provides clear evidence that, aside of EFE, the underlying distribution of returns is not normal.

In order to ensure that the previous result is valid, we conduct a test of normality called the Kolmogorov-Smirnov test. This test compares the values in the data vector with a standard normal distribution. The null hypothesis for the Kolmogorov-Smirnov test is that the data vector has a standard normal distribution. The alternative hypothesis is that the data vector does not follow that distribution. Table 2 reports the results of this test.

Table 2: Komogrov-Smirnov test results

	P-value
EW	0.0000
EE	0.0000
EUS	0.0000
EFE	0.0503
EEM	0.0000
BUS	0.0000
BEU	0.0000
BEM	0.0000

The test results of the Kolmogorov-Smirnov are highly significant and reject H_0 in favor of H_1 . Except for the EFE equity index, normality is rejected for all indices employed.

Table 3 shows the correlation matrix of our 10-day returns.

Table 3: Correlation matrix of returns

Correlation	EW	EE	EUS	EFE	EEM	BUS	BEU	BEM
EW	1.0000	0.8871	0.8879	-0.1472	0.2770	0.6932	0.2781	0.7511
EE	0.8871	1.0000	0.8536	-0.1465	0.4154	0.6865	0.4165	0.7959
EUS	0.8879	0.8536	1.0000	-0.2063	0.1721	0.6228	0.1722	0.7003
EFE	-0.1472	-0.1465	-0.2063	1.0000	0.4186	-0.0610	0.4163	-0.1567
EEM	0.2770	0.4154	0.1721	0.4186	1.0000	0.2712	0.9830	0.2424
BUS	0.6932	0.6865	0.6228	-0.0610	0.2712	1.0000	0.2644	0.7534
BEU	0.2781	0.4165	0.1722	0.4163	0.9830	0.2644	1.0000	0.2523
BEM	0.7511	0.7959	0.7003	-0.1567	0.2424	0.7534	0.2523	1.0000

Table 3 indicates that, while some indices are highly correlated, the majority presents low correlations with other indices, possibly giving room for diversification.

4.2 Methodology: Modeling the Portfolio Optimization Problem

4.2.1 Mean-Variance Model

The classical mean-variance framework introduced by Markowitz (1952) is the first proposed model of the reward-risk type. The expected portfolio return is used as a measure of reward and the variance of portfolio return indicates how well-diversified the portfolio is.

Suppose there are n financial assets with $n \geq 2$. Let the assets returns be the vector $X' = \{X, \dots, X_n\}$ in which X_i stands for the return on the i -th asset⁸. The returns are random and their expected rates of return or mean is $\mu' = \{\mu_1, \dots, \mu_n\}$, where $\mu_i = EX_i$.

The result of an investment decision is a portfolio whose composition is denoted by $w' = \{w_1, \dots, w_n\}$, where w_i is the portfolio weight corresponding to the i -th asset. Assume that all weights, $w_i \geq 0$, are positive and sum up to one⁹, $w_1 + w_2 + \dots + w_n = w'e = 1$, where $e' = \{1, 1, \dots, 1\}$. These conditions will be set as constraints in the optimization problem.

The expected portfolio return can be expressed by the vector of weights and expected assets returns,

$$Er_p = w_1\mu_1 + w_2\mu_2 + \dots + w_n\mu_n = \sum_{i=1}^n w_i\mu_i = w'\mu \quad (4.1)$$

The variance of the portfolio return, $\sigma_{r_p}^2$, can be computed through the variance of the stock returns and their covariances,

$$\sigma_{r_p}^2 = w_1^2\sigma_{X_1}^2 + w_2^2\sigma_{X_2}^2 + \dots + w_n^2\sigma_{X_n}^2 + \sum_{i \neq j} w_i w_j \text{cov}(X_i, X_j) \quad (4.2)$$

A more compact way of writing down the expression for $\sigma_{r_p}^2$ using matrix notation,

$$\sigma_{r_p}^2 = w' \Sigma w$$

Σ is $n \times n$ matrix of covariances,

⁸ We will use matrix notation to make the problem formulation concise.

⁹ Short-selling is not allowed.

$$\Sigma = \begin{pmatrix} \sigma_{X_1}^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{X_2}^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{X_n}^2 \end{pmatrix}$$

where σ_{ij} is the covariance between X_i and X_j . $\sigma_{ij} = cov(X_i, X_j)$ for $i \neq j$.

The mean-variance optimization problem can be restated as follows;

$$\max_w w'u \quad (4.3)$$

$$\text{Subject to: } w'e = 1$$

$$w'\Sigma w \leq R^*$$

$$w \geq 0$$

where R^* is the upper bound on the portfolio variance.

The main idea of Markowitz is that the optimal trade-off between risk and return should be the basis of financial decision making. However, the standard deviation of portfolio returns is contested as a true measure of risk, and is more considered as a measure of dispersion (ORSFB 2005). Further, mean-variance analysis is not consistent with second order stochastic dominance (RDM 2006) unless the distribution of returns is multivariate normal, which is a very restrictive assumption. As discussed in the literature, recent work has employed more “spectral” risk measures to study the optimal trade-off between risk and return. Thus, the framework of mean-variance analysis is extended into a mean-risk analysis.

4.2.2 Mean-Risk Model I: VaR as a measure of risk

The main optimization principle behind mean-risk (M-R) analysis is formulated as, from all feasible portfolios with a given upper bound on risk; find the ones that have maximum expected performance. The risk measure is denoted by $\rho(X)$ where X is a random variable describing portfolio return. Translating the formulation of M-R analysis into a linear program, we get the following general form,

$$\begin{aligned} \max_w \quad & w'u & (4.4) \\ \text{Subject to:} \quad & w'e = 1 \\ & \rho(r_p) \leq R^* \\ & w \geq 0 \end{aligned}$$

where R^* is the upper bound on the portfolio risk. In this section, the key input in (4.4), $\rho(X)$, will be substituted by the popular risk measure, VaR.

Historically, in the late 1980s, VaR was integrated by JP Morgan on a firmwide level into its risk management system called *Risk Metrics* (Risk Metrics Document, 1996), and which was later spun off into a separate company called *Risk Metrics Group*. VaR has been widely accepted as a measure of risk since 1990s. In the mid-1990s, the VaR measure was approved by banking regulators, mainly the Basel Committee on Banking Supervision, as a valid approach to calculating capital reserves needed to cover for risk exposure. Generally, regulators recommend that the capital reserve equals the VaR number multiplied by a factor between 3 and 4 thus linking the capital reserves for risk directly to the risk measure.

VaR is defined as the minimum level of loss at a given high confidence level for a predefined time horizon. The commonly used confidence levels are 95% and 99%.

Let $(1 - \epsilon)$ 100% denote the confidence level parameter of the VaR. Losses larger than the VaR occurs with probability ϵ . The probability ϵ is the tail probability. Computationally

defined by RiskMetrics Group. VaR at confidence level $(1 - \epsilon)$ 100% (tail probability ϵ) is the negative of the lower ϵ -quantile of the return distribution.

$$VaR_{\epsilon}(X) = -\inf_x \{x | P(X \leq x) \geq \epsilon\} = -F_X^{-1}(\epsilon) \quad (4.5)$$

Where $\epsilon \in (0,1)$ and $F_X^{-1}(\epsilon)$ is the inverse of the distribution function. The VaR figure can be expressed either in % or in dollar terms.

The approach of RiskMetrics Group assumes that stock and portfolio returns have a multivariate normal distribution. The $(1 - \epsilon)$ 100% VaR will appear as the negative of the ϵ % quantile of the $N(Er_p, \sigma_{r_p}^2)$. In order to calculate the portfolio VaR, we only have to calculate the expected return of r_p , Er_p (4.1), and the standard deviation of r_p , $\sigma_{r_p}^2$ (4.2). Due to the normal distribution properties, if $r_p \in N(Er_p, \sigma_{r_p}^2)$, then

$$Z_p = \frac{r_p - Er_p}{\sigma_{r_p}} \in N(0,1)$$

Finally, the $(1 - \epsilon)$ 100% portfolio VaR is represented as

$$VaR_{\epsilon}(r_p) = q_{1-\epsilon} \sigma_{r_p} - Er_p \quad (4.6)$$

where $q_{1-\epsilon}$ is the $(1 - \epsilon)$ 100% quantile¹⁰ of the standard normal distribution.

¹⁰ Note that $q_{1-\epsilon}$ is a quantity independent of the portfolio composition. It is merely a constant which can be calculated in advance. The parameters that depend on the portfolio weights are the standard deviation of portfolio returns σ_{r_p} and the expected portfolio return Er_p .

Plugging (4.6) into the general portfolio optimization problem (4.4), we get the following program,

$$\begin{aligned}
 & \max_w w'u & (4.7) \\
 & \text{Subject to: } w'e = 1 \\
 & q_{1-\epsilon}\sigma_{r_p} - Er_p \leq R^* \\
 & w \geq 0
 \end{aligned}$$

The VaR measurement is popular among financial institutions because of its relative straightforward interpretation as a single value. However, as the literature review chapter shows, VaR is not the panacea of risk measurement methodologies because it does not meet the subadditivity property of a coherent risk measure (Danielsson, 2002). For instance, the VaR of a portfolio maybe greater than the sum of the VaRs of the constituents,

$$VaR_\epsilon(X + Y) > VaR_\epsilon(X) + VaR_\epsilon(Y) \quad (4.8)$$

As explained by ADEH (1998), since VaR fails to satisfy the subadditivity property, then it fails to account for diversification which is a crucial element in portfolio management. Another downside is that VaR lacks information about the magnitude of the losses beyond than the VaR level. It only reports that losses larger than the VaR level occur with the relatively low probability ϵ , but it does not provide any information about the likely magnitude of such losses.

4.2.3 Mean-Risk Model II: CVaR as a measure of risk

The Conditional Value at Risk, CVaR, is introduced, as an improved risk measure, by RU (2002) and it provides the foundation for the analysis conducted in this paper. RU (2002) formally define CVaR¹¹ as the average of the VaRs which are larger than the VaR at ϵ . the tail probability. By construction, CVaR focuses on the losses in the tail which are larger than the corresponding VaR level. It is defined as;

¹¹ CVaR is also known as the average-value-at-risk or expected shortfall.

$$CVaR_{\epsilon}(X) = \frac{1}{\epsilon} \int_0^{\epsilon} VaR_p(X) dp \quad (4.9)$$

It is assumed that CVaR is only defined for all real-valued random variables that have a finite mean¹².

Pflug and Romisch (2007) reformulate CVaR through a minimization formula¹³,

$$CVaR_{\epsilon}(X) = \min_{\theta \in R} \left(\theta + \frac{1}{\epsilon} E(-X - \theta)_+ \right) \quad (4.10)$$

where $(x)_+$ denotes the maximum between x and zero. $(x)_+ = \max(x, 0)$; X describes the portfolio return distribution; $\theta \in R$. This formula has an important application in optimal portfolio problems based on CVaR as a risk measure.

Substituting the mathematical expectation by its formal definition, (4.7) can be polished further to derive a simpler minimization formula. If X is a random discrete variable with probability $p(x)$, then the mathematical expectation can be computed as the sample average¹⁴,

$$E(X) = \sum_i x_i p(x_i)$$

Suppose that we have a sample of observed portfolio returns and we are not aware of their distribution. Denote the observed portfolio returns by $r_1, r_2, \dots, r_n$ at time instants $t_1, t_2, \dots, t_n$.

¹² This should not be disturbing because random variables with infinite mathematical expectation have limited or no application in the field of finance. For example, if the random variable X is used for a modeling stock returns; then it is assumed that the common stock has finite expected return which is realistic.

¹³ Rockafellar and Uryasev (2002) also proves that equation (4.10) can be derived from the formal definition of CVaR (4.9).

¹⁴ This equation applies to a discrete distribution. It is used in this derivation assuming returns can be both continuously and discretely distributed.

Then,

$$\begin{aligned}
& E(-X - \theta)_+ \\
&= \sum_{i=1}^n \max(-r_i - \theta, 0) \cdot \frac{1}{n} \\
&= \frac{1}{n} \sum_{i=1}^n \max(-r_i - \theta, 0) \\
&\therefore \widehat{CVaR}_\epsilon(r) = \min_{\theta \in R} \left(\theta + \frac{1}{n} \sum_{i=1}^n \max(-r_i - \theta, 0) \right) \quad (4.8)
\end{aligned}$$

The minimization formula¹⁵ in (4.8) is more appealing because it can be calculated through the methods of linear programming. It can be restated as a linear optimization problem by introducing auxiliary variables $d_1, \dots, d_n$, one for each observation in the sample,

Let the value of θ be θ^* .

$$\text{If } -r_k - \theta^* < 0 \Rightarrow d_k = 0$$

$$\text{If } -r_k - \theta^* \geq 0 \Rightarrow d_k = -r_k - \theta^*$$

$$\therefore \theta + \frac{1}{n} \sum_{i=1}^n \max(-r_i - \theta, 0) = \theta + \frac{1}{n} \sum_{k=1}^n d_k$$

Finally, CVaR is computed using the following linear program:

$$\widehat{CVaR}_\epsilon(r) = \min_{\theta, d} \left(\theta + \frac{1}{n} \sum_{k=1}^n d_k \right) \quad (4.9)$$

$$\text{Subject to } -r_k - \theta \leq d_k, \quad k = 1, n$$

$$d_k \geq 0, \quad k = 1, n$$

$$\theta \in R$$

¹⁵ The "hat" above CVaR denotes that the number calculated by (4.8) is an estimate of the true value because it is based on a sample. This is standard notation in statistics.

Denote the scenarios for the assets returns by $r^1, r^2, \dots, r^n$ where r^j is a vector of observations. $r^j = (r_1^j, r_2^j, \dots, r_n^j)$, which contains the returns of all assets in a given time instant denoted by the index j . Thus the observations can be arranged in a $k \times n$ matrix,

$$H = \begin{bmatrix} r_1^1 & r_2^1 & \dots & r_n^1 \\ r_1^2 & r_2^2 & \dots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^k & r_2^k & \dots & r_n^k \end{bmatrix},$$

where the rows contain returns of all assets in a given moment and the columns contain all observations for one asset in the entire time period. Therefore, H contains historical data, k is the number of time instants observed and n is the number of assets observed.

Assuming we are currently dealing with just one portfolio, the observed returns of this portfolio with composition w are $r^1w, r^2w, \dots, r^kw$, or simply as the product Hw . Problem (4.9) can be restated employing matrix notation,

$$CVaR_\epsilon(Hw) = \min_{\theta, d} \left(\theta + \frac{1}{k\epsilon} d'e \right) \quad (4.10)$$

$$\text{Subject to } -Hw - \theta e \leq d$$

$$d \geq 0, \theta \in R$$

where $d' = (d_1, \dots, d_n)$ is the vector of auxiliary variables, $e = (1, \dots, 1)$, $e \in R^k$ is the vector of ones, and $\theta \in R$ is the additional parameter coming from the minimization formula given in (4.9). In (4.11), the first inequality concerns vectors and can be extensively interpreted as,

$$-Hw - \theta e \leq d \Rightarrow \begin{cases} -r^1w - \theta \leq d_1 \\ -r^2w - \theta \leq d_2 \\ \dots \\ -r^kw - \theta \leq d_k \end{cases}$$

The optimization problem in (4.10) calculates the CVaR of a portfolio with a given composition w . But we're not merely interested in finding CVaR. Our goal is to minimize the

portfolio CVaR by changing the composition w . Plugging (4.10) into the general portfolio optimization problem (4.4)¹⁶ and substituting $\rho(r_p)$ by $CVaR_\epsilon(Hw)$, we get the following program,

$$\begin{aligned}
 & \max_{w, \theta, d} w'u & (4.11) \\
 \text{Subject to: } & -Hw - \theta e \leq d \\
 & w'e = 1 \\
 & \theta + \frac{1}{k\epsilon} d'e \leq C^* \\
 & w \geq 0, d \geq 0, \theta \in R
 \end{aligned}$$

where C^* is a constant and represents the upper bound on the constraint.

Problem (4.12) has a simple structure whereby the objective function is linear and all constraints are linear equalities and inequalities. Hence, it can be easily solved using linear programming software.

Appendix A provides a clear example which highlights the difference between measures of risk and measures of uncertainty. It also presents different numerical illustrations to explain how each measure of risk (variance, VaR and CVaR) can be applied to a dataset. In addition, Appendix A presents how the mean-CVaR model produced in this chapter is applicable on a random set of returns. It further illustrates how each parameter of the model can be easily obtained and plugged into the optimization problem.

Appendix B provides examples of how the CVaR measure satisfies the ADEH (1998) four coherent risk measure properties unlike the variance and VaR.

¹⁶ The solutions of (4.4) and (4.10) with $\rho(r_p) = CVaR_\epsilon(Hw)$ coincide.

Chapter 5: Empirical Results

5.2 Mean-Variance Framework

The traditional approach used in portfolio optimization is the mean-variance framework, where returns are assumed to be normally distributed. This section uses variance of returns as a measure of risk. To solve the portfolio optimization problem, we follow the methodology introduced in section 4.2.1 for the mean-variance framework. In order to construct portfolios from the selected eight financial assets, we first calculate the expected return and standard deviation from different possible combinations. The solution that plots all alternative possibilities is shown in figure 4 below.

Figure 4: Mean-variance efficient frontier

Any point falling below or on the curve is attainable by a certain combination of assets, whereas points outside this region are not attainable. Since the investor regards higher expected

return as 'good' and a higher standard deviation of returns as 'bad', the objective is to achieve the highest possible expected return for any given level of risk. Minimizing the level of risk and maximizing the level of return implies that the investor will choose some point along the upper edge of the feasible region above the global minimum variance portfolio¹⁷, known as the efficient frontier. The efficient frontier shows the different risk-return combinations that are available to the investor. As we move upward along the efficient frontier, the composed portfolios become riskier but provide higher returns. A risk averse investor will choose a point on the efficient frontier with a low risk and a low expected return, and an investor who is less risk averse or more risk tolerant will choose a point on the efficient frontier with a higher risk and a higher expected return.

A further examination of the efficient frontier shows the distribution of weights across the different risk-return combinations which is presented in Table 4, and their graphical plot appears in Figure 5. The idea is to show/plot how the composition of assets varies with the riskiness of the entire portfolio.

Table 4: Mean-variance composition of different portfolios

Return	Std	Weights							
		EW	EE	EUS	EFE	EEM	BUS	BEU	BEM
0.0037	0.8525	0	0	0.1007	0	0.0008	0.8985	0	0
0.0558	0.9623	0	0	0	0	0.1021	0.847	0.0509	0
0.1079	1.2479	0	0	0	0	0.158	0.6414	0.2006	0
0.16	1.6407	0	0	0	0	0.2138	0.4358	0.3504	0
0.2121	2.081	0	0	0	0	0.2697	0.2302	0.5001	0
0.2642	2.5442	0	0	0	0	0.3256	0.0246	0.6498	0
0.3163	3.0907	0	0	0	0	0.4837	0	0.5163	0
0.3684	3.7746	0	0	0	0	0.6558	0	0.3442	0
0.4205	4.5351	0	0	0	0	0.8279	0	0.1721	0
0.4726	5.3394	0	0	0	0	1	0	0	0

¹⁷ The minimum variance portfolio is the portfolio of risky assets with the least variance of standard deviation. Graphically, it is the point at the kink of the efficient frontier.

Figure 5: Mean-Variance portfolio composition

Table 5 and Figure 5 show that the minimum risk portfolio has a standard deviation of 0.8049% and is mostly composed of the US bond index. As portfolio risk increase, the percentage of funds that are allocated to this bond index decreases. In parallel, the percentage of equity composition increases.

The mean-variance approach provides a framework for choosing among risky alternatives. However, the underlying assumption is that returns are normally distributed. In our case, however, and probably in most cases as the recent financial crisis proved, this assumption is violated. As a result, the standard deviation is likely to be an unsatisfactorily measure of risk in the mean-risk framework.

5.3 Mean-VaR Framework

5.3.1 Analysis of VaR

Given that the normality assumption is violated, we seek an alternative measure of risk in the mean-risk framework, or the VaR risk measure. Unlike the variance measure, VaR is valid in the face of a more general distribution. Using VaR, the return distribution is less restricted, and it focuses on the tail of the distribution. To illustrate the characteristics of VaR, this section will make use of the returns of one of the indices or BEU before using the mean-VaR framework analysis.

A graphical illustration of VaR for the BEU index is represented in Figure 6.

Figure 6: VaR of BEU plotted geographically

The figure shows a probability density function of BEU returns over a 10-day holding period. Using a confidence level of 95%, the VaR is the point on the x-axis that cuts off the top 5% of the area from the bottom 5% of tail observations. In the case of BEU returns, the corresponding x-axis value or 5th percentile is -3.5279%. This implies that the VaR for 10-day holding period BEU index is a loss of 3.5279%. Stated differently, over a period of 10 days, the probability of losing 3.5279 % of the original investment is less than or equal to 5%.

In interpreting the VaR numbers, one must consider the confidence level and the holding period. Without them, the interpretation of VaR numbers is meaningless. Two companies holding identical portfolios will come up with different VaR estimates if they make different choices of the confidence levels and the holding periods. Obviously, the loss that is suffered with a probability of only 1% will be a larger amount than the loss that is suffered with a probability of 5%¹⁸. The choice of the holding period can have an even larger impact because the VaR computed using a t -day holding period is approximately $\sqrt{t}$ times as large as the VaR computed using a 1-day holding period (Grootveld & Hallerbach, 2004). Different VaR values imply that banks need to allocate different provision amounts for contingent losses.

Figures 7 and 8 show the variation of VaR versus the confidence level and holding period respectively. Using the BEU index, VaR increases when the confidence level rises. For the BEU index, the VaR also rises with the confidence level at an increasing rate.

¹⁸ Under some assumptions, $VaR_{1\%} \approx 1.414 VaR_{5\%}$ (Grootveld & Hallerbach, 2004).

Figure 7: VaR versus confidence level

Figure 8 below plots 95% VaR against a holding period that varies from 1 day to 100 days. The positive association between VaR and the holding period increases at a decreasing rate and may eventually be reversed.

Figure 8: VaR versus the holding period (in days)

Since the use of VaR involves two parameters – the confidence level and the holding period – related assumptions are needed to solve the optimization problem. Most institutions generally prefer to set confidence levels in the region of 95% to 99% (Basel, 2007). The usual holding period is one day or one month; other institutions can also operate on other holding periods (e.g. one quarter or more), depending on their investment and/or reporting horizons (Basel, 2007). In the present analysis, we follow the Bank of International Settlements (BIS) capital adequacy rules which stipulate that banks should operate with a holding period of two weeks (or 10 business days).

5.3.2 Mean-VaR portfolio optimization

After solving the mean-VaR linear program as outlined in section 4.2.2, Figure 9 displays the mean-VaR efficient frontier; Table 5 lists possible optimal portfolios; and Figure 10 plots the distribution of funds across the employed indices graphically.

Figure 9: Mean-VaR efficient frontier

Table 5: Mean-VaR composition of different portfolios

Return	VaR	Weights							
		EW	EE	EUS	EFE	EEM	BUS	BEU	BEM
0.0184	1.4	0	0	0.0682	0	0.0350	0.8968	0	0
0.0503	1.5	0	0	0.0069	0	0.0942	0.8600	0.0389	0
0.0680	1.6	0	0	0	0	0.1152	0.7990	0.0859	0
0.0814	1.7	0	0	0	0	0.1295	0.7461	0.1244	0
0.0929	1.8	0	0	0	0	0.1419	0.7004	0.1576	0
0.1035	1.9	0	0	0	0	0.1532	0.6590	0.1878	0
0.1133	2	0	0	0	0	0.1637	0.6203	0.2160	0
0.1568	2.5	0	0	0	0	0.2105	0.4483	0.3413	0
0.1962	3	0	0	0	0	0.2526	0.2931	0.4543	0
0.2336	3.5	0	0	0	0	0.2927	0.1455	0.5618	0
0.2699	4	0	0	0	0	0.3317	0.0020	0.6662	0
0.3016	4.5	0	0	0	0	0.4352	0	0.5648	0
0.3283	5	0	0	0	0	0.5235	0	0.4765	0
0.3527	5.5	0	0	0	0	0.6039	0	0.3961	0
0.3756	6	0	0	0	0	0.6796	0	0.3204	0
0.3976	6.5	0	0	0	0	0.7521	0	0.2479	0
0.4189	7	0	0	0	0	0.8225	0	0.1775	0
0.4397	7.5	0	0	0	0	0.8912	0	0.1088	0
0.4601	8	0	0	0	0	0.9587	0	0.0413	0
0.4722	8.3	0	0	0	0	0.9987	0	0.0013	0
0.4726	8.31	0	0	0	0	1	0	0	0

Figure 10: Mean-VaR portfolio composition

Table 5 and Figures 5 and 6 show that the minimum risk portfolio has a VaR of 1.4% and, similar to the mean-variance framework, is mostly composed of the US bond index. Likewise, as portfolio risk increase, the percentage of funds that are allocated to this bond index decreases, and the percentage of equity composition rises. While the minimum (maximum) risk attached to the portfolio using VaR of 1.4% (8.31%) is larger than the minimum (maximum) reported standard deviation in Figure 5 of 0.85% (5.34%), the variance and VaR are not directly comparable. Still, this difference might point to a more restrained risk measure using VaR as compared to the variance. These suggest that the use of VaR may be preferred over mean-variance analysis because it provides a higher probability of loss range and have a more conservative estimate of losses.

As discussed in the previous chapters, VaR is a common, holistic, and probabilistic risk measure, but it has serious limitations as a risk measure. One important distinctive limitation of VaR is that it only tells us the most we can lose if a tail does not occur (e.g. it tells us the most

we can lose 95% of the time); if a tail event does occur, we can expect to lose more than the VaR. The VaR itself gives no indication of how much that might be. Another limitation is the violation of the subadditivity property when normality is not respected. However, these properties are satisfied using CVaR measure of risk.

5.4 Mean-CVaR Framework

The previous chapters established the superiority of CVaR as a coherent measure of risk because it does not impose normality on data distributions and satisfied all of the properties of ADEH (1998). To illustrate CVaR, the BEU financial asset is also used, and the graphical representation appears in Figure 11.

Figure 11: VaR and CVaR

Using equation 4.9 which is reproduced below, CVaR is computed as the average of VaRs over the region bound by $[0, \varepsilon]$.

$$CVaR_{\epsilon}(X) = \frac{1}{\epsilon} \int_0^{\epsilon} VaR_p(X) dp$$

Using a specialized software (MatLab), both VaR and CVaR are calculated. With a 5% confidence level and a 10-day holding period, the VaR for the BEU index is found to be 3.5279% (same as in section 5.3.1) and the CVaR is 4.4649%. This means that over a period of 10 days, the probability of losing 4.4649% of the initial investment is less than or equal to 5% while the corresponding loss using VaR is only 3.5279%. In order to validate the 4.4649% value of CVaR, Table 6 shows the average tail VaRs for different confidence levels¹⁹.

Table 6: Estimating CVaR as a weighted average of tail VaRs (in %)

Confidence Level	Tail VaR
95.0%	3.5279
95.5%	3.6415
96.0%	3.7658
96.5%	3.9035
97.0%	4.0583
97.5%	4.2363
98.0%	4.4471
98.5%	4.7087
99.0%	5.06
99.5%	5.6208
Average of tail VaRs	4.2969
Computed CVaR	4.4649

The computed average of the tail VaRs is found to be 4.2969% which is comparable to the actual CVaR of 4.4649% reported. While VaR tells us nothing about losses higher than the VaR itself, CVaR tells us what to expect in bad states.

¹⁹ Similar to VaR, CVaR is contingent on the confidence level and holding period.

We solve the mean-CVaR linear program following the procedure detailed in 4.2.3.

Figure 12 displays the mean-CVaR efficient frontier curve; Table 7 lists possible optimal portfolios; and Figure 13 plots the distribution of funds across the employed indices graphically.

Figure 12: Mean-CVaR efficient frontier

Table 7: Mean-CVaR composition of different portfolios

CVaR	Return	Weights							
		EW	EE	EUS	EFE	EEM	BUS	BEU	BEM
1.82	0.03	0.0000	0.0000	0.0611	0.0000	0.0574	0.7168	0.0947	0.0700
1.99	0.04	0.0000	0.0000	0.0435	0.0000	0.0618	0.6819	0.1428	0.0700
2.19	0.05	0.0000	0.0000	0.0259	0.0000	0.0663	0.6470	0.1908	0.0700
2.53	0.05	0.0000	0.0000	0.0083	0.0000	0.0707	0.6121	0.2389	0.0700
2.70	0.04	0.0000	0.0000	0.0182	0.0000	0.1716	0.4022	0.3380	0.0700
2.82	0.06	0.0000	0.0000	0.0145	0.0000	0.1619	0.4811	0.2725	0.0700
3.01	0.06	0.0000	0.0000	0.0576	0.0000	0.2222	0.4047	0.3155	0.0000
3.30	0.06	0.0000	0.0000	0.0218	0.0000	0.2302	0.4047	0.3432	0.0000
3.51	0.08	0.0000	0.0000	0.0191	0.0000	0.2588	0.3608	0.3613	0.0000
3.80	0.09	0.0000	0.0000	0.0164	0.0000	0.2873	0.3168	0.3794	0.0000
4.25	0.11	0.0000	0.0000	0.0343	0.3061	0.0000	0.2929	0.3666	0.0000
4.29	0.12	0.0000	0.0000	0.0414	0.0960	0.0000	0.6132	0.2494	0.0000
4.52	0.13	0.0000	0.2157	0.0236	0.4185	0.0000	0.1404	0.2019	0.0000
4.99	0.12	0.0000	0.1108	0.0272	0.2816	0.0000	0.4436	0.1369	0.0000
5.50	0.14	0.0000	0.2860	0.0442	0.3822	0.0000	0.2158	0.0719	0.0000
6.78	0.14	0.0000	0.4821	0.0270	0.4453	0.0000	0.0387	0.0069	0.0000
7.53	0.15	0.0000	0.4149	0.0276	0.3922	0.1538	0.0115	0.0000	0.0000
8.03	0.15	0.0000	0.4296	0.0272	0.4100	0.1224	0.0107	0.0000	0.0000
10.04	0.16	0.0000	0.4526	0.0000	0.4590	0.0785	0.0099	0.0000	0.0000
10.53	0.17	0.0000	0.4108	0.0000	0.4354	0.1446	0.0091	0.0000	0.0000
10.95	0.19	0.0000	0.0401	0.0000	0.3470	0.6046	0.0084	0.0000	0.0000
11.15	0.19	0.0000	0.0363	0.0000	0.3526	0.5992	0.0076	0.0043	0.0000
11.77	0.24	0.0000	0.0369	0.0000	0.3561	0.5959	0.0068	0.0043	0.0000
12.39	0.26	0.0000	0.0374	0.0000	0.3596	0.5926	0.0060	0.0043	0.0000
13.02	0.30	0.0000	0.0380	0.0000	0.3642	0.5883	0.0052	0.0043	0.0000
13.64	0.34	0.0000	0.0385	0.0000	0.3683	0.5844	0.0044	0.0043	0.0000
14.26	0.37	0.0000	0.0391	0.0000	0.3724	0.5841	0.0000	0.0044	0.0000
14.88	0.41	0.0000	0.0396	0.0000	0.3766	0.5838	0.0000	0.0000	0.0000
15.50	0.44	0.0000	0.0402	0.0000	0.3807	0.5791	0.0000	0.0000	0.0000
16.12	0.46	0.0000	0.0408	0.0000	0.3848	0.5744	0.0000	0.0000	0.0000
16.24	0.47	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000

Figure 13: Mean-CVaR portfolio composition

Table 7 and figures 12 and 13 show that the minimum risk portfolio has a CVaR of 1.8223% and, similar to the previous approaches, is mostly composed of the US bond index. Likewise, as portfolio risk increase, the percentage of funds that are allocated to this bond index decreases. In parallel, the percentage of equity composition increases; however, it is not until risk becomes very high for the mean-CVaR approach to suggest eliminating BUS from the portfolio.

5.5 Comparison of Mean-Variance, Mean-VaR and Mean-CVaR Frameworks

Having implemented the three different frameworks to solve the portfolio optimization problems, we now turn to a comparative assessment of the three methodologies. The first observation is related to the risk-return relationship. The three frameworks support the basic positive relationship between risk and return. Unlike the efficient frontier that corresponds to mean-variance which we eliminate the portion that lies below the global minimum variance

portfolio, the positive risk-return relationship is consistent across all possible weight compositions when using mean-VaR and mean-CVaR analysis.

Second, comparing the minimum risk portfolio of the three approaches, we deduce that CVaR is the most conservative measure of risk because it outputs the highest minimum (maximum) risk portfolio 1.82% (16.24%) compared to 1.4% (8.31%) for using VaR.

Third, we can notice that EW equity index is rarely included using all three frameworks. This result may be explained by the fact that EW is highly correlated with the other indices and the three frameworks respect diversification to a certain extent.

Fourth, the graphs of the distributions of weights show different patterns using CVaR as compared to the variance and VaR. The variance and VaR compositions of weights display definite trends of indices unlike the CVaR's graph that shows peaks in recommended weights. It is not clear at this stage whether this effect is arising from the size of the dataset or the fact that CVaR takes outliers heavily into consideration.

Fifth, the composition of bonds and equity varies as risk increases using each of the three measures of risk. Risky portfolios generated using the variance and VaR are mainly composed of equity indices. However, the use of CVaR maintains bond indices in high risk portfolios suggesting greater diversification potential. Furthermore, examining the portfolio compositions using the three measures of risk provides some interesting insights. In the region of higher expected returns, equity indices from different risky geographic locations such as the equity Emerging Markets and equity Far East are included in the portfolio. However, while variance and VaR suggest using the equity index of highest risk (EEM), CVaR recommends incorporating less risky equity indices such as EFE and EE.

Sixth, better diversified portfolios are likely to provide the investor with higher return and lower risk. It seems that CVaR provides the most diversified portfolio because at any risk level, the portfolio is divided among at least 5 indices; while using variance or VaR, the portfolio is generally composed of at most three indices.

5.6 Conclusion

This chapter has numerically implemented three alternative risk measurement frameworks to solve the portfolio optimization problem. The first, mean-variance framework, is adequate in the face of a limited set of assumptions (i.e. if we assume returns to be normally distributed). VaR is proposed as a better risk measure that can be used under less restrictive conditions. However, the VaR has serious flaws as a measure of risk mainly because it does not consider extreme cases of unlikely returns.

A more 'proper' or coherent risk measure is CVaR which is valid for all possible shapes of return distributions and is manifestly superior to VaR. Comparing the three approaches, CVaR is a more conservative measure than VaR because it provides higher expectations of probability losses in case of rare extreme events. One example of such events is the outburst of the financial crisis.

Chapter 6: Conclusion

This thesis examines mean-risk portfolio optimization problems using different measures of risk (variance, VaR and CVaR), and starting with the classical mean-variance analysis developed by Harry Markowitz. There are, however, several potentially significant caveats in mean-variance analysis. Portfolio theory presupposes that returns are normally distributed. Besides, variance is merely a proxy of risk; in fact, it is a measure of dispersion and volatility. A risk measure should provide information on the probability of loss.

VaR as a risk measure has steadily become prominent as a risk management tool, especially after the Basel Capital Accord recommended its use in estimating market risk. The focal point of the VaR concept is its estimation of probability distribution of returns and probability losses. However, after the recent chain of financial outbreaks, statisticians, financial managers and investors realized that it is about time to reconsider the consistency and usefulness of the conventional risk measures. We argue that VaR is a non-coherent risk measure because it violates the subadditivity property, and we show that it fails to account for risk above the VaR level. These limitations of the traditional risk measures are critical, and a proper and coherent risk measure or CVaR is needed to assess tail probability losses.

CVaR satisfies all four desired properties of a spectral risk measure (monotonicity, subadditivity, positive homogeneity and translation invariance) as set by ADEH (1998). We show that CVaR detects rare unexpected events and provides investors an idea about the magnitude of the possible losses in case such an event takes place.

Theoretical literature has established the superiority of CVaR compared to traditional risk measures using different portfolio optimization modeling techniques. In order to validate the superiority of CVaR over conventional risk measures (variance and VaR), we develop of a simple mean-CVaR linear program that is easy to estimate and implement in practice. We then collect recent data on eight indices over the period starting on January, 1st, 2003 and ending on June, 8th, 2009. When comparing the efficient frontiers and portfolio weight compositions of the three approaches, we find that the mean-CVaR portfolio optimization model produces more conservative estimates of the expected probabilities of losses both in “good” and “bad” states.

We also find that the Mean-CVaR framework recommends including a more diverse set of financial assets as compared to those recommended when using the variance or VaR as a measure of risk. The greater diversification effect using CVaR implies greater potential for risk reduction as opposed to solving the portfolio optimization problem using traditional measures of risk.

The empirical results also show that CVaR highlights outliers as opposed to the variance and VaR. Therefore, it is recommended that future proposals to adequately measure market risk (similar to those developed by the Basel Committee) consider CVaR as a serious alternative to the previously suggested VaR. Following the mean-CVaR model, financial institutions’ ability to withstand unexpected unfavorable outcomes or recessions may be improved. In light of the financial crisis, we need to make sure that financial institutions are not just doing well in good times but in bad times as well.

To conclude, we note that portfolio optimization is not purely a quantitative problem. At the heart of the subject is the notion of good risk management practice, and above anything else

this requires a good sense of judgment, an awareness of things that can go wrong, and an appreciation of market history. So while portfolio optimization involves quantitative methods, the subject itself rests on a foundation that is qualitative.

Future research should consider more interaction between the areas of operations research and finance. The mean-CVaR portfolio optimization model can be extended and used in several applications such as brokerage firms, hedge funds, pension funds, mutual funds, insurance industry etc... Additional work can also examine the implications of implementing CVaR as a risk measure.

In many ways, portfolio optimization is like engineering: it uses sophisticated tools, but context and judgment are everything. The key lesson in all of this is that it is much less important how we estimate risk measures; it is much more important that we estimate the right risk measure.

References

- Alderweireld, T., Garcia, J. and Léonard, L. (2006). "A practical operational risk scenario analysis quantification". *Risk*, Vol. 19 (2): 93-95.
- Alexander G. J. and Baptista A.M. (2004). "A Comparison of VaR and CVaR Constraints on Portfolio Selection with the Mean-Variance Model". *Journal of Management Science* 50 (9) : 1261–1273.
- Ammann, M. and Reich, C. (2001): "Value-at-Risk for Nonlinear Financial Instruments", *Financial Markets and Portfolio Management*, 15 (3):363-378.
- Artzner, P., Delbaen F., Eber J. M., and Heath D. (1998). "Coherent measures of risk," *Mathematical Finance* 6(3): 203 – 228.
- Bank for International Settlements. (2004). *Triennial Central Bank Survey of Foreign Exchange and Derivatives Market Activity in April 2004: Preliminary Global Results*. Basel: Bank for International Settlements. September.
- Basel Committee on Banking Supervision. (2007). *International Convergence of Capital Measurements and Capital Standards: A Revised Framework*. Basel: Bank for International Settlements.
- Beliakov G. and Bagirov A. (2006). "Non-smooth optimization methods for computation of the Conditional Value-at-risk and portfolio optimization," *Optimization* 55(5): 459 - 479.
- Billio, M. and Pelizzon, L. (2000): "Value-at-Risk: a Multivariate Switching Regime Approach", *Journal of Empirical Finance*, 7 (5):531-554.
- Birge J.R. and Louveaux F. (1997). *Introduction to Stochastic Programming*. Springer Series in Operations Research. Springer-Verlag, Berlin/Heidelberg.
- Boyle, P. and Ding B. (2006). *Numerical Methods in Finance*, Chapter 11: Portfolio selection with skewness. Springer. In Hatem Ben-Ameur and Michhle Breton, editors.
- Cox S. H., Lin Y., Tian R. and Zuluaga L. F. (2008). "Portfolio Risk Management with CVaR-like Constraints," Financial Management Association conference (September 8), Washington DC, USA.
- Danielsson, J. (2002). The emperor has no clothes: Limits to risk modelling. *Journal of Banking & Finance*, 26: 1273–1296.
- Duffie, D. and Pan J. (1997). "An Overview of Value-at-Risk", *Journal of Derivatives*, 4 (3) : 7-49.
- Einhorn D. (2008). Private Profits and Socialized Risk. *GARP Risk Review* (June/July - 2008).
- Gini, C. (1965). La dissomiglianza. *Metron* 24: 309-331.

- Gordon J. Alexander G. J. and Baptista A. M. (2004). "A Comparison of VaR and CVaR Constraints on Portfolio Selection with the Mean-Variance Model," *Management Science*, **50**(9): 1261–1273.
- Grootveld, H. and Hallerbach, W. G. (2004). "Upgrading value-at-risk from diagnostic metric to decision variable: a wise thing to do?", pp: 33 – 50 in G. Szego (ed.) *Risk Measures for the 21st Century*. Chichester: John Wiley and Sons, Ltd.
- Holton, Glyn A. (2004). "Defining risk," *Financial Analysts Journal* **60**(6): 19 – 25.
- Hoppe R. (1998). VaR and the unreal world. *Risk* **11**: 45 – 50.
- Ibragimov, R. (2004). "Portfolio diversification and value at risk under thick-tailedness," *Harvard Institute of Economic Research Discussion Paper* #2086.
- Knight F. (1921). *Risk, Uncertainty and Profit*. Boston, MA: Hart, Schaffner & Marx; Houghton Mifflin Co.
- Konno H., Waki H. and Yuuk A. (2002). "Portfolio Optimization under Lower Partial Risk Measures," *Asia-Pacific Financial Markets* **9**: 127–140.
- Krokhmal P., Palmquist J., and Uryasev S. (2002). "Portfolio optimization with conditional value-at-risk objective and constraints," *The Journal of Risk* **4**(2): 11-27.
- Künzi-Bay A. and Mayer J. (2006). "Computational aspects of minimizing conditional value-at-risk", *Computational Management Science* **3**: 3–27.
- Maracine V. (2002). "Interest – Rate Risk Management Using Income Gap Analysis". *Economy Informatics*, **1**: 124 – 126.
- Marshall, C. and Michael S. (1997). "Value at Risk: Implementing A Risk Management Standard", *Journal of Derivatives*, **4**: 91-110.
- Mulvey J. M. and Erkan H. G. (2006). "Applying CVaR for decentralized risk management of financial companies," *Journal of Banking & Finance* **30**: 627–644
- Nocera J. (2009). *Risk Mismanagement*, The New York Times Magazine (January 4) extracted from: http://www.nytimes.com/2009/01/04/magazine/04risk-t.html?_r=3&pagewanted=1
- Ortobelli S., Rachev S. T., Stoyanov S., Fabozzi F. J. and Biglovae A. (2005). "The proper use of risk measures in portfolio theory," *International Journal of Theoretical and Applied Finance (IJTAF)*, **8**(8): 1107-1133.
- Ortobelli, S., Rachev S. T., Shalit H. and Fabozzi F. J. (2006). "Risk probability functionals and probability metrics applied to portfolio theory", *forthcoming Latin American Journal of Probability and Mathematical Statistics*.
- Pflug G. and Romisch W. (2007). *Modeling, Measuring and Managing Risk*. WorldScientific Publishing Co., Singapore.

- Quaranta, A. G. and Zaffaroni A. (2008). Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance*, **32**: 2046–2056.
- Rachev S., Ortobelli S., Stoyanov S., Fabozzi F. J., and Biglovae A. (2006). “Desirable properties of an ideal risk measure in portfolio theory,” *International Journal of Theoretical and Applied Finance (IJTAF)* **11(1)**: 19-54.
- Riskmetrics Technical Document. 4th edn.* J.P. Morgan/Reuters, New York (1996)
- Rockafellar R.T. and S. Uryasev. (2000). "Optimization of Conditional Value-at-Risk," *The Journal of Risk* **2(3)**. (www.isg.nyu.edu/uryasev/cvar.pdf)
- Rockafellar R.T. and S. Uryasev. (2001). "Conditional Value-at-Risk for General Loss Distributions," *Journal of Banking and Finance* **26**:1443–1471. (www.isg.nyu.edu/uryasev/cvar2.pdf)
- Roman, D., Darby-Dowman, K.H. and Mitra, G. (2006). “Portfolio Construction Based on Stochastic Dominance and Target Return Distributions,” *Mathematical Programming Series B* **108 (2-3)** : 541-569.
- Romero R. and Laengle S. (2007). "Implementation of Conditional Value-at-Risk (CVaR): The Case of the Chilean Pension Funds," *working paper*.
- Sakalauskas V. and Kriksciuniene D. (2006). Short-Term Investment Risk Measurement using VaR and CVaR. Lecture Notes in Computer Science, Springer-Verlag, numb 3994, pages 316-323, Germany: Berlin.
- Shirletov, A. (2007). "Some Aspects of the Practical Problems Arising upon the Application of Value-at-Risk", *Economic Alternatives*, **1**: 3 - 7.
- Soyster, A.L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operation Research*, 1154–1157.
- Szego, G. (2002). Measures of Risk. *Journal of Banking & Finance*, **26**: 1253–1272.
- Taleb N. (1997). Against VaR. *Derivatives Strategy*, **2**: 21 – 26.
- Taleb N. (2007). *The Black Swan: The Impact of the Highly Improbable*. Random House.
- Uryasev, S. Conditional Value-at-Risk: Optimization Algorithms and Applications. Financial Engineering News, No. 14, February, 2000 (www.isg.nyu.edu/uryasev/pubs.html#1)

Appendix A: Numerical Illustrations

A.1 Risk versus Uncertainty

Here is an example of how a deviation measure possesses several of the characteristics of a risk measure, but it is a proxy of risk. Suppose that we have initially in our portfolio a common stock, X , with a current market value of \$95 and an expected return of 1% in a month (i.e. annual return is 12%). Let us choose one particular deviation measure, standard deviation for example. Assume that $\sigma_{r_x} = 20\%$ where r_x stands for the portfolio return. Assume that we add to our portfolio a risk-free government bond, B , worth \$95 with a face value of \$100 and a one-month maturity. The annual return on the bond equals $r_B = (\$100 - \$95)/\$95 = \$5/\$95 = 5.26\%$ and is constant and nonrandom. Our portfolio now consists of equal shares (equal weights) of the common stock and the bond. The portfolio return equals $r_p = \frac{r_x}{2} + \frac{r_B}{2} = 8.63\%$. Using the translation invariance axiom²⁰ of deviation measures, we obtain $\sigma_{r_p} = \sigma_{r_x + r_p} = \sigma\left(\frac{r_x}{2} + \frac{r_B}{2}\right) = \sigma\left(\frac{r_x}{2}\right) = \frac{1}{2} \cdot \sigma(r_x) = \frac{20}{2} = 10\%$. The deviation measure is informing us that the uncertainty of the portfolio return r_p decreases twice since the share of the risky stock decreases twice. However, please note the following: $r_x = 12\%$ and $r_p = 8.63\%$; $\sigma_x = 20\%$ and $\sigma_p = 10\%$.

²⁰ Rockafellar et al. (2006) provide an axiomatic description of dispersion measures. There are 5 basic axioms where $D(X)$ is denoted by the dispersion measure of the random variable X

- Positive shift: $D(X+C) \leq D(X)$ for all X and constants $C \geq 0$
- Positive homogeneity: $D(0) = 0$ and $D(\lambda x) = \lambda D(x)$ for all X and all $\lambda > 0$.
- Positivity: $D(X) \geq 0$ for all X , with $D(X) > 0$ for nonconstant X .
- Subadditivity: $D(X+Y) \leq D(X) + D(Y)$ for all X and Y
- Translation invariance: $D(X+C) = D(X)$ for all X and constants $C \in \mathbb{R}$

Half of the new portfolio earns a sure profit of 5.26%. Hence the return improved by $71.92\% = \frac{12-8.63-5.26}{12}$ whereas the risk decreases by 50%. The return gain exceeds the reduction in risk.

Therefore, in contrast to what the deviation measure indicates, the risk of r_p in fact decreases more than twice if compared to r_x . This effect is due to the translation invariance, which makes the deviation measure insensitive to nonrandom profit and to diversification.

If we use VaR instead of the standard deviation as a measure of risk and perform similar computations, we get an intuitive realistic answer of the value of risk. Assume that we calculate $VaR_\epsilon(r_x) = 12\%$. Recall that VaR respects the monotonicity, positive homogeneity and translation invariance coherent properties, but it fails to respect the subadditivity property. So, referring to the positive homogeneity and translation invariance properties, we get;

$$VaR_\epsilon(r_p) = VaR_\epsilon\left(\frac{r_x}{2} + \frac{r_B}{2}\right) = VaR_\epsilon\left(\frac{r_x}{2}\right) - \frac{r_B}{2} = \frac{1}{2} VaR_\epsilon(r_x) - \frac{r_B}{2} \approx 3.37\%$$

which is by far less than 12%, half of the initial risk obtained when using the standard deviation uncertainty measure. $r_x = 12\%$ and $r_p = 8.63\%$; $\sigma_x = 20\%$ and $\sigma_p = 3.37\%$. Similarly, half of the new portfolio earns a sure profit of 5.26%. Hence the return improved by $71.92\% = \frac{12-8.63-5.26}{12}$ and the risk decreases by 83.15%. The return gain and the reduction in risk are balanced in this case.

A.2 VaR Example

We demonstrate how the models of VaR and CVaR are applied in the following example. Suppose that the sample returns of a portfolio of 7 assets is -1.37%, -0.98%, -0.38%, -0.26%, 0.19%, 0.31%, 1.91% and our goal is to calculate the portfolio VaR and CVaR at the 30% tail probability. In this case, the sample contains 7 observations and they are equally probable.

Recall (4.6):

$$VaR_{\epsilon}(r_p) = q_{1-\epsilon}\sigma_{r_p} - Er_p$$

$$(4.1) \Rightarrow Er_p = \frac{1}{7}(-1.37\%) + \dots + \frac{1}{7}(-1.91\%) = -0.082\%$$

$$(4.2) \Rightarrow \sigma_{r_p}^2 = w_1^2\sigma_{X_1}^2 + w_2^2\sigma_{X_2}^2 + \dots + w_n^2\sigma_{X_n}^2 + \sum_{i \neq j} w_i w_j cov(X_i, X_j) = 1.39$$

where the covariance matrix is:

	A1	A2	A3	A4	A5	A6	A7
A1	26.60	9.27	14.18	7.25	3.00	1.63	8.26
A2	9.27	15.32	23.51	2.79	12.41	11.29	20.18
A3	14.18	23.51	17.59	8.43	18.32	1.68	13.04
A4	7.25	2.79	8.43	15.81	1.07	3.80	11.33
A5	3.00	12.41	18.32	1.07	0.85	16.58	16.01
A6	1.63	11.29	1.68	3.80	16.58	17.08	17.59
A7	8.26	20.18	13.04	11.33	16.01	17.59	14.59

$q_{1-0.3}$ is the 70% quantile of the standard normal distribution: $q_{0.7} = 0.214\%$

$$\text{Finally, } VaR_{0.3}(r_p) = 0.214\%(1.39) - (-0.082\%) = 0.38\%$$

This is interpreted as over the course of the next day, there is a 70% percent chance that the portfolio will not lose more than 0.38% of the initial investment.

A.3 Beyond VaR and Variance

This example shows how the variance does not distinguish between the risk of a loss (which is avoided) and the risk of a gain (which is preferred). It also shows how the VaR does not capture probability of losses beyond the VaR level or threshold.

Table 8: Variance and VaR

Return	Probability in Portfolio A	Probability in Portfolio B
-10	0.01	0.01
-7.5	0.04	0.04
-5	0.05	0.25
-2.5	0.1	0.25
0	0.5	0.3
2.5	0.15	0.1
5	0.15	0.05
μ	0.225	-1.775
σ	3.14	3.144
VaR1%	-10	-10
VaR5%	-7.5	-7.5

Table 8 indicates that portfolio A has a higher expected return from portfolio B, so there is a clear preference for portfolio A. However, both risk measures, the standard deviation and the VaR, fail to detect this preference because they both get the same values for portfolio A and portfolio B. This example shows that the VaR does not discriminate between severe losses as long as they are below the threshold.

A.4 CVaR Example

Recall (4.9):

$$\widehat{CVaR}_\epsilon(r) = \min_{\theta, d} \left(\theta + \frac{1}{n\epsilon} \sum_{k=1}^n d_k \right)$$

$$\text{Subject to } -r_k - \theta \leq d_k, \quad k = 1, n$$

$$d_k \geq 0, \quad k = 1, n$$

$$\theta \in R$$

Substituting the variables from the example into the model gives the following:

$$\begin{aligned}
 & \min_{\theta, d} \left(\theta + \frac{1}{7 \times 0.3} \sum_{k=1}^7 d_k \right) \\
 \text{Subject to} \quad & 0.98\% - \theta \leq d_1 \\
 & -0.31\% - \theta \leq d_2 \\
 & -1.91\% - \theta \leq d_3 \\
 & 1.37\% - \theta \leq d_4 \\
 & 0.38\% - \theta \leq d_5 \\
 & 0.26\% - \theta \leq d_6 \\
 & -0.19\% - \theta \leq d_7 \\
 & d_k \geq 0, k = 1, 7 \\
 & \theta \in R
 \end{aligned}$$

The solution to this optimization problem is the number 1.137%, which is attained for $\theta = 0.38\%$. We verify that the solution of the problem is indeed the number 1.137% by calculating the objective function in equation (4.8) for $\theta = 0.38\%$,

$$\begin{aligned}
 \widehat{CVaR}_\epsilon(r) &= \min_{\theta \in R} \left(\theta + \frac{1}{n\epsilon} \sum_{i=1}^n \max(-r_i - \theta, 0) \right) \\
 &= 0.38\% + \frac{0.98\% - 0.38\% + 1.37\% - 0.38\%}{7 \times 0.3} = 1.137\%
 \end{aligned}$$

This means that over the course of the next day, there is a 70% percent chance that the portfolio will not lose more than 1.137% of the initial investment.

Appendix B: CVaR as a coherent risk measure

In this appendix, we provide numerical illustrations that show how CVaR satisfies all the four ADEH (1998) coherent risk measure properties.

1) The Monotonicity Property

$$\rho(Y) \leq \rho(X), \text{ if } Y \geq X \quad \forall \text{ random variable } X \text{ and } Y$$

Let $X = \{-1.37\%, -0.98\%, -0.38\%, -0.26\%, 0.19\%, 0.31\%, 1.91\%\}$. Assume $Y = X + 0.5\%$, so that $Y > X$, then $Y = \{-0.87\%, -0.48\%, 0.12\%, 0.24\%, 0.69\%, 0.81\%, 2.41\%\}$.

$$\widehat{CVaR}_{0.3}(X) = 1.137\% \text{ at } \theta = 0.38\%$$

$$\widehat{CVaR}_{0.3}(Y) = 0.6428\% \text{ at } \theta = 0\% \text{ (Calculated after repeating the same procedure as } \widehat{CVaR}_{0.3}(X))$$

The results clearly show that $\widehat{CVaR}_{0.3}(Y) < \widehat{CVaR}_{0.3}(X)$ where $Y > X$. Hence, the first property holds for the risk measure, CVaR.

Unlike CVaR, the variance fails to respect the monotonicity property. For example, let $X = \{-1.37\%, -0.98\%, -0.38\%, -0.26\%, 0.19\%, 0.31\%, 1.91\%\}$ and $Y = \{-1\%, -0.6\%, -0.2\%, 0.2\%, 0.6\%, 1\%, 3\%\}$. In this case, $Y > X$. $\text{Var}(X) = 1.13\%$ and $\text{Var}(Y) = 1.75\%$ i.e. $\text{Var}(X) < \text{Var}(Y)$ which contradicts the monotonicity property.

2) The Positive Homogeneity Property

$$\rho(0) = 0, \rho(\lambda X) = \lambda \rho(X), \text{ for all random variable } X \text{ and all } \lambda > 0.$$

Let $X = \{-1.37\%, -0.98\%, -0.38\%, -0.26\%, 0.19\%, 0.31\%, 1.91\%\}$ and assume $\lambda = 1.2$, then $\lambda X = \{-1.64\%, -1.18\%, -0.46\%, -0.31\%, 0.23\%, 0.37\%, 2.29\%\}$.

$$\widehat{CVaR}_{0.3}(X) = 1.137\% \text{ at } \theta = 0.38\%$$

$$\widehat{CVaR}_{0.3}(\lambda X) = 1.3647\% \text{ at } \theta = 0.46\%$$

$1.3647\% \cong 1.2 \times 1.137 = 1.3644\%$, which means that CVaR satisfies the positive homogeneity property.

3) The Subadditivity Property

$$\rho(X + Y) \leq \rho(X) + \rho(Y), \text{ for all } X \text{ and } Y.$$

Let $X = \{-1.37\%, -0.98\%, -0.38\%, -0.26\%, 0.19\%, 0.31\%, 1.91\%\}$ and $Y = \{-0.87\%, -0.48\%, 0.12\%, 0.24\%, 0.69\%, 0.81\%, 2.41\%\}$. $Y + X = \{-2.24\%, -1.46\%, -0.26\%, -0.02\%, 0.88\%, 1.12\%, 4.32\%\}$.

$$\widehat{CVaR}_{0.3}(X) = 1.137\% \text{ at } \theta = 0.38\%$$

$$\widehat{CVaR}_{0.3}(Y) = 0.6428\% \text{ at } \theta = 0\%$$

$$\widehat{CVaR}_{0.3}(X + Y) = 1.77248\% \text{ at } \theta = 0.26\%$$

It is clear that $\widehat{CVaR}_{0.3}(X + Y) \leq \widehat{CVaR}_{0.3}(X) + \widehat{CVaR}_{0.3}(Y) = 1.137 + 0.6428 = 1.7798\%$. Consequently, CVaR satisfies the subadditivity property.

Similarly, we give a simple example which shows that VaR may not satisfy the subadditivity property. Suppose that X denotes a bond which either defaults with probability 4.5% whereby we lose \$50, or it does not default and in this case the loss is equal to zero. Let Y be the same bond. Assume that the defaults of the two bonds are independent events. Being the 5% quantile of the payoff distribution in this case, VaR fails to recognize losses occurring with probability smaller than 5%. Hence, the VaR of the two bonds at 95% confidence level (5% tail

probability) is equal to zero. $VaR_{0.05}(X) = VaR_{0.05}(Y) = \0 . A portfolio of the two bonds has the following payoff profile: it loses \$100 with probability of about 0.2%, loses \$50 with probability of about 8.6%, and the loss is zero with probability 91.2%. Thus the corresponding 95% VaR of the portfolio equals \$50 and clearly, $\$50 = VaR_{0.05}(X + Y) > VaR_{0.05}(X) + VaR_{0.05}(Y) = 0$.

4) The Translation Invariance Property

$$\rho(X + C) = \rho(X) - C, \text{ for all } X \text{ and } C \in \mathbb{R}$$

If we refer to the example presented in the 1st property. Let $X = \{-1.37\%, -0.98\%, -0.38\%, -0.26\%, 0.19\%, 0.31\%, 1.91\%\}$. Assume $Y = X + 0.5\%$, so that $Y > X$, then $Y = \{-0.87\%, -0.48\%, 0.12\%, 0.24\%, 0.69\%, 0.81\%, 2.41\%\}$, we get:

$$CVaR_{0.3}(X) = 1.137\% \text{ at } \theta = 0.38\%$$

$$CVaR_{0.3}(Y) = 0.6428\% \text{ at } \theta = 0\%$$

$1.137\% - 0.5\% = 0.637\% \cong 0.64\%$. This implies that CVaR respects the translation invariance property.