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**A PILOT STUDY OF AN INSTRUCTIONAL UNIT ON
FUNCTIONS USING MULTIPLE REPRESENTATIONS WITH
EMPHASIS ON PROBLEM SOLVING IN GRADE 10
LEBANESE PROGRAM**

A Thesis Presented to the Faculty of
the Division of Education and Social Sciences

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Master of Arts in Education

Emphasis: Mathematics

by

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Dedication

To my family with my love

Acknowledgment

I would like to express my special thanks to my advisor Dr. Iman Osta for her continuous support, guidance, advice and encouragement over the years during which we worked together. I would like also to thank Dr. May Hamdan and Dr. Samer Habre for being on my thesis committee.

Abstract

The purpose of this study is to develop and pilot an instructional unit emphasizing multiple representations of functions in Grade 10 Lebanese program. It also aims to investigate Grade 10 Lebanese students' use of multiple representations of functions while solving real-life problem situations related to linear functions during and after the completion the instructional unit. The objectives of the unit are a modified version of the objectives of the Grade 10 Lebanese Mathematics Curriculum related to functions. The unit emphasized students' construction of knowledge through a series of tasks and discussions. Technology is integrated in the teaching of the unit. The participants are 14 Grade 10 Lebanese students. An action research study was conducted. Data were collected using a pre-test and a series of four assessments tools administered during the unit. Qualitative and simple statistical analysis of students' work showed a high level of students' understanding of the concept of function which was reflected on students' use of representations and the frequent translations performed. In the pre-test, the most frequent representation mode that students used was the verbal one, while the graphical representation was not used at all. In the selected assessments, the most frequent representation mode that students used was the numerical one, while the algebraic mode was the least used by students among other modes. Further analysis showed students' ability to: (a) choose the convenient representation that helps them to solve the problem situation, and (b) perform several translations among the different representations of functions and move flexibly among and within these representations

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Chapter One

Introduction

The first educational reform in Lebanon after two decades of civil war took place in 1997. One of the outcomes of this reform was a new mathematics curriculum. In this new Lebanese Mathematics Curriculum (hereafter referred to as (LMC)), mathematics is considered as a utilitarian science that should be accessible to a larger number of citizens (ECRD, 1997 a). The main aim for this reform "is to form a citizen capable of critical thinking and intellectual autonomy" (ECRD, 1997 a, p. 287). This new reform in mathematics involved a reform in the mathematics curriculum itself on one hand, and a reform in the teaching of mathematics on the other hand (ECRD, 1997 a).

The new LMC has considered problem solving as one of its general objectives, but did not emphasize the role of mathematical representations in its general objectives (ECRD, 1997a). The National Council of Teachers of Mathematics (NCTM) does not totally share the same vision with the LMC because it has recognized problem solving and representations as two of the five process standards for school mathematics at all grade levels (NCTM, 2000). These two standards are important to school mathematics and to the study of mathematics because they help students learn about, develop, and deepen their understanding of mathematical concepts (NCTM, 2000).

Representations are effective tools in the study of mathematics (NCTM, 2000). Carpenter (1985) contends that representing a problem situation mathematically is the major objective of the mathematics curriculum and that the process of representing a problem and deriving its solution are integrally related to each other. At its most basic level, a problem can be solved once its appropriate representation is

found (Carpenter, 1985) because students "create and use representations to organize and record their thinking about mathematical ideas" (NCTM, 2000, p. 280).

On the other hand, one of the major goals of mathematics instruction is problem solving (NCTM, 1989; ECRD, 1997 a) and it is a major means for learning mathematics (NCTM, 2000). Being a good problem solver leads to great advantages in everyday life and in the workplace where there is an increasing demand for more problem solvers and decision makers. Consequently, students have to effectively use, adapt and extend the mathematics they know in order to meet the new challenges in work, school and life (NCTM, 2000).

To become successful problem solvers, students must understand and use mathematical content, must select, apply and translate among mathematical representations of the problem, and have a rich repertoire of strategies (Kilpatrick, 1985; NCTM, 2000).

In Lebanon, the LMC considers problem solving as "the most significant activity in the teaching of mathematics" (ECRD, 1997 a, p. 289) where students learn to use different strategies that help them tackle difficulties they face while solving problems (ECRD, 1997 a). Although this general objective is emphasized in the general objectives of the LMC, it is hardly reflected in the specific objectives and in details of the content of the official mathematics curriculum for the secondary level (ECRD, 1997 b).

Statement of the Problem

This study is concerned with the use of problem solving and multiple representations in teaching and learning linear functions in grade 10. The topic of linear functions is selected to be the object of this study because it is students' first formal encounter with the concept of function which is recognized as a central

concept and an essential component in secondary school and undergraduate mathematics curricula (Carlson, Oehrtman & Thompson, 2006; Dreyfus, 1990; Harel & Dubinsky, 1992). The importance of this concept is due to the fact that it is a unifying concept which enables students to understand other mathematics and connect ideas across different areas (NCTM, 2000), and it grows as one progresses in depth and breadth through one's understanding of the subject (Yerushalmy, 1993). On the other hand, modeling many real-life situations and analyzing these situations in order to make a decision or solve a certain problem is based on students' knowledge and understanding of functions (NCTM, 2000). Such a modeling process can't take place successfully without good representations of the situation.

Representations are fundamental in learners' understanding and use of mathematical ideas. They play an important role in mathematical problem solving (Schoenfeld, 1985). They are effective tools that help students organize their thinking and make their mathematical ideas more concrete and available for reflection (NCTM, 2000).

The concept of function can particularly be represented using different representational modes. Interpreting the different representations of functions and making connections among them are an integral part of understanding functions (Eisenberg, 1992). The grade 10 Lebanese mathematics curriculum mainly emphasizes the algebraic representations of functions and states, among its objectives, the graphical representations of functions as one of the principal aims of studying functions rather than a tool for understanding functions (ECD, 1997 b). A fundamental assumption of this study is the claim that such an emphasis on the algebraic representations of functions as tools and neglect of the role of many representations as tools make functions a difficult topic for many students.

Purpose and Research Questions

The purpose of this study is to develop and pilot an instructional unit emphasizing multiple representations of functions. It also aims to investigate the types of representations used by grade 10 Lebanese students in solving real-life problem situations related to linear functions, during and after the completion of the unit about functions. This unit focuses on: (a) the use of multiple representations as tools and not only as ends to help students understand and interpret functions, and (b) on solving real-life problem situations related to linear functions.

The research questions that guided this study are:

- How do the Lebanese mathematics curriculum and the grade 10 national mathematics textbook introduce functions and what types of representations do they use for that purpose?
- What are the representation modes that students use while solving real-life problem situations related to linear functions?
- Do Grade 10 Lebanese students decide on their own to shift from one mode of representation of functions to another while solving problems?
- What roles do students' interpretation of functions' representations play while solving real-life problems?
- To which extent are grade 10 Lebanese students able to interpret different representations of linear functions within the context of real-life problem situations related to linear functions?

Definition of Key terms

- *Representation:* In this study, the term representation will refer to the external manifestations of mathematical concepts (Pape & Tchoshanov, 2001).

- *Multiple External Representations*: will refer to at least two external modes of representation such as graphical representations, tabular (numerical representations), verbal representations and algebraic representations.
- *Problem Solving* as defined by NCTM "means engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p.52).
- *Lebanese textbook published by the government*: "Building Up Mathematics" is the Lebanese national textbook that reflects the objectives of the Lebanese Mathematics Curriculum (LMC).

Rationale and Significance of the Study

In *The Curriculum and Evaluation Standards for School Mathematics*, the NCTM has called for "a shift in emphasis from a curriculum dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that emphasizes conceptual understandings, multiple representations and connections, mathematical modeling, and mathematical problem solving," (NCTM, 1989, p.125) because conceptual understanding is fundamental to doing math meaningfully (NCTM, 1989). Research findings have also shown that using multiple representations in teaching mathematics improves students' problem solving ability and helps the students to develop a deeper understanding of concepts (Borba & Confrey, 1993; Yerushalmy, 1997). Since problem solving and multiple representations were the main approaches used in developing and implementing the instructional unit of this study, and since there is no study in Lebanon, so far, that investigates the use of multiple representations in teaching functions at grade 10, the results of this study will be relevant for curriculum designers and mathematics teachers since it will draw their attention to the following issues:

- The necessity of reviewing the general objectives of the LMC in order to account for the role of representations in mathematics education.
- The necessity of reviewing the grade 10 specific objectives of the curriculum and the content of the national text book in order to ensure the realization of the general objectives of the curriculum.
- The importance of the concept of function which sets the stage for more advanced work in school mathematics (Knuth, 2000 a).
- The importance of integrating problem solving approach in all mathematical activity because problem solvers are highly demanded nowadays.
- The necessity of designing a math curriculum that promotes conceptual understanding through multiple representations and connections.
- The importance of the role of multiple representations in teaching of the concept of functions and solving problems.
- The necessity of lessening the emphasis on the use of the algebraic representations of functions as tools and putting more emphasis on the role of other representations of functions as tools and not only as learning objectives to facilitate the learning and understanding of functions.
- The importance of modeling real-life situations using functions in students' understanding of functions.
- The importance of translating among different representations of functions and connecting them.

Moreover, it is hoped that this study will be a contribution to the increasing, yet still young, body of Lebanese studies related to teaching mathematics at the high school level.

Literature Review

Introduction

Studying functions and patterns is one of the central themes of mathematics (NCTM, 1989) and the fundamental object of algebra (Yerushalmy & Schwartz, 1993). A deep and rich understanding of this concept is considered crucial for the success of students in mathematics (Gagatsis & Mousoulides, 2004) and for their understanding of mathematics at the college level. However, research results indicate that understanding functions does not appear to be easy (Gagatsis & Mousoulides, 2004) and several studies were conducted to investigate this issue. The mathematics education community has suggested changes in curriculum and in instructional approaches to help solving this problem. This literature review addresses the curriculum changes and instructional approaches that are considered relevant to the purpose of this study. It begins by presenting the theoretical foundation on which this study is based. Next, it presents a review of research related to problem solving, multiple representations of functions, translation among multiple representations of functions, importance of the concept of functions, and difficulties related to the learning of the concept of functions.

Theoretical Framework

The present research is adopting a social constructivist approach as a framework. The main reason for this choice is the design of the instructional unit implemented in the study. It is designed in a way to realize the vision and the general objectives of the LMC that intends to foster individual construction of knowledge starting from real-life problem situations. This is also in line with the recommendations of the

National Council of Teachers of Mathematics (NCTM) which recommends that new mathematical knowledge should be built through problem solving (NCTM, 2000).

For many years, the math teacher was the transmitter of knowledge and the students were the passive absorbers of the transmitted knowledge. Research results have shown that learning does not take place by passive absorption only (Resnick, 1987) since "knowledge is not a transferable commodity and communication not a conveyance" (Von Glaserfeld, 1987, p.16). Consequently, constructivism has become a popular approach for learning within the mathematics education community (Cobb, Yackel, & Wood, 1992)

Constructivism is a general title that has been given to many theories about learning. In the twentieth century, these theories were developed through the work of several scholars such as Dewey, Piaget, Vygotsky and many others. The main guiding principles of constructivism as a theory of learning and knowing are:

- Knowledge is not transmitted to learners who passively receive it. Rather, learners are active constructors of their own knowledge
- Prior knowledge and experience have an impact on the learning process. Elby (2000) contends that students don't come to classrooms as "blank slates" that are ready to be filled with knowledge. The learners have their already existing ideas and experiences; learning takes place when the learners base the construction of their new knowledge on their prior knowledge and experiences (Selden & Selden, 1992).
- Learners are active organizers of their experiences (Borkowski & Muthukrishna, 1996) because learning is an organizational process based on making sense and constructing meaningful representations of one's experiential world.

- Motivation is essential to learning since learners' motivation affects their ability to learn.
- Knowledge is not only considered as an individual construction, but as a social situated activity as well. Knowledge can be acquired through the interaction with other individuals.

Social constructivism is an educational approach based on the main principles of constructivism. It has developed through the work of Vygotsky who agreed with Piaget's views in many ways but added and emphasized the social and cultural components of learning situations. From a social constructivist perspective, knowledge is considered as human product and learning is viewed as a social process. In a constructivist learning environment, learning does not take place within the individual only, but through the social activities in which the students are engaged. During such activities, students are given the chance to construct mathematical knowledge through the interaction with peers and teachers. Such interactions can take place through small group problem solving sessions or through class discussions to promote social collaboration and acquire better understanding of problems. In addition, students learn how to respect each others' thinking, and how to explain and justify to others the ways they used for solving problems. In such classroom, peer collaboration is emphasized as integral to the process of learning. Social interaction among the learners enables them to interpret situations, take others' perspectives in consideration, solve conflicts and negotiate shared meaning (Borkowski & Muthukrishna, 1996).

NCTM (2000) advocates that one of the major goals of school mathematics programs should be the creation of autonomous learners. Learning with understanding is necessary to achieve this goal. Consequently, the acquisition of a

deep and rich understanding of the mathematics learned at school is a major goal for all students (Hiebert & Wearne, 2003), and at the same time it has been the goal of many research and implementation efforts (Carpenter, T.P. & Hiebert, J, 1992). The constructivist approach emphasizes that development of understanding is not merely a result of information acquisition. Rather, it is the result of knowledge construction and transformation of information (Borkowski & Muthukrishna, 1996).

Problem Solving

Mathematical problems have always been a part of school mathematics curricula (Kilpatrick & Stanic, 1989). All Since antiquity, they have occupied a central place in the school mathematics curricula.

The terms "problem" and "problem solving" have various meanings in the literature, which makes it quite difficult to interpret (Schoenfeld, 1992).

According to the Webster's dictionary (1979), the term "problem" is defined as: a) A question that is perplexing or difficult (p.1434); b) in mathematics, anything required to be done, or requiring the doing of something (Webster Dictionary, 1979, p.1434).

In his attempt to comment on the meanings of the term "problem" as given by the Webster's dictionary, Schoenfeld (1992) has attributed the first definition of the word "problem" to the way it has been traditionally referred to in mathematics instruction as sets of mathematical tasks used as vehicles of instruction, as means of practice, and as yardsticks for the acquisition of mathematical skills" (P.337). In this traditional sense, a "problem" refers to routine problem that aims at allowing students to practice particular mathematical techniques that have already been demonstrated to them.

In its turn, the term problem solving has several descriptions, definitions and interpretations. In 1965, Polya has referred to problem solving as "a process of accepting a challenge and striving to resolve it" (p.117). On the other hand, Lester and Kehle (2003) defined problem solving as an "extremely complex form of human endeavor that involves much more than the simple recall of facts or the application of well-learned procedures" (p.509). NCTM has defined problem solving as "engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings" (NCTM, 2000, p.52).

Wide set of research studies and curriculum guidelines emphasized the role and the use of problem solving in teaching and learning mathematics (NCTM, 1989, 2000; LMC, 1997 a). NCTM *Standards* (1989) considers problem solving as: (a) the primary goal of all mathematics instruction, (b) an integral part of all mathematical activity, and (c) a process that should permeate the entire program and provide the context in which concepts and skills can be learned (NCTM, 1989, p.23), and by which students experience the power and the usefulness of mathematics in the world around them (NCTM, 1989, p. 75). Also, NCTM *Principles* (2000) considers problem solving as an integral part of all mathematics learning involved in all the content areas described in the *Standards*. NCTM contends that problem solving does not only help students solidify and extend their mathematical knowledge, but it leads to great advantages in the workplace and in everyday life (NCTM, 2000) and helps students become productive citizens (NCTM, 1989). With this spirit, NCTM recommends that all instructional programs at all levels should be designed in a way that enables students to:

- *build new mathematical knowledge through problem solving;*

- *solve problems that arise in mathematics and in other contexts;*
- *apply and adapt a variety of appropriate strategies to solve problems;*
- *Monitor and reflect on the process of mathematical problem solving.*

(NCTM, 2000, p.52).

Concerning the different roles of problem solving in teaching and learning math, Schroeder and Lester (1989) made a clear distinction between teaching for problem solving, teaching about problem solving and teaching through problem solving. Teaching for problem solving is identified as teaching mathematical content in order to use it later in solving mathematical problems. By teaching about problem solving, it is meant to teach the heuristic strategies that improve problem solving abilities. Teaching through problem solving, means to teach standard mathematical content using problems involving this content. In recent years, there has been a trend within the mathematics education community to adopt this teaching approach to help students "develop a deep understanding of mathematical concepts and methods. The key to fostering students' understanding is engaging them in trying to make sense of problematic tasks in which the mathematics to be learned is embedded" (Randall & Schoen, 2003). The main characteristics of this approach are related to the type of the used problematic tasks and to the teacher's role. The chosen tasks must be accessible and engaging to the students and have embedded in them the mathematics to be learned. The teacher's role is to ensure classroom norms that are supportive to students' learning in this way, and to ensure students' access to appropriate technological tools and intellectual tools (Randall & Schoen, 2003).

Along the same line but from a different perspective, Kilpatrick and Stanic (1988) identify three main themes that characterize the use and the role of problem solving

in teaching and learning mathematics: problem solving as context, (b) problem solving as skill and (c) problem solving as art.

When problem solving is considered as context, then it is used as means to achieve other valuable ends such as: justification for teaching mathematics, providing specific motivation for subject topics, recreation, vehicle that introduces students to new concepts or skills, and practice to reinforce skills, techniques and concepts that are taught directly (Kilpatrick & Stanic, 1988). From this perspective, problem solving is not considered as a goal by itself, but it is employed as a tool that serves other curricular goals (Schoenfeld, 1992).

The skill view of problem solving considers it as one of the many skills to be taught in the school curriculum and which is valuable in its own right. As a skill, problem solving has its implications on the role of problem solving in the curriculum. Kilpatrick and Stanic (1988) confirm it by stating that problem solving "has become dominant for those who see problem solving as a valuable curriculum end deserving special attention, rather than as simply a means to achieve other ends or an inevitable outcome of the study of mathematics" (p. 15). As a consequence, problem solving techniques are taught as subject matter to be mastered by the students and hierarchical distinctions were made, within the general skill of problem solving, between solving routine and non-routine problems. Non-routine problem solving is characterized as a higher-level skill to be acquired after the skill of solving routine problems is acquired (Kilpatrick & Stanic, 1988).

The third role that Kilpatrick and Stanic distinguish is problem solving as an art. The majority of their arguments related to this role are based on Polya's work. Polya considered the development of intelligence as a major aim of education. His experience in the teaching and learning of mathematics led him to conclude that the

deductively presented mathematics in textbooks and journals does not do justice to the subject because it requires demonstrative reasoning. On the other hand, mathematics in the making requires plausible reasoning that should be taught to students. According to Polya, (as cited in Kilpatrick & Stanic 1988), problem solving is a practical art that one learns by imitation and practice. For that purpose, he recommends that techniques of problem solving have to be discussed with the students, illustrated by the teacher, and practiced in a non-mechanical way. Finally, and as Kilpatrick and Stanic (1988) state it: "because teaching is an art, no one can program or otherwise mechanize the teaching of problem solving. It remains a human activity that requires experience, taste and judgment" (p. 17).

Representations

In the *Principles* (2000), NCTM referred to a representation as a process and a product. In other words, "it is the act of capturing a mathematical concept or relationship in some form" and "the form " (p. 67). Kaput (1991) refers to external representations as "notation systems" that people use to organize their mental structures, while Janvier, Girardon, and Morand (1993) consider external representations as external manifestations of mathematical concepts that act as stimuli on the senses and help us understand these concepts " (p. 81). Kaput (1991) refers to internal representations as "mental structures" and Pape and Tchoshanov (2001) share his view with him. They use the term "internal representations" to refer to the way the learners perceive a mathematical concept internally by developing cognitive schemata or abstractions of mathematical ideas through experience (Pape & Tchoshanov, 2001) in order to "allow the mind to operate on them" (Hiebert & Carpenter, 1992). One of the major differences between external and internal representations is the way they are communicated to other people. Internal

representations are not accessible to other people and cannot be communicated to others except through external representations, while external representations can be communicated to other people, and can be observed by others.

There are several forms of external representations. Possible forms of external representations suggested by Lesh, Post and Behr (1987) are: spoken language, written symbols, pictures or diagrams, and physical objects. In addition to the previously mentioned forms, Goldin and Kaput (1996) add more specific examples such as the tabular and graphical representations.

External mathematical representations have not been in their own right the focus of attention of the mathematics education community until the end of the last decade, because they were considered as a part of the mathematical communication process (NCTM, 1989; LMC, 1997 a). In the *Standards* (1989), representations are considered as a means of communicating mathematical ideas at all levels and their importance as being an integral part of the mathematics curriculum is highlighted among other key communication skills. Between 1986 and 2000, a bulk of studies investigated the role that external mathematical representations play in mathematics teaching and learning (Kaput, 1987; Hitt, 1998; Even, 1998; Cifarelli, 1993; Pape & Tchoshanov, 2001; Goldin, 1998). In 2000, NCTM reflected more emphasis on the role of representation in its document *Principles and Standards for School Mathematics* by including representations as one of the process standards (NCTM, 2000). The work concerning mathematical representations that was established during the last decade has set fundamental thoughts about the topic and was considered the basis for several studies that were conducted more recently about this topic (Gagatsis & Elia 2004; Elby, 2000; Gagatsis, Christou, & Elia, 2004; Cai & Lester, 2005).

The mathematics education community emphasizes the role of mathematical representations as mathematical tools that help students to: solve problems, communicate mathematical ideas, and support students' understanding of mathematical concepts (Dufour, Bednarz & Belanger, 1987; NCTM, 2000). As external manifestations of mathematical concepts, external representations help students understand those concepts by acting as stimuli on the senses (Janvier, Girardon, Morand 1993). Lesh, Post, and Behr (1987) contend that student's understanding of ideas increases if the student can "recognize the idea embedded in a variety of qualitatively different representational systems, flexibly manipulate the idea within given representational systems, and accurately translate the idea from one system to another" (p. 36).

In order for students to think about or communicate mathematical ideas, the letter must be represented in some way (Hiebert & Carpenter, 1992); consequently, representations are essential for learning and doing mathematics. They are the means by which the individuals can make sense of situations (Kaput, 1989). They help students organize their thinking, and make mathematical ideas more concrete and available for reflection. They help teachers gain access to the ways students interpret and think about mathematics (NCTM, 2000). They support the communication of mathematical approaches, arguments and understandings to one self and others (NCTM, 2000). Nevertheless, a representation does not represent by itself because it needs interpretations by an interpreter (Kaput, 1991; Von Glaserfeld, 1987 b).

Problem Solving and Representations

Representations don't only play a crucial role in mathematics teaching and learning, but in problem solving as well (Cai & Lester, 2005; Schoenfeld, 1985) because students frequently use many representational systems while solving

problem situations (Lesh, Landau & Hamilton, 1983). Lester and Kehle (2003) contend that students' success in problem solving depends on many factors among which the representations of the problems:

successful problem solving in mathematics involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve the tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity (p. 510).

Cai and Lester (2005) and Cifarelli (1993) support the idea that successful problem solving depends on the problem solver's ability to create the appropriate representations to represent the problem (Cai & Lester, 2005; Cifarelli, 1993). Greeno and Hall (1997) add to that the ability to translate among the different representations of the problem. Hitt (1998) relates this latter ability to understanding. He contends that students are able to reach a high level of understanding a concept when they are capable of articulating coherently the different representations of the concept during problem solving. Consequently, instructional programs should be designed in a manner that enables students to:

- Create and use representations to organize, record, and communicate mathematical ideas;
- Select, apply and translate among mathematical representations to solve problems (NCTM, 2000, p. 67).

As a result, those students who are able to apply, and translate among, different representations of the same problem situation will appreciate the beauty and

consistency of mathematics, and at the same time, they will acquire a flexible and powerful set of problem solving tools (NCTM, 1989).

Functions

The concept of function (COF) is one of the ancient concepts in mathematics that goes back to almost 4000 years ago (Kleiner, 1989) and that has been considered as primary theme in mathematics (Selden & Selden, 1992). Within the field of mathematics and mathematics education, it is recognized as a central concept and an essential component in secondary school and undergraduate mathematics curricula (Dreyfus, 1990; Harel & Dubinsky, 1992; Hitt, 1998; Kaput, 1992; O' Callaghan, 1998, Yerushalmy & Schwartz, 1993) that grows in importance as one progresses in the depth and the breadth of one's understanding of the subject. In the *Standards* (1989), NCTM considers the concept of function as one of its curriculum standards for grades 9-12 and recommended a mathematics curriculum that includes a continued study of functions.

The concept of function has acquired such an importance due to: (a) the fact that it is used for modeling and mathematically representing several real-world situations (NCTM, 1989; NCTM, 2000), (b) its unifying nature that connects ideas across different areas in mathematics (Dreyfus & Eisenberg, 1982; NCTM, 2000), and (c) the role it has in students' understanding of mathematical ideas (NCTM, 2000; Selden & Selden, 1992).

Difficulties in Learning Functions

One of the major goals of mathematics instruction is to help students understand mathematical concepts and use them in different situations. Searching for the reasons that hinder student's understanding of a certain concept helps mathematics educators to find better ways to improve mathematics teaching and learning. Several studies

were conducted to investigate the difficulties that students face while learning the concept of function. It was found that learning this concept is complex (Carlson, Oehrtman, & Thompson, 2006; Dreyfus & Eisenberg, 1982) and that most students have a limited knowledge about functions despite the fact that they learn algorithms and step-by-step procedures for using functions in problems (Vinner & Dreyfus, 1989). This is due to several reasons.

- First, the existence of many mathematical sub-concepts (also called functional concepts) associated with the concept of functions (Dreyfus & Eisenberg, 1982).
- Second, its integrative nature (Dreyfus & Eisenberg, 1982) that relates algebra, geometry and trigonometry together.
- Third, the diversity of representations needed for representing functions (Janvier, C., 1987; Dreyfus & Eisenberg, 1982). Results of studies have shown that when a given problem situation was presented in different representational modes, many students have shown inability of understanding these ideas and connecting the different representational modes (Dreyfus & Eisenberg, 1987, 1991; Vinner & Dreyfus, 1989).

Graphical and algebraic representations of functions are two communicative symbolic systems that are used to expand and understand one another and to define and construct the concept of function (Leinhardt, Zaslavsky & Stein, 1990). However, Piez and Voxman (1997) found that students tend to avoid graphical representations of functions because they face difficulties in reading information represented graphically. This was also confirmed by the results of a study conducted by Hitt (1998) on a group of 30 students taking a postgraduate course in

mathematics education. The findings of Hitt's study showed that these students faced difficulties in identifying sub-concepts of the function such as domain and image set that were not represented in algebraic form.

- Fourth, the connections between the different representations of a function are considered to be one of the difficulties in learning functions (Kaput, 1989). Moschovich, Schoenfeld, & Arcavi (1993) contend that understanding functions is associated with the ability of connecting the different representations of functions and many students face difficulties in making connections among different representations of functions (Eisenberg & Dreyfus, 1991; Sfard, 1992).
- Fifth, the emphasis on the algebraic symbolic representation in the teaching of functions rather than presenting it in practical, concrete contexts have made it difficult for many students to understand the concept of function (Demana, Schoen, & Waits, 1993; Leinhardt et al., 1990).
- Finally, the duality of the nature of function as a process and as an object represents one of the difficulties that face students' understanding of functions (Sfard, 1991). Moschovich, Schoenfeld, and Arcavi (1993) have defined the process perspective of a function as "linking x and y values: for each value of x , the function has a corresponding y value" and the object perspective of a function as "a function or relation and any of its representations are thought of as entities" (p.71). Students' inability to grasp and master both the object and process perspectives of functions make it difficult for them to understand functions and prevents them from

selecting an appropriate representation for a given problem (Moschkovich, Schoenfeld, and Arcavi, 1993).

Functions and Multiple Representations

The concept of function is by excellence a mathematical concept that can be represented in different forms. Each representation has advantages and disadvantages (Friedlander & Tabach, 2001) as it emphasizes one aspect of the concept and ignores other aspects. Consequently, learners are limited by the strengths and the weaknesses of the single representation (Ainsworth, 1999). To be able to understand the concept of function, it is necessary to understand the different representations of this concept and make connections among them (Cunningham, 2005). Using various representations of the same concept cancels out the disadvantages of a single representation because the various representations of the concept: complement each other by expressing different aspects of the concept (Ainsworth, 1999; Friedlander & Tabach, 2001), cater for students' individual styles of thinking (Friedlander & Tabach, 2001) and promote a deeper and thorough understanding of the concept (Carpenter & Hiebert, 1992; Kaput, 1989; Piez & Foxmann, 1997). When learners integrate information from the different representations of the concept, they become capable of achieving an insight that would have been difficult to gain only while using a single representation (Ainsworth, 1999). In both the *Standards* (1989) and the *Principles* (2000), NCTM promotes a mathematics instruction that uses multiple representations of mathematical concepts and advocates a mathematics curriculum that emphasizes the connections among these representations (NCTM, 1989, 2000). The use of multiple external representations (hereafter referred to as MER) is also recommended by many members of the mathematics education community (Hiebert & Carpenter, 1992; Lesh, Post & Behr, 1987). Dufour, Bednarz & Belanger (1987)

justify the necessity for using MER in mathematics teaching by considering representations as: an inherent part of mathematics, multiple concretizations of a concept, and a tool that reduces the difficulties that students face while solving problems.

As far as the concept of functions is concerned, some MER of this concept are necessary tools for treating this concept especially because it is hard to perceive it without them (Dufour, Bednarz & Belanger, 1987). Thus, NCTM (2000) encourages the idea that all instructional programs from 9-12 should enable students to "understand relations and functions and select, convert flexibly among, and use various representations for them" (NCTM, 2000, P.296). In order to develop a robust understanding of functions, NCTM (2000) fosters the use of multiple representations to make connections between different representations of a problem situation.

Despite the support that the use of MER receives in the mathematics education community (NCTM, 1989, 2000; Janvier, 1987; Hiebert & Carpenter, 1992; Lesh, Post & Behr, 1987), Dufour, Janvier et al. (1987) advance certain pedagogical considerations concerning the use of MER that may cause confusion rather than understanding. They argue that if MER are perceived as ends in themselves and students focus on the mastery of these representations rather than perceiving them as mathematical tools and seeing the connections between them, then students won't see these representations as an embodiment of the same situation, which creates misconceptions hindering later learning (Dufour, Bednarz & Belanger 1987).

Representations and Translations

Students' ability to translate from one representation of a concept to another is considered as a central goal of mathematics teaching (Hitt, 1998). By "*translation*", Janvier (1987) refers to "the psychological processes involved in going from one

mode of representation to another" (p.27). By restricting the representation modes to verbal, graphical, tabular and algebraic (formulae or equation), Janvier was able to represent the translation processes in detail using Table 2-1 (Janvier, 1987, p.28). The empty diagonal cells in the table refer to the translation taking place within the same mode of representation, and which is called transposition (Janvier, 1987).

Based on the translation processes described in Table 2-1 (Janvier, 1987, p.28), Janvier (1987) classifies translations in two types: direct and indirect. By direct translation, he means the translation that takes place from one mode of representation into another directly without passing by any other form of representational modes in the translation process, such as the translation from an algebraic representation to numerical representation.

Table 2-1

Translation Processes Among Different Representations

TRANSLATION PROCESSES				
To \ From	Situations, Verbal Description	Tables	Graphs	Formulæ
Situations, verbal description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulæ	Parameter Recognition	Computing	Sketching	

Note. From "Translation Processes In Mathematics Education", *Problems of Representation in the Teaching and Learning of Mathematics*, (p. 28), by C. Janvier, 1987, Hillsdale, NJ: Lawrence Erlbaum Associates.

By an indirect translation, Janvier (1987) refers to the process of translation from one mode of representation to another by passing through another mode in-between, as it is the case while translating from algebraic to graphical modes. In this case, the translation takes place from algebraic to tabular and then from tabular to graphical modes.

Translation among different representations of the same idea or of the same problem situation is one of the important factors that affect the understanding of a given idea or problem. Research findings show that students face difficulties in translating information from one representational system to another (Gagatsis, Shiakalli, & Panaoura, 2003). As a result, the relation between students' understanding of a concept and students' ability to translate among different representations of the same concept has been the subject of several studies over the past three decades (Goldin, 1998; Yerushalmy, 1997). Lesh, Post and Behr (1987) contend that among the criteria used for describing students' understanding of a given idea are students' abilities to (a) recognize the given idea when it is embedded in a variety of qualitatively different representational systems, (b) the flexible manipulation of the idea within the given representational systems, and (c) the accurate translation of the idea from one representational system to another. In the same sense, Eisenberg (1992) and Kaput (1992) relate student's understanding of functions to students' abilities to interpret the different representations of functions and to translate among different representational systems. Even (1998) conducted a study on 162 prospective secondary mathematics teachers to investigate the link between the translation from one representation of functions to another, and other aspects of knowledge and understanding. The results of this study showed that the flexibility of moving among different representations of the same concept, as well as

the ability to represent the same concept using different representations, are important for learning mathematics because they allow the students to develop a deeper understanding of concepts.

Several factors such as the direction of the translation from one mode to another and the sequence of appearance of the representations have an impact on students' translation ability among different modes of representations. As specified by Janvier (1987), the direction for the translation process that involves at least two different representational modes takes place from the source representation to the target representation. As there are different cognitive processes involved in the translation process, and in order to achieve directly and correctly a given translation, "one has to transform the source "target-wise" or, in other words, to look at it from target point of view and derive the results" (Janvier, 1987, p. 29). Specifying the direction of the translation affects the instructional design and the teaching strategies (Janvier, 1987). The direction of the translation among representations is related to the sequence of appearance of representations during instruction. Ainsworth (1999) argues that if all representations are not used at the same time, then decisions should be made about the time needed to add a new representation or switch between two representations. This is due to the differences in the cognitive demands of the translations among representations (Gagatsis, Christou & Elia, 2004). For some individuals certain representations are more difficult to articulate than others (Hitt, 1998). Yerushalmy and Schwartz (1993) also contend that the sequence of appearance of representations in the learning process affects students' understanding. As an example, they refer to the traditional process of learning algebra in the secondary schools which starts with numerical and symbolic manipulations of functions and ends with learning graphs of functions. Students learning functions with such a process faced certain difficulties

in the understanding of functions when the functions were represented using various representational modes.

Gagatsis, Christou and Elia (2004) support the importance of the sequence of appearance of representations on students' translation abilities. They have identified a hierarchical structure for multiple representations and the translation among them. Such a system is reflected on the sequence and direction of translation among representations of the same concept because some representations are considered as prerequisites for other complicated representations. Such prerequisite representations serve as a basis for connecting and understanding several representations of the same concept because they have a set of characteristics that are highly correlated with the other representations' characteristics. Consequently, and in order to facilitate instruction while using multiple representations, they recommend a correct sequencing of representations which implicitly indicates the direction of the translation among the different representations.

Summary

Several theories about learning mathematics have evolved during the last decades of the past century. As the literature confirms, constructivism has been the most popular one in the field of mathematics education. Within constructivism, problem solving has been promoted as an approach that helps students to actively construct their knowledge based on their prior knowledge and experience. When talking about problem solving, and as the literature indicates, it would be impossible not to talk about representations because representations are means of mathematical communication.

Several studies cited in this literature review investigated the role that representations play in students' problem solving abilities and students'

understanding of mathematical concepts and ideas and especially the concept of functions. As the literature confirms, many of the difficulties that students face while learning functions are related to: the diversity of representations needed to represent functions, the translation and the connection among the different representations of functions and to the emphasis on the use of one mode of representation on the expense of the other modes of representations of functions.

Between the late eighties and till the present time, a notable amount of studies are conducted to study the effect of using multiple representations in teaching functions. Although the use of multiple representations in teaching functions had received a strong support from the mathematics education community, there were few concerns about using multiple representations. However, this approach has been promoted by mathematics reform movements. In order to avoid difficulties related to the use of multiple representations in teaching functions, and as shown in the literature review section, mathematics education community recommended a bigger emphasis on the role of the translation among the different representations of functions and the direction of the translation from one representation to another to facilitate students' understanding of the concept of function.

As a conclusion, a curriculum based on the: implementation of real-life problem solving situations, the use of multiple representations of functions, and the successful translation among the different representations of functions will foster a deeper understanding of functions.

Chapter Three

Method

This study is an action research based on:

- The development and piloting of an instructional unit related to functions.
- The analysis of students' work while solving given tests and quizzes related to linear functions within a real-life context.

Participants

The participants in this study are 13 students from a private school in the eastern suburb of Beirut. This school offers two instructional programs: The Lebanese program and the American program. All participants are enrolled in the Lebanese program where math and sciences are taught in English. All participants are from the same grade 10 class. They are enrolled in a year-long mathematics course that is almost equivalent to Algebra II. Their age ranges from 15 to 16 years; six of them are females. Most of the 13 students are coming from families of middle socio-economic class. The math course is delivered in English, while the subjects' native language is Arabic and some of them face difficulties in understanding math in English. The mathematics course is taught by the researcher herself. All participants completed their intermediate level education in the same school.

Procedures

Several steps were undertaken to achieve the goals of this research.

First, the general objectives of the LMC, the grade 10 details of content (ECRD, 1997 b) related to functions and linear functions and the content of chapters 19, 20

and 21 of the grade 10 mathematics national textbook were analyzed for: (a) real-life problem situations, (b) the use of multiple representations and the translation among the different representations, and (c) the use of representations as ends and not as mathematical tools.

Second, an instructional unit covering the grade 10 objectives related to functions and linear functions was developed and piloted by the researcher who is also the teacher of the mathematics course. The unit was based on problem solving as a context for learning and use of multiple representations as tools.

Third, the instructional unit was implemented over 21 teaching sessions lasting for 50 minutes each.

To increase the validity of the results, data was collected using a pre-test, and four questions selected from students' formal assessments. In addition, a teacher's journal was kept throughout the implementation of the unit.

Pre-test

Before teaching the instructional unit related to functions, a pre-test was administered (Appendix A). In this pre-test, students solved a real-life problem situation related to linear functions. The main task was divided into several subtasks to help students solve the problem. In addition, more than one mode of representation of linear functions was involved in this test. The purpose of the pre-test was to investigate students' prior knowledge and skills of: modeling real-life situations using linear functions, interpreting information displayed in various representational modes, and translation among various modes of representations of linear functions. This pre-test took place in class and lasted for 90 minutes.

Development of the Unit

The unit about functions in the grade 10 Lebanese mathematics curriculum focuses on the algebraic representations of functions and the manipulation of these algebraic representations on the expense of other representations of functions. At the same time, straight lines are taught as graphs of linear equations of the form $y = ax + b$. There is no indication, neither in the grade 10 national mathematics textbook nor in the specific objectives of grade 10 mathematics curriculum, to the fact that linear equations of the form $y = ax + b$ are algebraic representations of linear functions and that straight lines are graphical representations of linear functions (ECRD, 1997 b; ECDR, 1998). In addition, following the sequence of grade 10 mathematics textbook, it is noticed that all the ideas related to straight lines such as: writing equation of a straight line in Cartesian and parametric form, graphing straight lines, studying the relative positions of two straight lines are treated quite before the chapter about functions. The chapter about functions is the first students' formal encounter with the concept of function and with the straight line as the graphical representation of a linear function. Solving real-life problem situations related to linear functions takes place in the chapter on systems of two equations with two unknowns and also before introducing the chapter about functions. In the chapter on systems of two equations in two unknowns, students have to find the linear system representing a situation then solve it using one of the algebraic methods given in the chapter with no indication that the system obtained is in fact a relation between two quantities and that one quantity depends on the other. No attention and is paid to the fact that these equations forming the system are algebraic representations of two linear functions.

Due to all the previously mentioned reasons, the researcher had developed and piloted a special instructional unit to teach functions. The unit had covered the

objectives of the grade 10 mathematics curriculum related to functions and was completed during 21 teaching sessions lasting for 50 minutes each. The emphasis in the unit about functions is on: (a) lessening the use of algebraic representations of functions as tools and emphasizing more the role of other representations of functions as tools and not as ends by themselves, (b) translations among different representations of functions, (c) real-life problem solving contexts, (d) development of processes in problem solving, and (e) group work instead of individual work.

Selected Assessments

To assess students' performance, several quizzes were administered during the teaching of the instructional, and a test was administered after its completion. Three questions were selected from these quizzes (Appendix B) and one question was selected from the test (Appendix C). All these questions were real-life problem situations related to linear functions. Two of them were similar to those presented in the pre-test, but of a higher degree of difficulty.

Researcher Notes

The researcher had kept a daily journal throughout the implementation of the unit. The journal documented students' main comments about the instructional unit, difficulties that students faced during the instructional unit, effect of the use of multiple representations, important questions asked by the students, students' preferred mode of representation and description of strategies that students use during problem solving. These notes were used for data analysis and for gradually adjusting the instructional unit to meet students' needs.

Data Analysis

The data collected from the pre-test, selected assessments were analyzed according to the following criteria:

- 1) Students' understanding of the problem.
- 2) Types and number of representations that students used in solving problems.
- 3) Students' use of the given representation of the problem.
- 4) Students' decision to translate from the given representation mode into another mode.
- 5) Students' ability to interpret different representations of linear functions.
- 6) Students' completion of the whole task.
- 7) Correctness of submitted answers.

In addition, the problem solving strategies used in solving the given problems and the relation between these strategies and the representation modes used for answering were also investigated.

Unit Development

The instructional unit about functions was a part of a year-long mathematics course. The duration of the unit was 21 teaching sessions each lasting for 50 minutes which is almost the same time allocated to the same unit by the LMC (ECRD, 1997

b). The researcher has developed all the curriculum materials related to this unit including homework, activities, quizzes and tests. The development and implementation of the instructional unit attributed a special emphasis to:

- Social constructivism as fundamental learning approach.
- The use of multiple representations both in teaching / learning activities and in problem solving situations, instead of the textbook's major focus on the algebraic representations of functions.
- The use of graphical representations of functions as teaching tools instead of being ends by themselves.
- Students' construction of their own knowledge starting from a real-life problem situation.
- Mathematical communication and reasoning.
- Connecting mathematics to real-life situations.
- The translation among the different representations of functions.
- The use of problem solving as a context for teaching and a tool for evaluation of learning.

Following are the main characteristics and educational foundations of the instructional unit:

Tasks

The unit emphasizes students' construction of knowledge through a series of tasks and discussions. The tasks were divided into activities and homework. Each lesson

in this unit starts with an activity representing the main idea characterizing the lesson. Each activity starts with a real life problem situation related to functions that students had to analyze. While developing the activities and homework, the following factors are taken into consideration: (a) the objectives to be covered, (b) students' mathematical and English abilities and (c) the type of problems. The solutions of the problems proposed in each activity or homework served a certain number of the learning objectives of the unit. Although the problems were challenging, each problem was divided into several parts to help students work it out. In addition, the proposed problems build on students' prior knowledge and stem from students' real-life experience to ensure students' familiarity with the proposed situations. In order to avoid problems related to difficulties in misinterpreting English text, all proposed problems were described using easy verbal representation of problem situations and easy vocabulary to make them accessible to all the students.

Group work

Group work was one of the teaching strategies, based on social constructivism, used in the implementation of this unit. In each session, students worked in groups of four with the exception of a single group that consisted of five members. The grouping schemes were not imposed by the researcher. In every session, students had the freedom to select the group they wanted to join. Members of each group varied from one session to another. Group members had different mathematical abilities. After distributing the activity sheets (Appendix D), group members had to read, interpret and solve the proposed problems. While working on the problems, the working groups were observed. The teacher encouraged discussions of the problems among group members and refused to give feed-back to the students

during their work unless group members reached a state that prevented them from moving forward in solving the problem. Group members were instructed to compare their results to the results of another group before having a public presentation and discussion of the results.

Mathematical Communication

NCTM contends that learning and teaching mathematics takes place through the interaction between the students and the teachers and among the students themselves (NCTM, 2000). During the implementation of this unit, teacher's approach fostered such type of interactions through discussions, questioning and writing journals.

Discussions

Activities were composed of different parts. After finishing each part of the activity, results of each group, whether correct or wrong, and the different processes used in solving the problems were written on the board by group representatives. A discussion of each proposed solution, in which all the class actively participated took place. Having more than one solution presented on the board had encouraged students to clearly express their ideas, justify their solutions, listen to the others and make decisions about the presented ideas. Most of the time and due to time limitations, the teacher oriented the discussions towards the new mathematical concepts that were introduced during the activities and used the solutions of the activity problems to formally present the new concept. Only in few occasions, the teacher left the discussions open. Between any two parts of the activity, the teacher took two minutes to ask students to give a quick reminder of the concepts taught.

Questioning

Questioning is one of the main approaches used in the implementation of this study. The researcher has ensured a safe environment in the classroom so that

students felt free to ask all the questions they wanted to ask. The teaching sessions were highly interactive and questions were not only between the teacher and the students, but also among the students themselves especially during the discussion of group work. An example about the ways questions were orchestrated in class is what used to happen in the first five or seven minutes of every session. Each session started by listening to students' questions about the material they learned in the preceding session. The teacher invited other students to answer their colleagues' questions. If students' presented answers were not clear enough, then the teacher would intervene either by asking additional questions that would lead to the correct answer or by directly answering students' questions. If students' proposed questions at the beginning of the session did not cover all the objectives taught in the previous session, then the teacher would pose additional questions to ensure the revision of all the taught material. Such a five to seven minutes revision of concepts through questioning and answering was an essential part of each teaching session.

Questioning was used for many purposes. First, the researcher used questioning to help students organize their thoughts, present their reasoning, gain insight into their thinking and draw out conclusions. Second, it helped the teacher to know more about students' thinking, get feedback about the taught material and tailor the instructional practices according to students' needs. Third, it was used for assessment purposes.

Open-ended questions that don't have a single correct answer were asked frequently to improve students' critical thinking abilities and deepen their understanding of the taught concepts. The teacher encouraged the students to solve such questions by giving them bonuses.

Journals and Summary

At the end of each session that introduced a new concept, each student had to summarize during three minutes only the main points learned during the session on a copybook specially designated for that purpose. The teacher collected these copybooks, reviewed them and gave them back to the students the next day with a written feedback

At the end of the unit, students had to develop a summary about the unit. This summary included the main objectives of the unit, examples clarifying each objective and a reflection about the unit. Such a summary was not only limited to this instructional unit, but was a part of the teacher's instructional strategy over the whole course. The summary was collected and graded for completeness, correctness and creativity and then distributed to the students who usually used it for revision purposes.

Homework

Most of the homework assigned in the unit was developed by the teacher. Samples of homework are available in Appendix E. Few problems and exercises were taken from the grade 10 national mathematics Lebanese textbook (ECD, 1998). There were two types of homework problems: one that directly tackled mathematical skills to reinforce them and the other was real-life problem situations similar to the ones treated in the activities. Homework was usually assigned at the end of each teaching session. In the following session, the teacher checked students' individual homework while one of the students was solving the homework on the board. If any student did not complete the homework, he/she would show his/her work to the teacher and tell her about the difficulties that prevented him/her from completely solving the homework.

When a problem is solved on the board, the teacher did not give an immediate feedback about the correctness of the solution. Rather, she opened the stage for the discussion of the presented solution and the difficulties that students faced while solving the homework based on her checking of the homework. Also, in cases where students used different approaches to solve the homework problems, she allowed the presentation of such different approaches and discussed with the students their advantages and disadvantages to adopt a solution at the end. It is worth noting that having only 13 students in the class had allowed for such discussions. If the number of students was bigger, those discussions would have been more demanding.

Technology Integration

Technology was integrated in the teaching of this unit. A laptop and an LCD were available in the class whenever it was necessary. The teacher used Excel, Graphmatica and Cabri for demonstration purposes. The teacher's choice of Graphmatica depended on two factors. First, Graphmatica is software that ~~presents~~ the symbolic, graphical and numerical representations of functions simultaneously. Second, students can download it for free from the internet and use it on their personal computers for graphing. Most computer activities were used as demonstrations and presented to the class by the teacher. Students did not have the chance to do activities in class using the computer due to time limitations.

During discussion time, the teacher used Excel in class to generate tables of values that were used for: displaying the numerical representations of the domain of definition of a function to help the students determine the domain of definition of a function, for studying the sense of variation of a function over an interval, and for comparison of functions.

Graphmatica was one of the main instructional tools used in the implementation of this unit. It was used for plotting graphs of functions and generating tables of values over a specified interval. It was used during activity time to display in a more dynamic way the same function shown on the activity sheet. The interactivity feature and the use of different colors helped students to clearly see different aspects of the problem. Graphmatica was also used during the discussion time in class to determine domain of definition and range of a function, to study the sense of variation of a function, to study the parity of a function, to study the vertical and horizontal translations of functions, and to compare functions graphically. Graphmatica helped the teacher in plotting different students' answers precisely and quickly and display these answers for discussion. Also, during some homework sessions, Graphmatica was used for displaying graphs because the graph constructed on the chalk board is not as precise as the one generated by graphing software. In addition, interactivity features of the software allowed for sparing instructional time and facilitated the process of translation among the different representations of a function that are displayed at the same time.

Cabri was used to introduce the notion of symmetry with respect to a point and with respect to a straight line. This notion of symmetry was necessary to study the parity of functions graphically.

Summary and Quizzes

Several quizzes were done during the instruction of this unit and one main test was administered at the end of the unit (Samples of quizzes are available in Appendix F). Some of the quizzes were corrected and graded in class by the students themselves (usually the correct answer was presented on the board and discussions of correctness of students' different answers took place).

Objectives of the Unit

The instructional objectives covered in this unit are the same grade 10 objectives related to the unit about functions as described in the details of content of the LMC (ECRD, 1997 b). In addition to the previously mentioned objectives, the teacher had added 11 objectives that she developed to serve the purpose of the study. Details of objectives are available in Appendix F.

Structure of the Unit

The structure of the instructional unit is summarized in Table 3-1.

Table 3-1
Description of the Instructional Unit

Session	Objectives	Class Work
1	1, 2.	Activity 1.
2	1, 2, 9	Activity 1
3	Applications of objectives 1 and 2	
4	Applications of objectives 1 and 2	Quiz
5	3, 5, 6, 7, 8, 9	Activity 2
		Graphmatica was used to display the functions given in activity 2 and to generate graphs similar to them used in the discussion.
		Activity 2
6	3, 5, 6, 7, 8, 9	Graphmatica was used to display the functions given in activity 2 and to generate graphs similar to them used in the discussion
7	Applications of objectives 3, 5, 6, 7	Graphmatica was used to display the homework correction
		Activity 3
8	4, 6, 9, 10, 11, 12, 13, 14, 15, 19, 20	Graphmatica was used to display the functions given in activity 3 and to

Session	Objectives	Class Work
8		generate graphs similar to them used in the discussion
9	4,6,9,10,11,12,13,14,15,19,20	Activity 3 Graphmatica was used to display the functions given in activity 3 and to generate graphs similar to them used in the discussion
10	Applications of objectives 4,6,9,10,11,12,13,14,15,19,20,16,17,18	Excel used for generating tables of values
11	Applications of objectives 4,6,9,10,11,12,13,14,15,16,17,18,19,20	Activity 4
12	24,25,27,28,29	Graphmatic was used to display students answers for the examples they gave
		Quiz
	24,25,27,28,29	Activity 4
13	Applications of objectives 24,25,27,28,29	Graphmatic was used to display students answers for the examples they gave

Session	Objectives	Class Work
14	Applications of objectives 24,25,27,28,29	<p>Activity 5</p> <p>Graphmatica was used to display graphs given on activity sheet and discuss them, draw students' suggested examples, generate table of values of constructed functions.</p> <p>Activity 5</p> <p>Graphmatica was used to display graphs given on activity sheet and discuss them, draw students' suggested examples, generate table of values of constructed functions.</p>
15	30,31,32,33,34,35,36,12,9,6,4	
16	30,31,32,33,34,35,36,12,9,6,4	
17	<p>Applications of objectives</p> <p>30,31,32,33,34,35,36,12,9,6,4</p>	
18	<p>Applications of objectives</p> <p>30,31,32,33,34,35,36,12,9,6,4</p>	<p>Quiz</p>

Session	Objectives	Class Work
19	21,22,26	Activity 6 Cabri was used to display graphs on activity sheets, discuss symmetry with respect to a point and with respect to a straight line.
20	Applications of objectives 21,22,26, 38	Activity 7 Graphmatica was used to generate graphs of functions and the graphs of translated functions ¹ .
21	Applications of objectives 21,22,26, 38	

Note. It is worth noting that only the part of the unit related to linear functions was considered in investigating students' problem solving strategies and representation modes.

¹ All detailed objectives of the instructional unit are provided in Appendix G

Chapter Four

Data Analysis and Results

To answer the first research question, the general objectives of the Lebanese Mathematics Curriculum, the content of the unit about functions as well as the content of the three chapters that form the unit about functions in the national mathematics textbook were analyzed.

Analysis of Lebanese Mathematics Curriculum

The latest reform of the Lebanese mathematics curriculum took place in 1997. As the curriculum text states it, the essential aim of this reform is to form a citizen who is capable of intellectual autonomy and critical thinking (ECRD, 1997 a). To achieve this aim, a new vision for mathematics was adopted. In this new vision, mathematics continued to be a field that helps the formation of mathematical reasoning, rigor, precision and objectivity, but at the same time, it has become a utilitarian discipline that aims at providing a more considerable number of citizens with the necessary knowledge and means to understand and explore the real world (ECRD, 1997 a). It has called for abolishing every theoretical overuse, and for implementing a new content of practical interest to the students (ECRD, 1997 a). To realize this new vision for mathematics, a change in the objectives, content and teaching practices was undertaken.

One of the major aspects of this change regarding the objectives and teaching practices is the shift from the traditional approach in teaching mathematics that focuses on the teacher teaching the "already made mathematics" (ECRD, 1997 a, p.288) to a new teaching approach that fosters individual construction of knowledge. This new curriculum has recommended that learner's construction of knowledge must start from a familiar real-life situation to give the students the opportunity to

see the connection between mathematics and every-day life and help them to raise questions, and pose problems (ECRD, 1997 a). As far as content is concerned, the new LMC has encouraged the choice of contents which are of practical interest to the students and that "respond to their need of formation and to their cultural development" (ECRD, 1997 a, p.288). It has also called for introducing calculators and computers in the teaching and learning of mathematics. Consequently, such changes will lead the students towards the "intelligence of conceptual models whose effectiveness will be understood by the transfer of successful teachings" (ECRD, 1997 a, p. 289).

The general philosophy of the new LMC is reflected in its five general objectives which are briefly described in the curriculum document:

1. Develop mathematical reasoning and training in the construction of arguments.
2. Solve mathematical problems.
3. Communicate mathematically.
4. Practice the scientific approach, improve research skills, and establish relations between mathematics and the surrounding reality.
5. Value mathematics.

The general philosophy and the spirit of the LMC are inspired from the NCTM vision of school mathematics (NCTM, 1989) and the general objectives of the LMC are almost the same general process standards suggested by the NCTM (NCTM, 1989). In its document, "*Curriculum and Evaluation Standards for School Mathematics*", NCTM had indeed considered problem solving, mathematical reasoning, communicating mathematically and mathematical connections as its first

four curriculum standards at all study levels (NCTM, 1989). These standards are the same four first general objectives of the LMC (ECRD, 1997).

In 1989, mathematical representations were not explicitly and separately considered as one of the curriculum standards of the NCTM (NCTM, 1989). Neither were they considered as one of the general objectives of the LMC which were developed in 1997 (ECRD, 1997 a). Few years later, the NCTM had published the "*Principles and Standards for school Mathematics*" (2000) in which it added representations as one of its process standards due to the important role that they play in the teaching and learning of mathematics. In Lebanon, there has not been any official curriculum review since 1997, and consequently, mathematical representations are still not considered as one of the explicit general objectives of the LMC.

Analysis of Content

Content is crucial to any curriculum because the reflection and realization of the general objectives takes place through content description. Following the release of the new LMC, and in 1997, the "*Details of Content*" for the first year of every cycle of the four school cycles was issued (ECRD, 1997 b). In the part of this document related to the unit about functions in grade 10 (hereafter referred to as UFG10), the LMC (ECRD, 1997 b), considers that:

The common functions form the essential object of the study of functions in first year secondary. It is preferable to apply all the rules of study first on these functions on a bounded and significant interval then on a function in general. The only functions to be studied are the ones deduced from common functions by translation or by symmetry.

The graphical representation of a function is the principal aim of the study of this function. The use of graphical calculator is desirable, in class, to control the drawing line done by the student. The use of appropriate computer program is beneficial in case of availability.

To motivate students, we have an interest to foresee real-life situations, in several domains, while avoiding possible complications in these situations.

(p.195)

The details of content of the UFG10 are divided into three main sections: (a) functions and graphical representations, (b) solving graphically equations and inequalities and (c) the study of functions (ECRD, 1997 b). In each section, there are statements of objectives to be taught and few comments related to these objectives. A copy of the grade 10 mathematics details of content related to the unit about functions is available in Appendix H.

In both the statements of objectives that describe the content to be taught in the UFG10 and the comments about these objectives, there is no indication about: the real-life situations that should be used to motivate students nor about problem solving, though the LMC considers problem solving as the "most significant activity in the teaching of mathematics" (ECRD, 1997 a, p.289).

In the comments about the objectives of the UFG10, the LMC recognizes the role of the graphical representation of a function as a tool to introduce different notions, while in the statements of objectives of UFG10, the representative curve of a function is considered as an end by itself and as tool used to recognize the parity of a function or its variation over an interval.

The translation from the symbolic representation of a function to the graphical one takes place by the translation from the symbolic to the tabular representation of a

function and then the translation from the tabular form to the graphical form through connecting all the plotted points on the system. Yet, there is no clear indication in the statements of objectives about the translation from the graphical representation of a function to its algebraic representation. However, the different representations of functions and the translation among them can be considered as means of communicating mathematically which is one of the general objectives of the LMC.

Analysis of the Chapters

Introduction

One of the goals of the latest mathematics reform movement that took place in Lebanon in 1997 was to modernize the mathematics curriculum and the methods by which it is delivered to the students. To accomplish that, new national mathematics textbooks ranging K-12 were published by the Lebanese Ministry of Education. All the public schools (supervised by the Lebanese Ministry of Education) and many of the private schools in Lebanon use these textbooks for teaching mathematics. These national mathematics textbooks are considered as one of the major means for the implementation of the LMC. The contents of these books are supposed to be aligned with the general objectives of the LMC, to cover all the specific objectives of the LMC, and to reflect the philosophy of the LMC. They are also supposed to prepare students for the national official exams. As a consequence, most of the Lebanese mathematics teachers use these textbooks as main guides for the content that they teach to their students, and deliver the contents of these books in the same way they are presented in the book. Hence, the contents and the ways these contents are presented in these textbook have their impact on students' learning.

Choice of Chapters

This section of the study focuses on the analysis of the three chapters on functions from the grade 10 national mathematics textbook *Building Up Mathematics* (hereafter referred to as BUM 10) (ECRD, 1998). The selected chapters are:

- Chapter 19: Functions.
- Chapter 20: Basic Functions.
- Chapter 21: Graphical Solutions of Equations and Inequalities.

Structure of Chapters

All the chapters in BUM 10 have the same structure. Each chapter is made of two main parts: the course part, and the exercises and problems part. The course part starts with a brief introduction that introduces the necessary prerequisites and the objectives of the chapter. It is usually followed by the *Activity* section. The number and nature of activities vary from one chapter to another depending on the objectives of each chapter. These activities are supposed to be done in group work in class.

Following the *Activity* section, there is the *Text* section that contains all the definitions, properties, theorems and their proofs. The *Focus* section follows the *Text* section. It summarizes the main points of the chapter. The last section in every chapter is the *Exercises and Problems* section that is divided into three parts. The first part of this section is devoted to exercises that are direct applications of the skills learned in the chapter. The *Exercises* section is divided into sub-sections. Each sub-section holds as a title the skill or property to be used in solving the exercises that constitute the sub-section. One of the disadvantages of titling these sub-sections is giving the student a quick access to the skill needed to solve the exercises. The second part of this section is the *Self-Evaluation* part that consists of few exercises whose answers are available at the end of the book. This section is for students to evaluate their acquisition of skills on their own. The exercises in the *Self-Evaluation*

part are not titled according to the skills needed to solve them because each exercise involves one or more skills. The third and last part of the *Exercises* and *Problems* section is the part that consists of problems. The number of problems in this section is quite less than the number of exercises, and each problem requires more than one skill to be solved.

Criteria for Analysis

Each of the chosen three chapters will be analyzed according to the following criteria:

- Modes of representation used to represent functions.
- Use of mathematical representations of functions as teaching tools.
- Use of mathematical representations of functions as ends.
- Use of real-life problem situations in the teaching of functions.

The three chapters (19, 20 and 21) are not presented as a unit in BUM10. Rather, they are presented individually. However, they can be grouped in a unit according to the sequence of their appearance in the book.

Modes of Representation Used to Represent Functions

According to the *Standards* (NCTM, 1989), there are four different modes to represent a function: graphical (curve of a function), tabular (table of input-output values), algebraic (algebraic formula and a written statement) and verbal (relation describing the variation between two variable quantities). One of the common intentions of the LMC (1987 a) is to form students who are able to use these different representational modes. In BUM 10, the graphical, tabular and algebraic representational modes of functions are used in addition to two other modes which are: "the table of variations" mode and the *two-point* mode. The table of variations is a mode in which the variations of a function are summarized symbolically and

presented in a table. The *two-point* mode is a representation form that consists of the coordinates of two points. These coordinates are either presented separately or in ordered pairs. It is only used to represent linear functions since a linear function can be defined only by using two points. This representation mode is similar to the tabular mode except that the given two values are not arranged in a table.

Modes of representations in the Activity Section

In all the activities that introduce each chapter, functions are represented algebraically with the exception of *Activity 1* in chapter 19 (Appendix I, p. 193) that introduces functions through a table of input-output values.

Verbal and graphical representations of functions do not appear at all in the activities that introduce each of the three chapters

Modes of representations in the Text Section.

In the body of the text of the three chapters (19, 20 and 21), the algebraic representations of functions is the most dominant mode. The other frequent mode that is used in the text of the three chapters with a lesser frequency than the algebraic mode is the graphical one. In addition to the algebraic and graphical representations, the tabular mode appears only in the body of the text of chapter 20 where three tabular representations of functions derived from the algebraic representations of the functions (Appendix I, pp.209, 210) are present in the text of this chapter.

As a mode for representing functions, the "table of variations" mode is used in the body of the text of chapters 19 and 20 (Appendix I, pp. 197, 208, 209, 210). This mode is not used to define a function on its own; rather, it is deduced from the algebraic or graphical representations of a function.

Modes of representations in the Exercises and Problems Section.

In the *Exercises* and *Problems* sections of the three chapters, the graphical, tabular, algebraic and "table of variations" modes are used in the given of the problems and exercises to represent functions. Counting representations for this analysis has taken place on the following basis: all different representations of functions existing in an exercise or a problem are counted separately. If an exercise involves two representations of the same type for two different functions, then the number of representations is considered as two. For instance, in *Exercises 1* in chapter 21 (Appendix I, p.223), there are two algebraic representations representing two different functions, consequently, two representations are counted. If an exercise involves two different representations of the same function, then two representations are counted as it is the case in *Exercises 5* in chapter 21 (Appendix I, p.223) where the given function is represented both algebraically and graphically.

The *two-point* mode appears only in the *Exercises* and *Problems* section of chapter 20 (Appendix I, pp. 214, 215). This mode forms 3.36% of the total representations used in the *Exercises* and *Problems* sections. The details concerning the number and types of representations used to represent functions in the *Exercises* and *Problems* section of each chapter are presented in Table 4-1.

Table 4-1

Number and Types of Representations Used in the Exercises and Problems Section

Chapter	Alg	Grp	Tpr	Tab	Tov	Total
19	39	12	-	0	1	52
20	52	3	5	-	-	60
21	23	3	-	-	-	26
Total	114	18	5	0	1	138

Note. Alg = algebraic representation; Grp = graphical representation; Tpr = two point- mode representation (used for linear functions only); Tab = tabular representation; Tov = table of variations.

Although in the *Exercises* and *Problems* section of each chapter different representations of functions are used in the given of these exercises and problems, the emphasis in this section is mostly on the algebraic mode. In the *Exercises* and *Problems* section of chapter 21, 91.3% of the given representations are algebraic. In the *Exercises* and *Problems* section of chapter 20, 86.67% of the given representations are algebraic and in the *Exercises* and *Problems* sections of chapter 19, 75 % of the given representations are algebraic. According to the results of the analysis shown in Table 4-2, it is noticed that 82.61% of the total given representations in the *Exercises* and *Problems* sections of the three chapters are algebraic. Despite that the second frequent mode in the *Exercises* and *Problems* sections of the three chapters after the algebraic one is the graphical mode, the difference between the percentage of algebraic representations and graphical representations is 69.57% which is quite a big difference.

Table 4-2

Percentages of Each Type of Representations in the Exercises and Problems Section

	Alg	Grp	Tpr	Tab	Tov
percentage	82.61	13.04	3.62	0	0.72

Note. Alg = algebraic representation; Grp = graphical representation; Tpr = two point- mode representation (used for linear functions only); Tab = tabular representation; Tov = table of variations.

The graphical mode is not equally used in all the chapters. It is used more frequently in chapter 19 than in the other chapters where the percentage of the given graphical representations in this chapter is 23.07%. This percentage decreases to become 11.53% in chapter 21 and 5% in chapter 20.

As representations of functions, the input-output tables of values are totally ignored in the *Exercises and Problems Section* and this is justified by the total absence of tables of values that define functions in this section. The case with table of variations is not a lot better since there is only a single table of variations used to define a function in *Exercise 18* in chapter 19 (Appendix I, p.202).

The previous statistics shows the dominance of the algebraic representations over all other forms of representations of functions.

Use of Mathematical Representations of Functions as Teaching Tools or as Ends

Different forms of representations of mathematical concepts and ideas have been part of school mathematics for a long period of time. However, many of these representational modes are not recognized as mathematical tools that are used to communicate mathematically and support students' understanding of concepts. Rather, they are considered as ends by themselves that have to be taught (NCTM, 2000).

In BUM10, the representational modes of functions are used as both tools and ends. Despite that the most dominant form used as a tool is the algebraic; it is rarely used as an end. This is not the case with the other representation forms that are used as tools and ends, but with different degrees.

Use of Representations in the Activity Section.

The algebraic representations of functions in all the activities are used as tools for: defining functions, studying behavior of functions, and constructing graphical and tabular representations of functions.

Being one of the objectives of each of *Activity 3* of chapter 19 (Appendix I, p. 194), *Activity 1* and *Activity 2* of chapter 20 (Appendix I, pp. 205, 206) and *Activity 1* of chapter 21 (Appendix I, p. 219), constructing a graphical representation of a

function is obtained by plotting a given number of points and joining them. *Activity 2* in chapter 19 (Appendix I, p. 194) is the only activity in the three chapters in which the construction of graphical representation of functions is not among its objectives. As a teaching tool, graphical representations of functions are not efficiently used in the *Activity* section of each of the three chapters. In *Activity 3* of chapter 19 and *Activity 2* of chapter 20, students are requested to find an element of symmetry of the curve constructed by hand using specific numerical values provided by the book. In particular, *Activity 3* of chapter 19 (Appendix I, p. 194) is considered as the students' first encounter in the mathematics curriculum with non linear functions. From previous experience in teaching these chapters, it is noticed that the first tendency that students have in such a situation is to join the plotted points by a straight line and this is one of the concerns of BUM10 because it is clearly stated in the book that "the graph of a function is not always a straight line" (ECD, 1998, p.197). A better approach would have been the study of the properties and the behavior of non linear functions starting from their given graphical representations. Such an approach familiarizes students with non-linear functions, helps them at a later stage to plot the curves of such functions by hand. A second reason that prevents students from using the curves that they had constructed by hand in the previously mentioned activities as learning tools is the lack of the exact specification of the domain of definition of the given functions. Some students tend to plot the curve of the given function and limit it only to the given points, while others tend to extend it towards infinity. In addition, having used rational numbers for plotting without specifying a scale creates differences among students' plotted curves. Consequently such curves can't be used to study symmetry correctly.

Activity 2 of chapter 20 (Appendix I, pp. 206) is also students' first encounter in the mathematics curriculum with discontinuous rational functions that have an infinite limit at a point and a finite limit at infinity. The given points in this activity that are used for plotting the curve of the given function neither show the behavior of the given function near infinity nor give a clear idea about the behavior of this function in the neighborhood of the point excluded from its domain of definition. Students' curves of such a function can't be used as a teaching tool to study symmetry or construct table of variations of the given function. A better alternative would have been studying symmetry graphically using already plotted curves by graphing utilities. In *Activity 1* of chapter 21 (Appendix I, p. 219), students are asked to use the curves of functions that they plotted to obtain approximate solutions of given equations and inequalities graphically. In such a case, students' graphs won't be accurate enough to be used for that purpose. A better approach to this issue would have been using a precise graph constructed by computer to draw information out of it and that replaces the one plotted by hand.

Use of representations in the Text Section.

The strong emphasis in the body of the text of the three chapters is on use of the algebraic representation of functions as teaching tools that help studying the properties of functions, determining domain and range of a function, creating a table of input-output values and drawing graphical representations of functions. Almost all other representations of functions are deduced from the algebraic one. This emphasis limits the role of the other modes of representations of functions as teaching tools. For instance, students' first encounters with the notions related to: the study of variations of a function, finding the extrema of function, and to the horizontal and

vertical translations of a curve of a function are all done algebraically or starting from the algebraic representation of functions (Appendix I, pp. 197, 198, 206, 207)

In the body of the text of the three chapters, tabular representations of functions are mainly created to be used as a tool for plotting curves of functions. Their role as a teaching tool to find domain of definition of a function, to study continuity, variations of function, symmetry of the graph of a function and translation of curves of functions is totally ignored as the tables on pages 201, 202, 203, 209, 210 and 216 show (Appendix I, pp. 201, 202, 203, 209, 210 & 216).

In the body of the text of the three chapters, graphical representations of functions are used as objectives and tools in each chapter. The construction of graphical representations of basic functions is among the objectives of chapters 19 and 20, consequently many of the graphs constructed in both chapters are considered as ends (Appendix I, pp 196, 198, 208, 209, 210). As teaching tools, graphical representations of functions do not play an efficient role in the teaching of functions in BUM 10 due to the following reasons: (a) their position in the text, (b) their size, (c) the ways used to present them, (d) the lack of connection among them and their algebraic and tabular representations, and (e) their deduction from their algebraic expressions.

Most of the graphical representations in the body of the text of the three chapters are presented in the margins of the pages of the text of the three chapters. The only exceptions are two graphs presented within the body of the text of chapter 19 (Appendix I, p.196) and two other graphs presented within the body of the text of chapter 20 (Appendix I, p.207). Placing the graphs in the margins of the pages diminishes the role that these graphs could possibly play in the teaching of concepts, more so because especially that all of the graphs in the margins are relatively of

small size. Most of the graphs presented in the body of the text are neither captioned nor referred to in the text. As an example, while introducing the notion of extremum of a function in chapter 19, the algebraic definition of the extremum is presented and highlighted in the middle of the page; while the graphical representation of a maximum of a function is presented on the top of the right margin of the page, without being adjacent to the algebraic representation that it represents and without any indication about the nature of the extremum in question (Appendix I, p.198). Another example from the same chapter consists of the graphs of the parabolas presented in the left margin of page 199. In the text, the parabolas used as examples have different algebraic representations than the graphical ones presented in the margin of the page (Appendix I, p.199). Such a lack of connection between the graphical and algebraic representations diminishes the role of the graphical representations as tools that display information about functions and that can't be easily seen or deduced from algebraic representations.

Correctness of graphical representations is also another issue that marginalizes the role of these representations as teaching tools. Some of the graphs of functions presented in these chapters are not drawn correctly such as the graph constructed in chapter 20 (Appendix I, p.220). According to the text adjacent to this graph, it is supposed that the curves of the functions f and g defined by $f(x) = x^2$ and $g(x) = 0.5x + 3$ should be constructed over the same interval $[-1.5, 2]$. However, the presence of the solid part of these two curves over the interval $[2, +\infty[$ contradicts the algebraic given. Such a lack of matching between the algebraic representation of a function and its graphical one implies that either the graph is wrong or it is not the graph that represents the given algebraic function. In both cases, such a graphical representation can't be considered as a teaching tool. The previously mentioned problems related to

the graphical representations of functions are not only applicable to the stated examples, but they are applicable to many of the graphical representations of functions throughout the three chapters.

The colors of the graphs, the lack of grids and scales on the graphs has also its impact on the teaching role of the graphical representations. For instance, the graph presented at the bottom of page 220 (Appendix I, p 220) is supposed to show that the graphs of two functions f and g defined over the set of positive real numbers by $f(x) = x-1$ and $g(x) = |x-1|$ are not equal if there exists at least one point of abscissa a in their domain for which $f(a)$ is not equal to $g(a)$. The graphs of these two functions are represented on the same system using close colors (black and dark blue), the common part of the two graphs appears as if it belongs to one function because it is drawn using a single color and both graphs are neither labeled nor captioned.

Another example is the graph shown at the bottom of page 221 (Appendix I, p.221).

This graph is used for solving an inequality of the form $f(x) \leq K$ where $f(x)$ is the expression of a function and K a given constant. This graph is confusing because instead of directly highlighting the points of intersection of the two curves on the system and labeling them, three additional segments are used for that purpose, and the graph of a parabola with a downward concavity is used to represent the curve of any function. In such a case, and in the lack of a graphing utility in class, a better approach for using graphical representation as a teaching tool would have been providing the students with the curves of the basic functions studied in the previous chapter to study their intersection with horizontal straight lines and with each others.

Use of representations in the Exercises and Problems Section.

To speak about the role of representations in the *Exercises and Problems* sections of the three chapters (19, 20 and 21), it is better to distinguish between the given

representations of functions and the requested representations of functions due to the differences in their roles. All the given representations of functions appear in the given of the exercises and problems. The role of these representations is restricted to being tools that solve these exercises and problems. In this analysis of chapters, if a certain given representation is used to answer more than one question in the exercise or the problem, then its role as a tool is counted once only. For instance, the algebraic representation given in *Problem 1* of chapter 19 (appendix I, p. 203) is used to prove that the given function is increasing on a given interval in the first question, and is also used to prove that the same function is decreasing over another interval. Hence, the role of this representation as a tool is counted once only and not twice. On the other hand, all the requested representations of functions are all the representations that do not appear in the given of exercises and the problems, but are answers to the given questions. These requested representations are of two types. The first type is the one that is explicitly requested in the statement of the exercise or the problem such as "*plot the graph of the given function*". The other type is the implicit one that is implied by the given of the problem. This implicit type exists in chapter 21 whose main objectives are to compare functions and to solve graphically equations and inequalities. There are 11 exercises in this chapter that refer to representations that are implicitly requested and they are counted among the requested representations. For instance, in *Exercise 13* (Appendix I, p. 224), the question states: "*solve graphically $0 < (1/x) < 3$* ". Such a demand for a graphical solution requests the construction of the graph of $(1/x)$ in the first place in order to be able to solve the exercise. Concerning the construction of the horizontal or vertical translates of graphs which is an objective of chapter 20, it is directly stated in the question to plot the graph of the translated function without any reference to the

parent graph. These translated graphs can be either constructed using a table of values or by constructing the parent graph and deducing from it the translated graph. Since there is not any indication in the exercises and problems about constructing the parent graph and deducing from it the requested curve, and since the student has the freedom to choose the construction method, then all requested representations of this type will be considered as explicitly requested representations. The only exception to this is *Problem 3* in chapter 20 (Appendix I, p. 216). In this problem, the construction of the curve of the parent function is requested in *part c* of the problem and the graphs of the translated functions are requested in *part d*.

Table 4-3

Number and Type of Given and Requested Representations in the Exercises and Problems Section

	Chapter 19		Chapter 20		Chapter 21			Total		
	GR	ERR	GR	ERR	GR	ERR	IRR	GR	ERR	IRR
Algebraic	39	1	52	9	23	1	0	114	11	0
Graphical	12	8	3	45	3	13	11	18	77	11
Tabular	0	8	0	3	0	0	0	0	11	0
TOV	1	5	0	1	0	3	0	1	9	0
TPMR	0	0	5	0	0	0	0	5	0	0
Total	52	20	60	58	26	17	11	138	95	11

Note. GR = given representation; ERR = explicitly requested representation; IRR = implicitly requested representations.

The role of requested representations can either be an objective on its own as it is the case with the graphical representation requested in *Problem 5* in chapter 20 (Appendix I, p. 216), or an objective in the first question that is used as a tool in the following question as it is the case with the tabular representation requested in the previously mentioned problem. The tabular representation is requested in *part c* of

Problem 5 and is used as a tool for plotting the curve of the given function in *part d* of the same problem. Further details concerning the number, type and role of representations are represented in Table 4-3 and Table 4-4.

Concerning the *two -point* mode representation, all representations in this mode are not requested in any of the exercises or the problems because there is not any requested representation in this form as Table 4-3 shows. Consequently all the given five representations in this mode are used as tools to write five algebraic representations of five linear functions that are defined in this mode. Further details about the use of the representations as tools are displayed in Table 4-4.

Table 4-4

Use of Representations as Tools in the Exercises and Problems Sections of the Three Chapters

	GR	RR ¹	RRAT	TRAT ²
Algebraic	114	11	2	116
Graphical	18	77	63	81
Tabular	0	11	10	10
TOV	1	9	0	1
TPMR	5	0	0	5

Note. GR = given representation; RR = requested representation; RRAT: number requested representation considered as tools; TRAT = total number of representations considered as tools

¹ The number of requested representations is the sum of the explicitly and implicitly requested representations.

² The total number of representations is the sum of numbers of representations used as tools whether given or requested.

The second less requested form of representation is the table of variations mode. Nine tables of variations are requested in the *Exercises and Problems Sections* of the three chapters. As Table 4-2 and Table 4-3 show, the table of variations mode is used only once as a tool, while it is requested as an objective nine times. In addition to

their traditional role that shows the behavior of a function, table of variations are efficient in showing the domain and range of a function, studying continuity of a function, studying signs of functions and determining number of solutions to equations. This mode is never used for such purposes in BUM 10.

The completion of 11 tables of values is requested in all the *Exercises and Problems Sections* of the three chapters. Almost all the requested table of values are used as tools that are used for plotting curves except for the requested table of values in *Exercise 5 of the Self-Evaluation* in chapter 19 (Appendix I, p. 203). This table of values is not used later in the exercise for any purpose. Limiting the role of the tables of values for graphing purposes only, deprives the students from seeing the changes in one quantity with respect to another which is an effective teaching tool because it enables the students to see this change in a quantified way.

As a requested form, the algebraic mode is equally requested as the tabular form despite that the given representations of these modes can't be compared. There are 114 given algebraic forms, while there is not any given tabular representation that defines a function. Five of the 11 requested algebraic representations are deduced from the five exercises represented in the *two-point* mode, three others are deduced from three real-life problem situations related to linear functions. This shows that finding the algebraic expression of a function is a minor objective of the three chapters and it is mainly restricted to linear functions. However, only two of these requested representations are used as tools to study the variations of functions (Appendix I, pp. 204, 216). In all the exercises and problems of the three chapters, there is not a single exercise that is presented in graphical, tabular or verbal form and requests the deduction of the algebraic form out of these modes. This limits students' ability to move flexibly among representations of functions, and as a result it would

limit students' understanding of concepts. The translation between representations in the exercises and problems of the three chapters is taking place in most of the cases from the algebraic form to the other forms. Chapters 19 and 20 would have been a natural place for presenting functions in their tabular, verbal and graphical forms and asking the students to find the corresponding algebraic form. Although 12 graphical representations are given in chapter 19, none of the questions in which these representations are involved have requested to find the algebraic expression of the represented function. None of them was used to study the parity of a function even though the parity of a function is translated graphically in terms of symmetry either with respect to the origin or with respect to the axis of ordinates. All the exercises that involved studying the parity of a given function started from the algebraic representation of a function. In chapter 20, the translation of curves always starts from the algebraic expression of a function. There is not a single exercise in this chapter where the translated graph is given graphically or in tabular form and that requests the algebraic form of the function. Putting such an emphasis on the direction of translations among representations of functions from the algebraic form to other forms prevents the students from solving problems related to functions whenever the functions are not presented in algebraic form.

By comparing the number of given representations in the *Exercises and Problems* sections of the three chapters, the dominance of the algebraic representations is clear. Out of the 138 given representations, there are 114 algebraic ones. All the given algebraic representations are used as tools to: determine the domain of definition of a function, study the variations of functions, construct tables of values, compare functions, and study the parity of functions. By comparing the number of requested representations, the dominance of the graphical representations is clear. There are 77

requested graphical representations out of which 63 are used as tools to: study variations of functions, and solve equations and inequalities graphically and these are basically the objectives of chapters 20 and 21.

Use of Real-Life Problem Situations in the teaching of Functions

The concept of functions is a mathematical concept that has wide applications and connections with real life situations, since it is one of the most popular tools used for modeling situations.

The study of functions must start informally through activities describing real world functional relationships that students frequently encounter in their world. The sub-concepts related to functions such as domain and range of a function should be a natural extension of students' initial informal experience of functions which is not the case in BUM 10. The only real-life informal functional relationship used to introduce functions is in activity 1 of chapter 19 (Appendix I, p.193). Despite the fact that *Activity 2* of chapter 19 (Appendix I, p.194) looks like a real life situation, it is rather a formal geometrical exercise rather than a real-life situation. The introduction of the concepts of the domain of definition of a function, the sense of variation of a function, the comparison of functions, and the translation of graphs of functions are introduced through pure mathematical definitions and not as a smooth extension of a real-life problem situation, as shown in *Activity 1* of chapter 20 (Appendix I, p. 205) and *Activity 1* of chapter 21 (Appendix I, p. 219).

In the body of the text of all three chapters, no real-life problem situations are used.

In all the exercises and problems of the three chapters, functions are not used as tools to model real-life situations mathematically. The only exception takes place in *Exercise 20* of chapter 19 (Appendix I, p.203), *Exercise 3* and *Problem 2* of chapter

20 (Appendix I, p.p214& p.216). Although *Problem 2* of chapter 20 is a real-life situation presented in the problem section of this chapter, it can only be categorized as an exercise represented verbally instead of algebraically like the other exercises in this chapter, and which requires a single skill to be solved.

Conclusion

The traditional method for teaching functions emphasizes the algebraic representations of functions. This method is still adopted by the BUM10 despite LMC recommendations. In fact, one of the LMC reform suggestions concerning the teaching of mathematics and which was supposed to have its benefits on students' mathematical formation is the possibility of using computers in teaching mathematics (ECRD, 1997a) and especially when representing a function graphically (ECRD, 1997 b). Such a change was also supposed to have an influence on the mathematical content to be taught (ECRD, 1997 a). However, in BUM10, the construction of a graphical representation of a function is one of the main objectives of the unit about functions that is based on the point-wise approach done by hand. Within this approach, students use the algebraic representation of a function (expression) to find a "certain number of points whose coordinates can be easily calculated" (ECRD, 1998, p. 197) and then connect them by a smooth curve. The first tendency that students have is connect these points using a straight line. As a consequence, BUM10 draws students' attention to the fact that "the graph of a function is not always a straight line" (ECRD, 1998, p.197). Such an approach is discouraged by NCTM (NCTM, 1989) and it has several disadvantages because it prevents the students from accepting the following ideas easily:

- There are discontinuous functions whose graphs are not connected at all points.

- There are non differentiable functions whose graphs are not smooth.
- There are functions that can be represented by points or segments or arcs only, without connecting them.

In addition, none of the graphical or tabular representations of functions in any of the parts of the three chapters is used to represent a real-life problem situation. Such an issue has its impact on students' problem solving abilities. Lebanese students, who have been learning math for years by the traditional method that emphasizes the algebraic representations of functions, have a natural tendency to represent any problem algebraically. Varying the modes of representations of problems helps them to use more than one mode of representations in solving problems and allows them to think of representations as tools that organize and communicate mathematical ideas to the others, and model and interpret phenomena (NCTM, 1989).

Consequently, a review of BUM10 is a must. The need for such a revision is even more imposing because it was published 12 years ago and educational approaches have significantly changed during this period due to rapid technological advancements.

Pre-test Results

The pretest was administered before the instructional unit. It was done in class. It lasted for 60 minutes and was monitored by the math teacher. Students were neither allowed to ask their teacher nor each other any question. Students were expected to write their answers in the space devoted for that purpose on the question sheet. Eight students submitted their answer sheets after 50 minutes of the test. Scientific calculators were available for students' use.

The test consists of a real-life problem situation about the cost of two vacuum cleaners and their dust bags (Appendix A). It was taken from the RITESMATH project (Rites, 2005). The first part (Part A) was kept the same as in the source, while the second one (Part B) was slightly modified. The first part was presented verbally and was divided into three sub-questions, while the second part was presented graphically and was also divided into three sub-questions. Question "a" of *Part A* (Appendix A) involved a translation from the verbal description of a real-life situation to its algebraic representation. Question "1" of *Part B* (Appendix A) involved an interpretation of the slope of two given constructed straight lines within the context of a real-life problem situation. Questions "b" of *Part A* and "2" of *Part B* (Appendix A) were about finding the coordinates of the point of intersection of two straight lines representing two linear functions within the context of a real-life problem situation. Questions "c" of *Part A* and "3" of *Part B* (Appendix A) were two optimization situations related to two offers of vacuum cleaners with their dust bags to determine which one is the best buy.

The problems in the pre-test were chosen for the following reasons:

- the chosen situations are presented within a real-life context,
- the chosen situations are familiar to the students,

- the language used for presenting the situations is simple,
- both parts are almost of the same degree of difficulty,
- both parts tackle almost the same idea represented verbally and graphically, which makes easy the comparison of the results related to the two different modes of representation,
- both parts are related to linear functions,
- both parts can be solved by more than one strategy,

All the participants in this study had learned to plot a straight line given its equation, and to write the equation of a straight line given: two points, a point and a slope or the graph of the straight line, since these skills are among the objectives of the grade nine Lebanese mathematics curriculum and since they are a part of almost every Lebanese brevet exam. However, such skills are taught in isolation from the concept of linear functions. Consequently, the pretest neither focused on the translation from the equation of a straight line to its graph nor on the translation from the tabular representation of straight line to its graph. Creating a table of values using the equation of a straight line and using such a table is considered as a part of the process of plotting a straight line, and consequently, this issue was also not addressed in the pre-test. In the pre-test, straight lines were treated as graphical representations of linear functions presented within a real-life context.

The aims of the pre-test were to assess participants':

- pre-conceptions about linear functions,
- understanding of the givens of problem situation,
- ability to interpret verbal and graphical representations of linear functions,

- ability to translate from verbal representation of linear functions to algebraic representations,
- ability to interpret graphical representation of linear functions,
- ability to solve real-life problem situation related to linear functions.

In addition, the pretest aimed to investigate the strategies as well as the representations that the participants use in solving real-life problem situations. The pre-test was marked manually by the teacher.

In order to answer the second, the third, the fourth and the fifth research questions, students' answers to the pretest were analyzed qualitatively to determine the modes of representations and translations among representations that students use in solving the problem, and to investigate students' abilities to interpret different representations of functions. Details about the criteria involved in each question of the pre-test are shown in Table 4-5.

Table 4-5

Criteria Involved in Each Question in the Pre-test

Criteria involved in the questions	Questions
Submitted work reflects understanding of the given (UG)	a,b,c,1,2,3
Submitted work reflects understanding of the verbal given (UVG)	a, b, c
Submitted work reflects understanding of the graphical given (UGG)	1, 2, 3
The given representation form is used for solving the problem (UGF)	b, c, 1, 2, 3
Adopted representation form for solving the problem guided the student towards the solution (RF→S)	b, c, 1, 2, 3
Develops an algebraic representation of the given of the problem(G→AL)	a

The qualitative results of the analysis are quantified by assigning to each criterion an index of fulfillment (CIF). This index is calculated as follows: first, it is calculated for each individual student by dividing the sum of the number of times that a given criterion is fulfilled in the corresponding questions by the number of questions in which the criterion is embedded; and then by finding the average of the individual indices of the given criterion for the whole class. If a given criteria is fulfilled for a certain question, then its score of the question is 1, otherwise it is zero. The range of the CIF is a positive number between 0 and 1. A given criterion is highly fulfilled if its CIF is close to 1. If the CIF of a given criterion is less than 0.5, then this given criterion is considered as weakly fulfilled.

Table 4-6
Pre-test Results

Criterion	Criterion Index of Fulfillment (CIF)
UG	0.45
UVG	0.45
UGG	0.45
UGR	0.27
(RF→S)	0.39
(G→AL)	0.5

Note. UG = submitted work reflects understanding of the given; UVG = submitted work reflects understanding of the verbal given; UGG = submitted work reflects understanding of the graphical given; UGR = used given representation form to solve the problem; (RF→S) = adopted representation form guided the student towards the solution; (G→AL) = develops an algebraic representation of the problem.

Understanding the given was one of the priorities of the researcher especially because the participants were learning mathematics in English, which is a second language for them. In addition, real-life problem situations are not very frequent in

the national Lebanese mathematics books that were used by the participants for learning math for several years. Students' understanding was reflected through their submitted work. For questions "a" of *Part A* and 2 of *Part B* (Appendix A), the criterion UG was considered fulfilled if the submitted answer is correct. For questions "b" and "c" of *Part A* and 1 and 3 of *Part B* (Appendix A), the criterion UG was considered fulfilled if the submitted justification to the question is correct.

According to the analysis of the pre-test, there was no single question that was understood by all the students. Understanding the given of the problem varied from one student to another, and even for the same student, it varied from one question to another. Only two participants' work reflected understanding of the given of all the questions, three participants' work reflected the lack of understanding of any question of the given, and two other participants' work reflected an understanding of the given of one question only. The pre-test results presented in Table 4-6 show that the index of fulfillment of the criterion UG was 0.45 which is a weak index.

To see the relation between the mode of representation in which the given is represented, and students' understanding of the given, students' understanding of verbal and graphical given are compared because they are the only modes used to represent the given in the test. The CIF of the criterion UVG is 0.45 and it is equal to the index of fulfillment of the criterion UGG. The equality between these two indices revealed that both graphical and verbal representations led to the same level of understanding. However, this level is a weak level of understanding of the given of the problem. It was noticed that *Question 1* of *Part B* (Appendix A) which was represented graphically was difficult to understand for the majority of the students. Figure 4-1 is one example of the submitted answers where the student was comparing the price of a vacuum cleaner with 7 dust bags to its price with 15 dust

bags. This student did not understand that the comparison has to take place between two different brands. It is worth mentioning that at the end; the student wrote "*I don't know*". Only three

Which brand has cheaper dust bags at this store? Explain how you found the answer? Wizard has cheaper dust bags.
 For 125\$ the 1st person got a vacuum & 7 dust bags
 for 150\$ he got a vacuum & 15 dust bags
 (I don't know)

Figure 4-1. A sample of student's work showing lack of understanding of the given. participants were able to understand it. The answer to this question requested the comparison of the prices of the dust bags of two different brands that were represented by the slopes of the two graphically given straight lines. The majority of the students were not able to identify the slope of the straight line when it was presented within the context of a real-life problem situation. Rather they considered the price of the vacuum machine with the dust bags as the price of the dust bags and were not able to identify that this price comprised the price of the machine itself in addition to the price of the dust bags as shown in Figure 4-2 and Figure 4-3. It is worth noting that Figure 4-3 shows the answer of one of the high achievers in math in this class.

The Wizard has cheaper dust bags.
 because you can get 15 bags for 150\$ (Wizard)
 but 15 bags for more than 150\$ (Miecle)

Figure 4-2. A sample of a student's answer (Question 1 of Part B).

Figure 4-3. A sample of a student's answer (Question 1 of Part B).

The analysis of the pre-test also revealed that understanding of *Question c* of *Part A* (Appendix A), whose given was represented verbally, was difficult for many students. Out of the 14 students, only 5 were able to understand it. A sample of students' answers to this question is shown in Figure 4-4.

Figure 4-4. A sample of a student's answer (Question "c" of Part A).

Correctness of solutions was considered as an indicator of students' problem solving ability. In this analysis, an answer to a given question was considered correct if it had an accompanying correct justification or explanation. The only exception to this are question "a" of *Part A* and question 3 of *Part B* that can be answered directly from the given of the question. Students' weak ability to understand and interpret the given of the problem was reflected on their ability to solve the problems. For instance, in *Question 1* of *Part B* (Appendix A), out of the three students who understood the question, only two were able to solve it correctly as shown in Table

4-6. Another example is Question "a" of *Part A* (Appendix A) which required finding an algebraic expression that represents the given verbal situation. Although the participants had mastered in grade 9 writing the equation of a straight line given two of its points or a point and a slope, 7 out of the 14 students were not able to write this equation correctly when it was represented verbally within the context of a real-life problem situation as shown in Figure 4-5. According to Table 4-7, question "c" of *Part A* is the second least understood question among all the questions. This is also confirmed by the qualitative analysis of the pre-test.

Figure 4-5. A student's answer (Question "a" of *Part A*).

Table 4-7
 Number of Correct Answers to Each Question in the Pre-test

Question	a	b	c	1	2	3
Number of correct answers ¹	7	7	5	2	9	7

Note.¹ Total number of students is 14.

Concerning the analysis of types of representations used in solving the problems, the following procedure was adopted. An answer is considered represented in algebraic form if the justification and explanation of the answer are represented by equations or using symbols. An answer is considered represented in verbal form if the justification and explanation are written verbally. An answer is considered represented in numerical form if the justification and explanation are expressed by numbers or involved calculation of numbers. An answer is considered represented in

graphical form if the justification and explanation are represented by graphs. This classification of representations was applied to all the questions in the pre-test except for question "a" of *Part A* in which the algebraic mode was imposed on the students and question "2" of *Part B* whose answers were numbers deduced from a direct graphical reading.

According to the pre-defined criteria about representation forms, students' answers in the pre-test were presented in algebraic, verbal or numerical forms. Graphical forms were not used at all. The number and types of representations that students used in the pre-test are presented in Table 4-8.

Table 4-8
Types and Number of Representations Used in Students' Answers in the Pre-test

Type of representation	Number of representations
algebraic	12
verbal	18
numerical	12

Concerning students' adoption of the representation form in which the question is represented for answering the question, the CIF of the UGR was 0.27 which is a very weak index of fulfillment. This index shows that a minority of students used the given representation form for answering, while the majority of them had translated the given form of representation into another form and used it to solve the problem. Question "c" of *Part A* that involved an optimization situation presented verbally was difficult for many students to answer. Six students were not able to translate the given of this question to any other mode and failed to understand and consequently solve this question as shown in Figure 4-6. Out of the six students that used the given verbal form to answer the question, three answered it correctly. By comparing this

optimization situation to the other one that is represented graphically in Question 3 of *Part B*, a similar case appears where five students did not try to solve this question and seven out of the nine students who translated the graphical given into verbal, algebraic and numerical had succeeded in solving the question as shown in Figure 4-7.

If a friend wanted to buy either a MIRACLE or a WIZARD vacuum cleaner from this store, which one would you suggest they buy? Explain your answer.

A wizard because the wizard is cheaper and you can buy extra bags than the miracle.

Figure 4-6. A sample of students' answer to question "c" of *Part A*.

In one year, Janila needs 12 dust bags.

$\Rightarrow$ Wizard: $y = \frac{3}{5}(12) + 124 = 7.2 + 124 = 131.2 \$$.

Miracle: $y = 6(12) + 90 = \$162$

So, it's cheaper for her to buy the Wizard.

Figure 4-7. Sample of translations done in Question 3 of *Part B*.

This shows the necessity of teaching mathematics with multiple representations with an emphasis on the flexible translations among the different representations. Such an approach in teaching mathematics will improve students' problem solving abilities due to the effect that the types of representation that students use in problem solving on the students' problem solving ability.

By considering the total number of answers to questions *b*, *c* 1, and 3 which is 56 = 4 questions multiplied by 14 students, it is noticed that 13 answers were blank as shown in Table 4-9 which means that the students neither understood the given of

Table 4-9

The Number of Representations Used in the Given of Questions b, c 1, and 3 and in the Answers to These Questions

Question	RMG	Blank answer	Representations used in students' answers		Total
			URMG	DURMG	
b	verbal	2	0	12 (9 alg, 3 num)	14
c	verbal	6	6	2(2 num)	14
1	graphical	2	1	11(7 vb, 1 alg,3 num)	14
3	graphical	3	0	11 (5 vb, 2 alg, 4 num)	14
Total		13	7	36 ¹	56 ²

Note. RMG = representation mode of the given; URMG = used representation mode of the given to answer the question; DURMG = did not use the representation mode of the given for answering; vb = verbal representation; alg = algebraic representation; num = numerical representation. ¹36 is the total number of answers in which the students did not use the given representation form for answering. Hence it represents the number of translations that took place. ²56 is the total number of answers and it is obtained by multiplying the number of questions by the number of students (4 x 14).

these questions nor were they able to translate this given to any other mode to solve the problem. Out of the remaining 43 answers, there were 7 answers in which the students had used the same mode of representation as the one used in the given of the corresponding question, and 36 answers in which the representations used for answering differed from the ones used in the given of the corresponding questions. This justifies the weakness of the CIF of the criterion UGR and shows at the same time that there are 36 translations that took place.

The types and directions of translations that took place from the given representation form into other forms is presented in Table 4-10. It is noticed that the most frequent translation is the one that takes place from the given in graphical form

to the verbal form where 12 translations of this type took place, while the least frequent translation is the one that took place from the given in graphical form to the algebraic form as shown in Table 4-10.

Table 4-10

Type and Direction of Translations That Took Place From Given Representation Mode to Other Modes in the Pre-test

	(vb→alg)	(vb→num)	(grp→alg)	(grp→num)	(grp→vb)	Total
Number	9	5	3	7	12	36

Note. (vb→alg) = translation of given in verbal form to algebraic form; (vb→num) = translation of given in verbal form to numerical form; (grp→alg) = translation of given in graphical form to algebraic form; (grp→num) = translation of given in graphical form to numerical form; (grp→vb) = translation of given in graphical form to verbal form.

The effect of the mode of representation that students used in solving the problem on the process of problem solving is also studied. If the mode of representation that the student uses has guided the student towards the solution of the problem, then the criterion (RF→S) is considered as fulfilled even though the final answer might not be correct as shown in Figure 4-8.

$$(M): y = 5x + 80 = 5(12) + 80$$

$$= \cancel{5(5) + 80} = 140 \$$$

$$(W): y = \frac{25 \times (12)}{13} + 120 = \cancel{113.07} \quad \frac{300 + 1560}{13}$$

$$= \frac{1860}{13} = 143.07 \$$$

 The miracle is better since it's cheaper

Figure 4-8. A student's answer showing a wrong answer due to a calculation mistakes despite the correctness of the solving process (Question 3 Part B).

The CIF of (RF→S) is 0.39 which is a weak index. This shows that despite the translations that the students performed, many of them were still unable to solve the

problem. For instance, in Question "b" of Part A, nine translations were performed from the verbal form to the algebraic form, but only five of them guided the students towards the solution as shown in Figure 4-9. Another example is the seven translations that students performed from the graphical form to the verbal form in Question 1 of Part B. Only two of these representations helped the students to solve the problem. An exception to the previously mentioned translations was the translation of the verbal given into numerical form. Out of the performed five translations, four helped the students to get the right answer. However, that was not the case of the seven translations that took place from the graphical mode to the numerical mode in Part B where three of them only had helped the students to get the right answer as shown in Figure 4-9.

Both paid the same amount of money and got the same number of dust bags.

How many dust bags did each get?

$$X + Y = Z$$

$$H + Y = Z$$

$$X + Y = H + Y$$

$$X + H = Z$$

$$X + H = 2Y$$

$$246 = 2Y$$

$$Y = 123$$

$Z = \text{total cost}$
 $X = \text{Mircea}$
 $Y = \text{dust bags}$
 $H = \text{Wenzel}$

Figure 4-9. A student's answer showing inefficiency of algebraic form (Question b Part A).

The relation between the types of representation that students use for answering and the strategy used to solve the problem is also investigated in this analysis. It was found that with 9 out of the 12 algebraic representations used, *solving a system of two equations in two unknowns* was the strategy used as shown in Figure 4-9. The use of numerical representation was associated with the *guess-and-check* strategy as shown in Figure 4-10. The *guess-and-check strategy* was used with 6 numerical representations out of 12.

Figure 4-10. A student's answer showing the use of the guess-and-check strategy with the numerical mode (Question *b Part A*).

Logical reasoning was the main strategy used the 17 verbal representations as shown in Figure 4-11. Consequently, the type of representation used for answering has its impact on problem solving strategies.

Figure 4-11. A student's answer showing the use of logical reasoning as a strategy with the verbal mode (Question *c Part A*).

As a conclusion, it is found that the participants had difficulty in identifying mathematical concepts and ideas whenever they are represented within the context of real-life problem situations. They also faced difficulties in interpreting graphical and verbal representations representing linear functions within the context of a real-life problem situation. Seven out of the 14 students failed to translate the verbal given of a situation into an algebraic expression. Many of them failed to solve certain questions because they were unable to translate the given of the question into another mode that they can use. One of the unexpected behaviors of these students was that despite all the training they received in Grade 9 in drawing a straight line given its

equation, the majority did not try to translate the graphical representation of a straight line into algebraic and use it for solving the problem. Rather, the major translation that took place from the graphical mode was into the verbal mode.

Analysis of Selected Assessment

Selection and Presentation of Assessments

Several quizzes were administered during the teaching of the instructional unit and one major test was administered upon its completion. The questions in these quizzes and in the test were of two types. The first type gave an immediate feedback about students' acquisition of the taught mathematical skills and assessed students' knowledge about different types of functions (polynomial functions, square root functions and rational functions). The second type involved solving real-life problem situations related to linear functions. Samples of the quizzes and a part of the test are provided in Appendix F, while the selected questions to be analyzed to answer the research questions of this study are available in Appendix B and Appendix C, and are arranged in the chronological order in which they were given to the students. The sample of analyzed problems includes only assessments, not in class activities or homework.

Four questions involving different types of representations were selected for analysis. The first question (Appendix B) requires the completion of a table of values that represents a linear function using the verbal given of the problem. This table of values is then used to find the algebraic expression of the function. It is a part of a quiz administered after completing objectives 1, 2, 3, 4, 6, 10, and 11 (Appendix G). Its degree of difficulty is low. It took the students about 10 minutes to complete it. At this level, students hadn't been introduced to the concept of function yet. All the work done at that stage focused on exploring functions informally within the context

of real-life situations that are familiar to the students. The representations involved in this question are the tabular and the algebraic representations.

The second question (Appendix B) involves a given verbal representation of piecewise function describing the temperature of an oven during baking that students have to represent graphically. It is a part of a quiz administered after completing objectives 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 27, 28, 29 and 37 (Appendix G). It took the students about 20 minutes to complete it. Some students submitted their work after 15 minutes. At this level of the course, students were introduced to the notion of function, had studied variations of functions using different representation forms and had experienced the translation from graphical to verbal forms, the translation from algebraic to graphical form, as well as the translation from numerical to algebraic and from numerical to graphical form. All the previously mentioned translations took place in both directions. Also, students were introduced to piecewise functions. The representations involved in this question are the verbal and the graphical and the students' task is to translate the verbal representation to a graphical one.

The third question (Appendix B) is a problem represented graphically that shows the savings of one person and the expenses of another one during the period of one month. It is a part of a quiz administered after the completion of all the objectives from 1 to 37 (Appendix G). Some students started submitting their papers after 25 minutes. The given functions in this problem are: a strictly decreasing linear function and a piecewise linear function composed of three parts two of which are strictly increasing, while the third is constant (Appendix B). In this question, and as in the pre-test, students had to study and compare the variations of linear functions presented within the context of a real problem situation. In addition, students had to

solve the equality between two functions, and interpret the meaning of the intercepts of the graphs in order to be able to answer the given questions. Solving this problem is based on the interpretation of the given that was represented graphically.

The fourth question (Appendix C) is an optimization situation involving three linear functions, one of which is a piecewise function. It is presented verbally as in the pre-test. However, it is of a higher degree of difficulty than the one presented in the pre-test. It took the students about 40 minutes to complete it and it was done after the completion of the unit.

The first three questions used for analysis are developed by the researcher, while the fourth problem was taken from E. Phillips (personal communication, June, 2006) who sent it to the researcher in a private e-mail with permission to use it.

All selected four questions are real-life problem situations related to linear functions and are familiar to the students. Questions III (Appendix B) and IV (Appendix C) are similar to the questions given in the pre-test, but in some of their parts, they are of a higher degree of difficulty than those given in the pre-test. In addition, students had the chance to use more than one strategy and more than one mode of representation to solve these two questions.

The aims of the selected assessments were to assess participants':

- understanding of the givens of problem situation,
- ability to interpret different representations of linear functions,
- ability to translate from one representation of linear function to another
- ability to solve real-life problem situation related to linear functions.

In addition, the selected assessment investigated the strategies that the participants used in solving real-life problem situations and the role that the different representations played in solving problems related to linear functions.

Analysis of students' work according to the criteria

Table 4-11

Criteria involved in each question of the selected assessments

Criteria involved in the question	Question
Submitted work reflects understanding of the given (UG)	All questions
Submitted work reflects understanding of the verbal given (UVG)	I-a, II-a, II-b, IV-A, IV-B, IV-C
Submitted work reflects understanding of the graphical given (UGG)	All parts of question III
The given representation form is used for solving the problem (UGR)	I-c, II(a,b) III(1,2,3,4,5,6,7), IV(a,b,c)
Adopted representation form for solving the problem guided the student towards the solution (RF→S)	All questions
Develops an algebraic representation of the given of the problem (G→AL)	I-c
Develops a tabular representation of the given of the problem (G→TAB)	I-b
Develops a graphical representation of the given of the problem (G→GRP)	II-a

The selected assessments were marked manually by the teacher. Non-programmable scientific calculators were allowed during the administration of these questions.

Students' answers to the selected assessment are analyzed qualitatively in the same manner as the pre-test following the same criteria used for analyzing the pre-test. Two additional criteria related to graphical and tabular representations are used in the analysis of the selected assessments due to the use of more representational modes in these assessments than in the pre-test. Details about the criteria involved in each question are presented in Table 4-11.

In the first two parts of Question I (Appendix B), the criterion (UGR) is not involved because the students had to use the given representation form to solve the problem.

According to Table 4-12, students' ability to understand and interpret the given of a real-life problem regardless of its mode of representation has improved as its index of fulfillment was 0.91 which is a high index. In general, almost all the students were capable of understanding all the questions. There was not a single question in all the selected assessments that was not understood by the participants. The mode of representation of the given did not affect students' ability to understand and interpret the given since both students' ability to understand the verbal given and students' ability to understand the graphical given had almost equal high indices as shown in Table 4-12.

The verbal given of Question I (Appendix B) is simple and was understood by all the students. The verbal given of question II was understood almost by all the students except by students S8 and S9. Also the verbal given of Question IV part C (Appendix C) was understood by all the students except students S8 and S9. Figure 4-12 shows that although S9 understood the mathematical concepts, still this student faced difficulty in the language of the problem. For instance, S9 assumed that the time elapsed between heating the oven and taking the cakes out of it is 35 minutes,

while the right interpretation implied that the cake was left in the oven for 35 minutes. In fact, S9 had complained in several occasions about real-life situations because his English was weak and he was not able to understand and interpret the given of the problem that was represented verbally.

Table 4-12
Results of selected assessments

Criterion	Criterion index of fulfillment (CIF)
UG	0.91
UVG	0.90
UGG	0.908
UGR	0.436
(RF→S)	0.906
(G→AL)	1
(G→GRP)	0.87
(G→TAB)	1

Note. UG = submitted work reflects understanding of the given; UVG = submitted work reflects understanding of the verbal given; UGG = submitted work reflects understanding of the graphical given; UGR = used given representation form to solve the problem; (RF→S) = adopted representation form guided the student towards the solution; (G→AL) = develops an algebraic representation of the problem; (G→GRP) = develops a graphical representation of the problem; (G→TAB) = develops a tabular representation of the problem

The problem with S8 was different. S8 did not have difficulties in the English language as S9 did, but always had an incomplete interpretation of the given of the problem as shown in Figure 4-13 that shows S8's answer to Question IV part C. In this part, S8 did not interpret the whole given of the problem.

Figure 4-12. An answer to Question II revealing a problem in understanding the English language.

As mentioned in the pre-test analysis, correctness of submitted answers has a relation with students' problem solving abilities, which in turn is related to students' understanding of the problem. Table 4-13 shows the percentage of correct answers in each question.

Figure 4-13. Student's answer showing inability to interpret the given of the problem.

Table 4-13

Percentages of correct answers to each question in the selected assessments

Question	I	II	III	IV
% of correct answers	100	87	88	90.4

Although Question IV (Appendix C) is an optimization problem of a higher degree of difficulty than the one presented in the pre-test, 90.4% of the answers submitted to this question are correct. The high percentages shown in Table 4-13 are indicators of improvement of the participants' problem solving abilities which is related to the types of representation that these participants used to answer the selected assessments. The participants showed ability to use graphical, algebraic, verbal and numerical representations as tools for solving problems and to translate flexibly among these representations.

Types of representations used by students

Figure 4-14. Sample of a student's answer showing the use of graphical representation to solve the optimization situation (Question IV part C).

Graphical representations are used for different purposes in these selected assessments. They are used to solve the optimization situation presented in Question IV (Appendix C). Five students solved this question by modeling the situation graphically as seen in Figure 4-14.

Graphical representations are also used to determine the time that Raji needed to spend all his money in Question III -3 (Appendix B). Eight students reproduced the given graphs on their answer sheets in order to find the x-intercept of the straight line representing Raji's expenses as seen in Figure 4-15. Graphical representations were used by two students to solve Question II-a. They were also used to solve many parts of Question III such as parts 5, 6 and 7 as seen in Figures 4-16 where the students compared the variations of two functions using slopes or steepness of straight lines or their slopes.

Figure 4-15. Sample of a student's answer showing the use of graphical

representation in solving problems. (Question III, part 3 Appendix B).

Draw $(BC) \parallel (OA)$, (BD) is steeper than (BC) . $a_{BD} > a_{BC}$. I mean, that Sami saves money more from day 20 to day 30.

$$a_{DE} = -20 \text{ and } a_{0B} = +20$$

$$a_{0B} = a_{0H}$$

Sami saved the same amount daily as the one Raji spent per day from day 0 till day 15.

Figure 4-16. Students' answers showing the use of graphical representation to solve the Question III(5, 7) (Appendix B).

Numerical representations are also used in Questions II, III and IV. In Question II (Appendix B), many students translated the verbal given to a numerical one and used the numerical representation for plotting the required curve and answering part "a" of this question as seen in Figure 4-17.

In Question III (Appendix B), numerical representations are used for answering many parts of this question such as parts 3, 4, 5 and as seen in Figure 4-18. Two students answered part 3 using numerical representations, 4 students answered part 4 using numerical representations, 10 students answered part 5 using numerical representations, and 10 students answered part 7 using numerical representations. In Question IV (Appendix C), numerical representations were used by: 6 students to solve part A of this question, 14 students to solve part B and by 6 students to solve part C of this question as seen in Figure 4-19.

Figure 4-17. A student's answer showing the use of numerical representation to solve the Question II (a, b).

t	m
0	0
5	100
10	200
15	300
20	300
30	600

Same neither option is best for $t > 15$ and $t < 20$ because money did not change

Figure 4-18. A student's answer showing the use of numerical representation to solve sense of variation of a function (Question III-4, Appendix B).

* Less than 30 calls:

Option I: \$10.00 for someone who make less than 30 calls must
 Option II: \$30.00 for
 Option III: \$18.00

If someone makes less than 30 calls Option I is the best for him to choose.

* 60 calls & more:

(60 calls)

Option I: $\$10 + 0.2(30) = \16 (min)
 Option II: \$30.00
 Option III: \$18.00 (min)

* more than 60 calls

~~$\$10 + 0.2(30) = \16~~ Option I: (70 calls) $\$10 + 0.2(40) = \18
 Option II: \$30
 Option III: $\$18 + 0.05(4) = \18.05 \$18.5

* (80 calls):

Option I: $\$10 + 0.2(50) = \20
 Option II: \$30
 Option III: $\$18 + 0.05(20) = \19

* (90 calls):

Option I: $\$10 + 0.2(60) = \22
 Option II: \$30
 Option III: $\$18 + 0.05(30) = \19.5

* (100 calls): Done before

Option III is the best $\Rightarrow$ \$20

If a person makes 60-70 calls Option I is the best
 If a person makes 80-90 calls Option II is the best
 If a person make 200 calls Option III is the best
 Unlimited = \$30.

Figure 4-19. Students' answers showing the use of numerical representations to solve the optimization situation in Question IV (Appendix C)

Verbal representations were used in Question III part 4, and Question IV parts and C and some parts of Question II to analyze the given of these questions and submit an answer as seen in Figure 4-20.

the slope of a horizontal line is zero, means rate of change is zero when $15 \leq t \leq 20$. then Sam ~~spent~~ ^{neither} saved any money for 5 days

Figure 4-20. A student's answer showing the use of verbal representation (Question III-3, Appendix B).

Algebraic representation was basically used as a tool by 5 students to find the equation of the first part of the graph given in Question II and by two students to solve *part a* of the same question. One student had used algebraic representation to solve parts 3, 5 and 7 of Question III and another student had used it to solve part 3 of the same question as seen in Figure4-21.

3) m = Raji's money

t = time

$$y = at + b \quad b = y - \text{intercept} = 900$$

$$a = \text{slope} = \frac{\text{rise}}{\text{run}} = -20$$

$$m = 20t + 900$$

$$0 = -20t + 900$$

$$t = 45 \text{ days}$$

after 45 days, Raji will not have any money.

4) Sami neither spent nor saved any money means money

is constant. Five days, from $t = 15$ till $t = 20$.

$$5) m_r = -20t_r + 900$$

-20 means spends 20 dollars daily

- means lots of money

$$m_s = 100t_s + b$$

$$m_s = 20t + b$$

$$m_s = 20t$$

$$0 \leq t \leq 15$$

Sami saves 20 dollars per day if $0 \leq t \leq 15$

Figure 4-21. A student answer to Question III parts 3 and 4 using algebraic representation.

As mentioned earlier, the algebraic, tabular and graphical modes that are requested in Questions I and II are not taken in consideration while counting the

representations that students used for answering the questions. However, each representation that the students used to construct a part of the piecewise function defined in Question II is counted. For constructing the graph of Question II that consisted of 5 pieces, many students translated the given of this question into other representation modes and used them to plot the graph. In many of the cases, the same student translated the verbal given associated to two different pieces of the graph of this function into two different intermediate representation modes. For this reason, the representation used to construct each part of the graph is counted independently. Consequently, a total of 238 answers were counted and consequently 238 representations are associated with the selected assessments out of which 9 are left unanswered by S8 and S9. The most dominant representation mode used by the students is the numerical one followed by the graphical mode as Table 4-14 shows. The algebraic mode was the least used among all other forms.

Table 4-14

Types and number of representations used in students' answers in the selected assessments

Type of representation	Number of representations
algebraic	17
verbal	31
graphical	69
numerical	98

Concerning students' adoption of the representation form in which the question is represented for answering the question, the CIF of the UGR was 0.43 which is a weak index of fulfillment showing that many translations from the given representational mode to other modes had taken place. The number of given

representations that are adopted by students to answer Questions II, III and IV as well as the number and direction of translations that took place from the given representation mode to other modes is shown in Table 4-15.

Table 4-15

The Number of Representations Used in the Given of Questions II, III and IV of the selected assessments and in the Answers to These Questions

Question	RMG	Representations used in students' answers			Total
		Blank answer	URMG	DURMG	
II	verbal	4	20	60(12 alg, 46 num, 2 gr)	84
III	graphical	5	62	31(5 alg, 26 num)	98
IV	verbal	0	11	31(5 gr, 26 num)	42
Total		9	93	122 ¹	224 ²

Note. RMG = representation mode of the given; URMG = used representation mode of the given to answer the question; DURMG = did not use the representation mode of the given for answering; vb = verbal representation; alg = algebraic representation; num = numerical representation.¹122 is the total number of answers in which the students did not use the given representation form for answering. Hence it represents the number of translations that took place.²224 is the total number of counted answers and it is obtained by multiplying the number of counted parts of the questions by the number of students (16 x 14).

Translations Among Representations

122 translations took place. The majority of these translations was related to Questions II and IV and took place from verbal form to numerical form. This result shows students' tendency to translate the verbal given in the first place to a numerical one. However, this is not the case with Question III where 63 out of the 98 representations related to this question were in the same mode as the given graphical

representation mode. These results not show only the familiarity of the students with the graphical mode, but their ability to use this mode in solving problems.

Concerning the translation from verbal mode to the tabular mode that took place in Question I and was not mentioned in Table 4-15, it was accomplished correctly by all the participants. Also, all the students, without any exception, were able to develop correctly an algebraic representation of the given situation of Question I –c using the tabular representation developed in Question I-b. Moreover, six students out of the 14 had defined the variables that they used in developing the requested algebraic expression as shown in Figure 4-22.

Figure 4-22. A sample of a student's answer to Question I-c.

Defining the variables in the algebraic expression indicates that students were thinking of the algebraic expression as an algebraic representation of a function and not only as an equation or formula. The first tendency that students usually have is to represent the independent variable by x and the dependent variable by y . This is confirmed by the tendency that this student had to use x for the independent variable, but he had changed it later to n as shown in Figure 4-22.

Unlike the pre-test where students were not able to solve certain questions because they were not capable of translating the given of the question to another representation form that they could have used, there was no single question in these selected assessments that the students did not solve due to lack of ability to translate into another representational mode. In addition, most of the representations adopted for answering the questions of the selected assessments helped the students to reach

a correct solution. This is confirmed by the high index of fulfillment of (RF→S) which was equal to 0.9. For instance, in Question II, 4 answers were blank and out of the 80 representations used for answering, 74 led the students to the correct answers. The numerical form was used as an intermediate form between the verbal form and the graphical form as seen in Figure 4-23.

over at ~~Room~~ Room temp = 25°

$$\begin{array}{l}
 16 \longrightarrow 10 \\
 x \longrightarrow 10 \\
 25 \longrightarrow 160 \xrightarrow{\text{same}} 160^T \longrightarrow 300 \longrightarrow 300 \\
 0 \xrightarrow{+10} 10 \xrightarrow{+20} 30 \xrightarrow{+10} 40 \xrightarrow{+35} 75 \\
 \hline
 300 \longrightarrow 25 \quad 82.25 \\
 75 \longrightarrow x \leq 82.25 \\
 \quad + 8.25 \\
 x \text{ of } 300 = 75 \times 25 \\
 x = \frac{25}{75} \times \frac{1}{8.25} = 57.75 \\
 \hline
 \text{300} \\
 \hline
 \end{array}$$

Figure 4-23. A student answer showing the process for solving Question II using numerical representation.

It is also worth noting that in Question IV, 5 translations from the verbal to the graphical mode took place and helped the students to solve the situation correctly, as was seen in Figure 4-14. The translation from the verbal form to the algebraic form in Question IV was not successful because the students who adopted the algebraic form to solve part C of this question changed it to the numerical form almost in the middle of the process of solving or even earlier as seen in Figure 4-24.

c Let x be the n^o of calls made
add. to the

$$\$10 + 0.2x = 18 + 0.05x$$

$$-8 = -0.15x$$

$$\Rightarrow x = \frac{-8}{-0.15} = 53.3$$

So if ~~the~~ ~~the~~ people speak
53 calls, option 1 and option ~~2~~ 3
are the same.

If the n^o of additional calls is 55,

In option 1 : $10 + 0.2 \times 55 = \$21$ ~~cents~~

In option 3: $18 + 0.05 \times 55$
 $= \$20.75$

So option 3 is better than option 1
if the n^o of additional calls
is greater than 53.

If the n^o of additional calls is 50.

In option 1 : $10 + 0.2 \times 50 = \$20$

Figure 4-24. A student answer showing the shift from the algebraic mode to the numerical mode in the middle of the solving process (Question IV-C, Appendix C).

Also such a situation took place while solving part b of Question II. Six students had started solving the question using the translation from the verbal mode to the algebraic mode and used it to graph the first part of the required piecewise function, but they had changed the translation to the numerical mode starting from the second part of the graph to be constructed.

Even when no translations took place, students were able to use the given representation mode to answer the question, especially when it the question is represented graphically. In part c of Question III, 8 students had reproduced the given graph on their answer sheets and extended the line that represented Raji's expenses to find the x-intercept of the graph which represented the number of days that Raji needed to spend all his money as was seen in Figure 4-15. Also, to compare Sami's savings, two students had constructed a semi straight line parallel to the first part of the graph starting at the point (24, 300) and compared its steepness to the last part of the graph and drew out correct conclusions as was seen in Figure 4-16.

Concerning students' ability to develop an algebraic representation or a tabular representation of the verbal given, all the students had succeeded in doing that. The CIF of $(G \rightarrow AL)$ and $(G \rightarrow GRP)$ were both 1. Although not all the students succeeded in developing a complete correct graphical representation of the given of Question II especially S8 and S9, its index of fulfillment was 0.87 which is a high index. There were another two students who failed to draw the last part of the graph correctly because they did not pay attention to the fact that cooling the oven meant a decrease in its temperature and not an increase. Instead of constructing a decreasing function to represent the cooling, they had constructed an increasing one as seen in Figure 4-25.

Figure 4-25 A student's answer showing a mistake only in the last part of the graph requested in Question II (Appendix B).

Solving strategies as related to types of representations

There was a number of strategies used for solving the questions of the selected assessments. It was noticed that the strategies used were related to the representation form used for answering and to the given of the problem. For instance, in Question IV, *substituting numerical values* to calculate the answer was a strategy adopted with 20 numerical representations out of 26. It was the main strategy adopted to solve Question IV (Appendix C). This was not the case in Question II part a where *solving a proportion* was the major strategy used with 8 numerical representations out of 12. In Question III, *comparison of slopes* was the main strategy associated with numerical representations where calculation of slopes using the graph took place as a first step and the comparison happened later.

Logical reasoning was the strategy used with the verbal representations that students used to answer question IV, while the *interpretation* of either the verbal

given or the graphical given were the strategies that students used when their answers were submitted verbally.

The strategies used with graphical representation differed from one question to another depending on the nature of the question. For instance, in Question IV part C, the strategy that students used was the *comparison of the values* of different functions and *drawing out conclusions* based on the graphical representations. In Question III part 3 (Appendix B), the strategy that students used was the *extension of a straight line* to find its x-intercept. To calculate rates of change, students *calculated the slope of a straight line* as rise/run from the graph and then *compared* the obtained numerical values. Another strategy used for calculating rates of change was to *draw from the same point parallels to the given lines* and study their steepness. In Question II-a (Appendix B), two students *constructed the graph* of the requested function as a first step, and then *used it* to find its intersection with horizontal lines.

The strategies used with algebraic representations that students used for answering varied also from one problem to another depending on the given of the problem. In Question II (Appendix B), *developing an algebraic representation* of the given and *use it to plot* the graph was mainly associated to the first part of the graph, and *substituting x by 10* was the strategy used for calculating the temperature T . In Question III (Appendix B), one student *found slopes of straight lines* using the equation of the straight line, and another student *found the x-intercept of a straight line* by setting $y = 0$ in the algebraic equation of the straight line.

Summary

Briefing the selected assessment results, conclusions can be drawn about students' ability to use multiple representations as tools and as ends within the context of real-

life problem situations. Students were able to understand and interpret the different forms of representations which led them to perform successful translations from the given representation mode to other representation modes. By translating the given of a real-life question into a representation mode of their own choice, the students' capabilities to solve the given situation had improved. Also, they were able to represent real-life situations in tabular form and graphical form. The most dominant form that students used for answering was the numerical mode, and the most dominant form that students preferred to translate it to other modes before solving the problem was the verbal mode. The least representation mode that was used as a tool to solve problems was the algebraic mode. Finally, different strategies were associated to the same representation mode depending on the given of the problem.

Chapter 5

*Discussion and Concluding Remarks**Summary*

The purpose of this study was to I develop and pilot an instructional unit that multiple representations of functions in Grade 10 Lebanese program. In addition, it aimed investigating the types of representations that Grade 10 Lebanese students used while solving real-life problem situations related to linear functions during and after the completion of the instructional unit about functions. This unit's objectives are a modified version of the objectives of the grade 10 mathematics curriculum related to functions. However, all the content of the unit is developed by the researcher who also is the participants' of mathematics. In this unit, there is a main focus on the use multiple representations as tools to solve real-life problem situations related to functions in general and to linear functions in particular. To address the first research question, the philosophy and the general objectives of the Lebanese mathematics curriculum, the objectives of the grade 10 Lebanese mathematics curriculum related to functions, and the content of three chapters that constitute that unit about functions, from the grade 10 national mathematics textbook, were analyzed. To answer the second, the third, the fourth and the fifth research questions, data were collected using a pre-test administered before the implementation of the instructional unit, and a series of four assessment tools that were selected from the quizzes and test administered during and after the completion of the instructional unit. Students' work was analyzed qualitatively using simple statistics.

The results of the pre-showed that the main representation form that the students used for solving problems was the verbal form followed by the algebraic and numerical forms. Students performed many translations while solving the pre-test.

The main translations of the graphical given were into its verbal form in the first place and into its numerical form in the second place. The main translations of the verbal given were to its algebraic form in the first place and to its numerical form in the second place. However, most of these translations did not help the students to solve the problems.

Presenting the given of the problems in verbal and graphical forms was an obstacle for most of the participants who failed to understand it, or even to translate it into other forms. Many students were unable to identify mathematical concepts or ideas that are not presented in algebraic form such as the slope of the straight line or the equation of a straight line.

The results of the pre-test also showed that guess-and-check was used as a strategy when students used numerical representation for answering, while logical reasoning was the strategy adopted by most of the students while using the verbal form for answering.

The results of the selected assessments showed that the most frequent mode that students used for solving problems was the numerical mode, followed by the graphical mode, then the verbal mode and the least frequent one was the algebraic mode. The graphical and the verbal forms used to represent the given of the problems were not an obstacle for the students who were able to understand the given, interpret it and translate it into other forms. The students performed many translations while solving the selected assessments. The main translations of the given, whether presented in graphical or verbal forms were into its numerical form. Most of the representation forms that students used for answering helped them to solve the problems correctly. Students showed a mastery of transforming verbal given into tabular one as well as transforming a tabular given into algebraic one.

The results of the selected assessments revealed students' capabilities of using different representation modes as tools to solve real-life problem situations. One part of the role that many representations played in the selected assessments was either an intermediate representation between the source representation and the target one to facilitate solving the problems or translating the verbal given into a numerical one was one of the tools used to produce the graphical representation of the given. The other aspect of the role of representations was to display the given of the problem in a manner that helped students to analyze and develop a plan for solving the problem. Translating the graphical given into a numerical one was one of the tools used to study variations of functions. The graphical translation of the verbal given helped students visualize the situation and make decisions in the problem about optimization.

In the selected assessments, there was no single strategy that could have been associated to a specific representation form because for the same representation form different strategies were used in different problems depending on the given of the question.

Discussion

The instructional unit developed by the researcher and the unit about functions in the national mathematics text book share the same objectives; however, their contents and approach are different. The instructional unit emphasizes the use of multiple representations which are fundamental to conceptual learning. Although in the national mathematics textbooks, multiple representations of functions are used, the major emphasis is on the use of algebraic representations as tools. Almost all other representations of functions such as the tabular, the graphical or the table of variations are developed starting from the algebraic representations of functions.

Knuth (2000 b) contends that emphasizing the translation whose source representation is the algebraic one and whose target representation is the graphical one does not help the students to see graphical representations as tools to be used in problem solving. In the instructional unit, different representations of functions were used as tools because that helps the students to choose the most convenient representation mode to be used in a problem situation and understand the reasons that justify that appropriateness of the chosen mode (Hines, 2002). In addition, in the instructional unit, the translation between any two representation modes takes place in both directions and consequently, any representation mode is considered as source and target at the same time because students' ability to translate among the different representations of a concept is considered as an indicator of students' understanding of that concept (Gagatsis & Shiakalli, 2004, Gagatsis & Mousoulides, 2004, Lesh, Post, & Behr, 1987).

One of the important remarks about the exercises and problems related to functions in the national mathematics textbook is that almost all of them are presented within a pure mathematical context. Such a fact is not aligned with the philosophy of the Lebanese mathematics curriculum that calls for connecting mathematics to real life to in order to help students value mathematics and experience its applications (ECD, 1997 a). In addition, almost all of the exercises and the problems presented in the unit about functions in the national mathematics textbook are drill exercises that program the learners to follow a certain algorithm to obtain solutions. Such types of exercises and problems have their implications on instructional approaches. It is known that most of the Lebanese teachers depend on the national mathematics textbook to plan their teaching because the national textbook is supposed to mirror the objectives and the philosophy of the curriculum,

and supply the mathematical content and the pedagogical approach. The Lebanese mathematics teachers use the exercises and problems in this book to assign class-work and homework. The type of exercises and problems presented in the unit about functions in this book fosters the traditional method of teaching where the teacher is the transmitter of knowledge and the student is the receiver because they reduce the teaching of the concept of function to a set of rules and procedures taught in class, which has to be followed and applied in order to solve the routine and drill exercises and problems. Moreover, such type of exercises and problems does not give students the opportunity to use their own approaches and strategies to solve problems, thus reducing their problem solving abilities. This is in contradiction with the philosophy of the Lebanese mathematics curriculum that calls for an active construction of knowledge and for the promotion of problem solving.

The researcher tried to align the instructional unit with the general objectives and the philosophy of the mathematics curriculum. In this unit, mathematics is connected to every-day life situations. The majority of the exercises and problems in this unit are real-life problem situations of various degrees of difficulties. All representations of functions are used mainly as tools and not only as ends with no special emphasis on a particular type of representation. The instructional design of the unit is reflected on the instructional approach that was also in harmony with the philosophy of the Lebanese mathematics curriculum. Active construction of knowledge is the basis of the problem-solving approach adopted as an instructional approach for teaching this unit. The presentation of all concepts and ideas in this unit starts from real-life problem situations. The safe class environment insured the application of this approach. The students did not fear ridiculing or embarrassment upon expressing any wrong answers. Rather, they learned to respect each others' ideas and discuss them.

In fact, group work, as one of the aspects of the instructional approach, helped students to appreciate each other's input, taught them to work together, helped them to improve their mathematical communication, and showed them that sharing ideas with others helps them to refine these ideas and defend them at the same time.

Since teaching took place through a problem solving approach, and strategies to solve problems were not imposed by the teacher, several strategies for solving the same problem were suggested in class. Discussion of these strategies helped students' acquisition of knowledge and improved their problem solving skills. In fact, class discussion is one of the corner stones of the instructional approach adopted for teaching this unit. It taught the students that the teacher is not the only one in class who has the right answers and each one of them is capable of getting the right answer. They also learned not to wait for teacher's input and to value their own. Unlike the traditional teaching method where students' misconceptions are identified through tests and quizzes, students' misconceptions were identified through class discussions and were corrected by the students themselves in most of the cases due to the positive and safe class environment.

Students' journals were means of private communication between the teacher and her students. In these journals, students not only summarized what they learned, but expressed their feelings and attitudes about what they were learning as well. These journals helped the teacher to improve the instructional design of the unit so that students enjoyed the mathematics they were learning.

The use of computer helped the implementation of multiple representations in this unit. First, the researcher was able to generate multiple representations of functions and print them out to be used in homework assignments, worksheets, quizzes and activities. In addition, the ability to bring a computer to class granted a quick access

to the generation of multiple representations of functions and helped the students to work within the same representation mode or between two different modes of functions because it allowed the visualization.

Another difference between the instructional unit and the national mathematics textbook is the focus in the instructional unit on the use of linear functions to introduce the concept and all sub-concepts of function. Such an emphasis on linear functions is due to the fact that linear functions represent students' first formal encounter with functions and they involve all the various aspects of functions. Moreover, understanding this family of functions can be considered as a foundation to understanding the concept of function in general (Moschkovich, 1998). In addition, students are more familiar with linear functions than with other types of functions since they encounter linear functions frequently in their everyday-life.

The main concern of this study is to investigate students' use of multiple representations while solving real-life problems related to linear functions. By comparing the results of the pre-test to the results of the selected assessments, the difference between students' results in these two assessments reveals the effect of the instructional unit on students' use of multiple representations while solving problems. Students' understanding and interpretation of the situations in the pre-test is a poor one to the extent that some students left many questions unanswered because they were neither able to use the given mode to solve the problem nor were they able to translate the given of the questions to other representation modes that they could have used. This result was not a striking one because the findings of several studies showed that students face difficulties in interpreting graphical representations of functions (Beichner, 1993) especially when they have to shift

between representation modes (Moschkovich, Schoenfeld & Arcavi, 1993; Yerushalmy, 1991).

Hines (2002) contends that understanding in mathematics grows when students become capable of translating from one mode to another. This is confirmed by the results of the selected assessments that revealed quite a good improvement in students' understanding of the situations. Students were capable of performing successfully several translations within the same question. For instance, in *Question 2* of the selected assessments, there were many students who translated the verbal given into algebraic one at the beginning, then to numerical form at a later stage. Two students were even capable of performing three types of translation in this problem which are from the verbal to the algebraic form, then from the verbal to the numerical form, followed by the translation from the verbal to the graphical form. These results assert previous findings about the relation between the levels of understanding a concept and the problem solver's ability of articulating coherently the different representations of the concept during problem solving (Hitt, 1998). Another aspect that reveals students' understanding of linear functions is based on viewing linear functions "as generalized relationships involving systematic co-variation of variables" (Hines, 2002, p.358). In the pre-test, students were unable to identify the notion of co-variation between two variables. Twelve students out of the 14 were unable to see the co-variation between two variable quantities when they were asked about that in *Question 1 Part B*. A surprising result showed up in the selected assessments where the students were able to see the co-variation of the variable quantities in more than one exercise. For instance in *Question III*, students displayed their understanding of the co-variation between the variable quantities by studying the slope of the given in straight lines using different methods that varied

from calculating the slopes using formulas, or directly from the graph or by comparison of steepness of straight lines. It is worth mentioning that in the selected assessments interpretation and construction of graphs are both requested.

In the pre-test, students used algebraic, verbal and numerical representations to solve the problems and the majority used the verbal representations, whereas in the selected assessments algebraic, verbal, graphical and numerical modes were used and the most dominant mode was the numerical mode. These results match the results of previous studies that report students' frequent use of many representational systems while solving problem situations (Lesh, Landau & Hamilton, 1983).

In the pre-test, students failed to interpret the graphical representation of a linear function. Students' abilities to understand and interpret graphical representations had improved after the instructional unit. In the pre-test, students faced difficulties in interpreting linear functions, whereas in the selected assessments the majority of the students were able to interpret the graphical representation of piecewise linear functions. In the pre-test, no prediction task was associated with the graphical representations. In the selected assessments, there was a prediction task related to the graphical representation of *Question III*, and all of the students succeeded in that task, and eight of them were able to do the task using the graphical representation itself. No construction of graphical representation of functions was requested in the pre-test. In the selected assessments, majority of students succeeded in the construction of the piecewise function requested in *Question II*. Normally, graphing starts from a set of data and the graphing task is usually limited to the match between the given data and the plotted points. In the selected assessments, students were given a verbal real-life situation that they had to transform into a graphical one without being given a data set as in all construction tasks. Such a translation of the

verbal situation into its graphical form requires a higher abstraction level than the one related to plotting a graph when given a set of points.

There was a visible difference between students' ability to develop an algebraic representation of the problem situation before the instructional unit and during its implementation. Half of the students failed to develop an algebraic representation of the situation in the pre-test, while all of the students were capable of developing an algebraic representation of the situation in the selected assessments.

Students' ability to use the given mode of representation of the situation to solve a problem situation had improved. The index of fulfillment of this criterion was 0.27, while it has become 0.43 in the selected assessments. This improvement is an indicator of students' translation abilities within the same mode of representation. However, it is not a high index due to the number of translations between the different representation modes that the student performed. This is compatible with the results of previous studies which suggest that one mode of representation is not sufficient to describe all aspects of a mathematical concept (Gagatsis & Shiakalli, 2004) because the different representations of this concept complement each other and contribute to the understanding of this concept (Gagatsis & Shiakalli, 2004). I

By comparing the efficiency of the representations used in the pre-test and in the selected assessments in guiding the students towards the correct solution, it is noticed that students' ability of choosing the correct mode of representation that helps in solving problems improved. The index of fulfillment of this criterion had increased from 0.39 in the pre-test to 0.906 in the selected assessments. This shows that the students were not arbitrarily using representation modes to solve the problems; rather, they were aware of the properties of the mode of representation that they were using and that helped to solve the problems correctly. Such results

support previous findings about the relation between the problem solver's ability and his/her ability to create and use the appropriate representations to represent the problem (Cai & Lester, 2005; Cifarelli, 1993).

Teaching students functions using a mathematics curriculum that emphasizes multiple representations of functions does not always lead to students' use of multiple representations in their problem solving. Knuth (2000 a) reported students' tendency to use algebraic representation in their submitted work even though these students had studied functions using a curriculum that emphasized multiple representations of functions. Yershalmy and Schwartz (1993) attribute this tendency to the emphasis on the manipulation of the algebraic representation during instruction. Referring to the results of this study, it is noticed that the students used 12 algebraic representations in the pre-test out of 42 representations and most of the representations they used were verbal ones. As the results show, this due to the difficulties that students faced in understanding the given of the questions and translating it into other forms. Whereas, in the selected assessments, where the students showed a good understanding of the given, the students had used 17 algebraic representations out of 215. By comparing these results as percentages, 28.5 % of the representations used in the pre-test were algebraic and 7.9% of the representations used in the selected assessments were algebraic. Such a minimal use of algebraic representation in the selected assessment could be attributed to the fact that no emphasis was put on any particular type of representation during the instructional unit. Nonetheless, this does not mean to ignore the algebraic representations of functions because they are important modeling tools. This shows that the instructional approach can be considered as of the factors that affects on

students' choice of representations during problem solving and special attention should be paid to instructional approaches in any curricular reform.

Finally, the integration of real-life problem situations in the instructional unit had contributed to students' understanding of the mathematics involved in the unit. Real-life situations were used to introduce new topics and to demonstrate to the students the ways the mathematics they are learning can be applied. Such an environment in which real-life situations were integrated in the teaching of mathematics had increased students' motivation. In addition, students' familiarity with the presented situations was an important factor that enabled them to reason correctly, correct their mathematics using the logical conclusions drawn from the real-life situations, make decisions and model such decisions mathematically using different representation modes.

Limitations of the study

Several factors were identified which limit the generalizability of this study.

This study is limited by the number of participants which is relatively small. The school where the study was conducted is a private one where the number of students in any class ranges from 10 to 20 students.

One of the limitations of this study is the presence of one section of each grade level at that school. The singularity of the section in the same grade level prevented the researcher from comparing the results of the experimental group to that of a control group that could have been students of the other section.

Despite the fact that a computer was available in class during the teaching of the instructional unit to generate multiple representations of functions; it would have been fair to ensure students' an access to graphing utilities during assessments. The participants did not have graphic calculators. The use of a graphic utility would have

given students a wider choice of representations and allowed them even to check their answers.

Another limitation of this study was the instruction of math in English. Few participants had a weak English background and faced difficulties in interpreting the content of real-life problems despite the fact the researcher was careful to use simple English during teaching and assessment.

The fact that functions can be represented using various representation modes, and that the instructional unit is about functions, might limit the generalizability of the results to other mathematical contents.

Finally, having the researcher as the mathematics teacher of the course can be considered as one of the limitations of this study, because it may have incurred some bias especially concerning teacher's preference of a certain mode of representations to other modes that could have been transmitted to students.

Perspectives for further research

The use of graphing utilities is a must with a mathematics curriculum that emphasizes multiple representations. This study was conducted on an experimental group that did not have access to graphing utilities during testing. Future studies can be conducted on larger number of participants that are divided into experimental and control groups and that have access to graphing utilities in class, at home and during testing.

In this study, the use of multiple representations in the unit about functions was limited to students from grade 10. Future studies can be conducted on students from other grades such as Grade 7 or grade 9 where concepts related to functions are treated in isolation from the notion of functions such as proportionality and straight lines.

In this study, the use of multiple representations was limited to the concept of function. Further studies can investigate the use of multiple representations to teach several concepts at the same grade level and compare its efficiency with the different concepts.

As a final remark, a further study can be conducted to investigate the appropriateness of a certain selected mode of representation as related to the problem situation.

Recommendations for teaching

As the results of this study indicated, integration of real-life situations played an important role in students' understanding of the concept of function as well as the use of multiple representations. As a consequence, curriculum designers are kindly requested to review the national mathematics textbooks and check their alignment with the general objectives and philosophy of the Lebanese mathematics curriculum.

Any curricular reform can not be effective if the teachers are not convinced by it. Teachers' attitudes and beliefs about mathematics and about pedagogical approaches should be developed. Consequently, teachers' professional development is highly recommended to allow for any curricular change to succeed. In addition, it is noticed that the one of most efficient ways that urges teachers to change their instructional approaches is the changes that take place in the official exams. Any change in the official exams is automatically followed by a change in teachers' practices.

Conclusion

Finally, students' understanding of mathematics is a major goal for curriculum designers, teachers, schools and parents. Students' understanding of mathematics is related to several factors, one of which is the representation modes used to represent mathematics. Emphasizing one particular mode on the expense of other modes does

not cater for individual learning styles. Hence, more than one of representation must be used in the teaching of mathematics. Real-life problem situations should be integrated in the mathematics curriculum in order to increase students' motivation, and teachers' instructional approaches should be developed to ensure the application of the suggested changes.

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Appendix A

Pre-test (Ritemaths project)

Part A

A department store sells two brands of vacuum cleaners. The MIRACLE vacuum cleaner price is \$136, but to use it you must also buy disposable dust bags that are \$3 each. The WIZARD vacuum cleaner price is only \$110 and its dust bags are \$4 each.

- a. Write an algebraic expression that helps you to calculate the cost in dollars of a WIZARD vacuum cleaner and some dust bags in terms of the number of dust bags.
- b. Mary bought a MIRACLE vacuum cleaner and some dust bags. William bought a WIZARD vacuum cleaner and some dust bags

Both paid the same amount of money and got the same number of dust bags.

- How many dust bags did each get?
 - How much money did each spend?
- c. If a friend wanted to buy either a MIRACLE or a WIZARD vacuum cleaner from this store, which one would you suggest they buy? Explain your answer.

Part B

Another department store has different prices for the vacuum cleaners and their dust bags.

The graph hereafter shows the relationships between the cost

and the number of dust bags bought with the vacuum cleaner in this store.

1. Which brand has cheaper dust bags at this store? Explain how you found the answer?

2. For the same cost a customer could buy a MIRACLE or a WIZARD vacuum cleaner together with the same number of dust bags. How many dust bags would the customer get? About how much money would the customer spend?
3. Jamila needs one dust bag each month for her vacuum cleaner. After a year, which vacuum cleaner brand would have been the best buy for Jamila? Explain your answer.

Appendix B

Selected Assessments

(Developed by the researcher)

Question I

A particular type of account earns a simple interest at the rate $r = 0.07 = (7\%)$ per year.

a) If you save \$1000 in this account for one year, what would be your balance at the end of the year?

b) Complete the following table

Number of years	Amount earned each year	Balance at the end of the year
1		
2		
3		
4		
5		
6		
7		

c) Develop a formula that enables you to calculate the amount you have in your account after n years?

Question III

Mama's Oven

All functions involved in this problem are linear.

Mom is preparing cookies and pies. She heated the oven at the rate of 16° per minute for 10 minutes to reach the temperature T needed for baking the cookies. She left the cookies in the oven at temperature T for 20 minutes before taking them out. Then,

she increased the temperature to 300° to bake the pies. The temperature needed 10 minutes to increase from temperature T to 300° . The pies needed 35 minutes in the oven at 300° . After taking the pies out of the oven, mom turned it off and it cooled down at the rate of 10° per minute.

- What is the time needed for cooling the oven?
- Use the given grid to represent the changes in the oven's temperature graphically. Match each part of the graph with its verbal description.

Labeling of the axes is requested.

Question III

The given graph shows the amount of money that each of Sami and Raji has each day for a period of one month.

- 1) What is the amount that each of them had at the beginning of the month?
- 2) What is the amount of money that Sami had at the end of the month?
- 3) How long would Raji's money last if he keeps spending it with the same pattern described in the graph? Show your work.
- 4) For how many days Sami neither spent nor saved any money? Explain.
- 5) For how many days did Sami save the same amount daily as the one that Raji spent per day? Explain.
- 6) When did Sami and Raji have equal amounts of money? Explain.
- 7) During which period of the month was Sami's daily savings greater? Explain.

Appendix C

Selected Assessment

(Taken with permission to use from E. Phillips)

Question IV

Your phone company offers three different monthly billing options for local phone service.

Option I: \$10.00 for up to 30 calls, plus \$0.20 for each additional call.

Option II: \$30.00 for an unlimited number of calls.

Option III: \$18.00 for up to 60 calls, plus \$0.05 for each additional call.

- A. If Rami makes about 23 local calls each month, what would be the best option for him? Explain your thinking
- B. If Nadia makes about 100 local calls each month, which would be the best option for her? Explain your thinking.
- C. Suppose you want to help other people decide which the best option for them is, what would be your recommendations?

Appendix D

Excerpts from the Activities

Excerpt from Activity 2

The Parking Lot

A parking charges \$2 for the first hour of parking or portion thereof. The charge increases to \$3 per hour or portion thereof for parking beyond 30 minutes up to a daily maximum of \$15. The table to the right helps to clarify the given of the problem.

1. Draw the graph of the given function for a 24-hour time period.
2. Write the algebraic representation of this function.
3. What are the implications of these results for someone parking in this lot?

Time Parked	Money Charged(in dollars)
17 mins	2
30 mins	2
31 mins	3
1 hr	3
1 hr 1 min	6
2 hrs	6
2hrs 1 min	9
4 hrs	12
4 hrs 1 min	15
5 hrs	15
12 hrs	15
22 hrs	15

Activity 3

The *American Heritage Dictionary* defines **function** as something closely related to another thing and dependent upon it for its existence, value, or significance.

Independent – Not subject to control by others, self-governing.

Dependent – Relying on or subject to something else for support.

Write sentences that represent a function. Determine the independent and dependent parts. Include several different words such as depends, determines, and 'is a function of' in your statements.

1. How much money I make depends on the number of hours I work
 - a. I control the hours I work – independent
 - b. The money depends on the hours – dependent

Exercise 1

Read each statement below. Determine the two variables (in words) in each situation and identify each as independent or dependent.

1. How fast the grass grows depends on how much rain we get.
 _dependent
 _independent
2. The number of problems missed on a test determines your grade on the test.
 – dependent
 – independent
3. How long I talk on my cell phone depends on the number of minutes on my calling plan.
 – dependent
 – independent
4. The amount of money I make is a function of the number of hours I work.
 – dependent
 – independent
5. The number of cakes sold in a bake sale determines the amount of money made.
 – dependent
 – independent

Exercise 2

<p>3 × 3 design 4 dark tiles</p>	<p>4 × 4 design 8 dark tiles</p>	<p>5 × 5 design 12 dark tiles</p>
--	--	---

A tiling company specializes in multi-color tile patterns. A small hotel is interested in the pattern above for its square-shaped reception area.

- a) How many dark-colored tiles will there be if the reception area needs 10 tiles on each side?
- b) How many dark-colored tiles will there be if the reception area needs 18 tiles on each side?
- c) Complete the following table :

# of tiles forming the side of the square	# of dark tiles in the square shaped reception area	# of white tiles in the square shaped reception area

- d) The independent variable is
- e) The dependent variable is
- f) Use the grid given here after to plot the number of dark tiles versus the number of tiles forming the square area. (Label your axes)

The x-axis is usually used to represent the independent variable and the y-axis is used to represent the dependent variable.

- g) Write an algebraic expression that helps you to find the number of dark tiles in terms of the numbers of tiles forming the side of the square.

Exercise 3

Identical tooth picks are used to form the squares shown here after in such a way that each tooth pick is used to represent a side of the square.

Find a way to determine how many toothpicks there would be in the 50th term of the pattern. Give reasons for your answer.

Visual Description	Verbal Description	Algebraic Description

Represent the relation you have found using the following grid and don't forget to label the axes.

Exercise 4

Pizza Hut charges a fixed amount for a basic pizza plus an additional cost per topping. Pizza Hut charges \$13 for its 4-topping pizza and \$16.75 for its 7-topping pizza.

- Determine the cost of a deluxe 12-topping pizza. Show the details of the strategy you have used.
- Express the price of the pizza as a function of the price of its toppings.
- Find the price of a pizza with 6 toppings.
- As the number of toppings increase by 1, the price of the pizza increases by
- What are the possible values for the number of toppings on the pizza and what are the associated possible costs?
- Can one topping have two different prices?

Excerpt from Activity 5

Exercise I

The above graph describes the weights of Rami and Samia who are twins over several years.

- 1) At which age did Samia and Rami have the same weight?
- 2) Who was gaining weight faster than the other during the first four years?
- 3) How much weight did Samia gain during the first 4 years?
- 4) How much weight did Rami gain during the first 4 years?
- 5) What is the average increase in Rami's weight per year? Give a geometric interpretation?
- 6) What is the average increase in Samia's weight per year? Give a geometric interpretation?
- 7) Represent Samia's weight as a function of time algebraically.
- 8) Represent Rami's weight as a function of time algebraically.
- 9) Find algebraically the age at which Rami and Samia had the same weight. Match your algebraic results with the graphical one.
- 10) Find algebraically the interval of time over which Samia was gaining more weight than Rami?

Match your algebraic results with the graphical one.

Appendix E

Excerpts from Homework

Exercise

Imagine that we are in the year 2025, Ali is married and has two kids.

Ali's story

Sunday started a bit cloudy. The temperature was about 13°C , but I thought I'd keep to the original plan and go cycling with the kids around the harbor. My wife preferred to stay at home. The eight-year-old has got his own bike; I can hire a bike for me with the four-year-old on a seat on the back when we get there. What we usually do is to cycle from the bike-hire place to a "cafee" where we can sit outside and have some lunch – burger and fries probably. The "cafee" is not far – about 5 km from the start and it takes us about 25 minutes to get there. We usually stop for about 45 minutes. After lunch, we go on another 2 kilometers, stop for about 15 minutes and then head back the way we came. I guess our average cycling speed between stops is about 10 km per hour. On the way back, we usually stop at a playground for half an hour. From there it is 3 km to the bike-hire shop which takes us about 15 minutes.

- a) Use the information in the story and the relationships between distance, average speed and time to complete the following table:

Time (from start) in minutes)	Distance (from start) in kilometers	Comments
0	0	Leave cycle hire-shop
25	5	Reach "cafee"

- b) What is the distance travelled after the lunch break? At which speed was this distance covered?
- c) How many minutes does Ali need to reach the playground?
- d) Complete the given distance-time graph for the journey back to the cycle hire shop. What is the average speed for this part of the ride?
- e) Write the algebraic expressions that represent the graph you have obtained.

Exercise II

The velocities of car A and car B, starting out side by side and traveling along a straight line road, is given by $v = f(t)$ and $v = g(t)$ respectively, where v is measured in Km/min (see accompanying graph).

- At $t = 2$, which car was moving faster than the other?
- At $t = 8$, which car was moving faster than the other?
- At what time did the two cars have the same velocity?

- d) Specify the interval of time over which car A was moving faster than car B.
- e) Specify the interval of time over which car A was moving slower than car B.

Exercise I

Ali and Serhal have a factory that produces mops. Every month, they have to pay \$1500 as rent and salaries and each mop costs them \$1 to be produced.

1. What is the total cost of producing n mops per month (the \$1500 that they pay as rent and salaries are taken in consideration).
2. If they produce n mops per month and they sell them all for \$3 each, what is the sum that they will collect?
3. Represent your results on the given grid.
4. When 300 mops are produced and sold, is the cost of producing them more than the cost of selling them?
5. When 700 mops are produced and sold, is the cost of producing them more than the cost of selling them?
6. For which number of mops would the cost of the mops be more than the amount of money obtained from selling them?
7. For which number of mops would the cost of the mops be less than the amount of money obtained from selling them?
8. In your opinion, what is the minimum number of mops that Ali and Serhal need to produce and sell in order to able to gain money?

Appendix F

Sample Quizzes

I'm 190
cm tall

Question

These are the courageous members of EWC basket ball team.

Walid is 5 cm taller than Rahal.

Mehdi's height is half of Rahal's height increased by 85 cm.

Salameh's height is equal to Walid's height decreased by 9 cm.

Hassan's height is half the sum of Rahal's and Mehdi's heights decreased by 3

Ali's height is 17 times the difference between Rahal's height and Mehdi's height increased by the ratio of Mehdi's height to Hassan's height.

Find the height of each one.

Question

What is Koma doing in Saudi Arabia?

During his summer vacation, Koma decided to work as a sales person in Saudi Arabia. On the first part of his 317 kilometer trip, Koma was driving at the rate of 58 Km per hour. On the second part of the trip, Koma was driving at the rate of 52 Km per hour because of an increased volume of traffic. The total time of the trip was 5 hours and 45 minutes. Find the time that Koma spent on each part of his trip.

Question

To the right is the graph of a function g .

- Determine domain of definition and range of g . Express your answer in interval

notation.

- b) Write the algebraic expressions that represent g .
- c) Is g even or odd? Justify your answer
- d) Solve $f(x) = 4$
- e) Solve $f(x) > 2$

Question

Mohamed intends to take a bath. The following graph represents the water level (**not the volume of the water**) in the bath tub measured in cm versus time measured in minutes..

1. Describe the events described by each section of the graph.
2. Write the algebraic representations of the first two sections of the graph

Appendix G
List of Objectives of the Unit

Appendix G

List of Objectives Covered in the Instructional Unit.

Objective	Source of objective
1. Translate key words and sentences into algebraic expressions and equations.	R
2. Solve word problems..	R
3. Read simple graphs.	R
4. Represent real-life situations graphically and numerically using linear functions.	R
5. Represent real-life situations graphically and numerically using piecewise-linear functions.	R
6. Translate verbal representations of functions into algebraic representation of a function.	R
7. Translate graphical representations of functions into verbal representations	R
8. Translate graphical representations of functions into algebraic representations.	R
9. Explore the concept of the slope of a straight line as average rate of change of one quantity with respect to the other.	R
10. Find patterns.	R
11. Determine dependent and independent variables.	R
12. Model real-life problem situation using linear functions.	R

-
13. Know the definition of a function	LMC
14. Identify a real valued function of a real variable.	LMC
15. Represent a function graphically point by point.	LMC
16. Recognize if a given curve represents a function or not.	LMC
17. Know that the domain of definition of a function can be found from an explicit formula.	LMC
18. Know that the domain of definition of a function can be deduced from the representative curve of a function	LMC
19. Construct a table of values of a function f , representing the points $(x, f(x))$ of this table in a system and join the different points.	LMC
20. Recognize if a point $M(x, y)$ of the plane belongs to the representative curve of a function.	LMC
21. Recognize a part of $\mathbb{R}$ centered at 0.	
22. Recognize analytically an even function and link it to the symmetry with respect to the axis of ordinates in an orthogonal system.	LMC
23. Recognize analytically an odd function and link it to the symmetry with respect to the origin of axes.	LMC
24. Recognize analytically an increasing or a decreasing function on an interval.	LMC
-

-
25. Recognize graphically an increasing or decreasing function on a given interval.	LMC
26. Recognize graphically the parity of a function.	LMC
27. Find according to the representative curve the intervals where the function is increasing or decreasing.	LMC
28. Identify graphically a relative minimum and maximum of a function on interval.	LMC
29. Identify graphically an absolute minimum and maximum of a function on interval.	LMC
30. Compare graphically and analytically two functions on an interval.	LMC
31. Solve graphically an equation of the form $f(x) = a$ or an inequation of the form $f(x) \leq a$ (resp. $f(x) \geq a$) where a is a given constant.	LMC
32. Recognize graphically a positive function over an interval.	LMC
33. Recognize graphically and analytically the equality of two functions on an interval.	LMC
34. Compare graphically and analytically a function f on an interval I with a constant function, an affine function and another function g .	LMC
35. Solve graphically the equation $f(x) = 0$, and the inequations $f(x) > 0$ and $f(x) < 0$.	LMC
36. Study and represent graphically a function.	LMC
37. Read the representative curve of a function and reconstitute its table of variations.	LMC
-

38. Study the functions defined by $y = ax + b$, $y = x^2$, $y = \sqrt{x}$, $y = |x|$. LMC

39. Deduce the representative curves of function defined by $f(x) + a$, $f(x+a)$, and $-f(x)$ starting by LMC that of f .

R = objective developed by the researcher

LMC = Grade 10 Lebanese mathematics curriculum objectives related to the unit about functions.

Appendix H

Details of Content

CALCULUS (NUMERICAL FUNCTIONS) (20 h)

1. DEFINITIONS AND REPRESENTATION (20 h)

The common functions form the essential object of the study of functions in first year secondary. It is preferable to apply all the rules of study first on these functions and on a bounded and significant interval then on a function in general.

The only functions to study are the ones that are deduced from common functions by translation or by symmetry.

The graphical representation of a function is the principal aim of the study of this function. The use of graphical calculator is desirable, in class, to control the drawing line done by the student. The use of an appropriate computer program is beneficial in case of availability.

To motivate the students, we have an interest to foresee real-life situations, in several domains, while avoiding possible complications in these situations.

The analytic comparison of two functions on an interval must be done in very simple cases, and must not lead to equations and inequations that are hard to solve.

CONTENTS	OBJECTIVES	COMMENTS
1.1. Functions. Graphical representation.	<ol style="list-style-type: none"> 1. Identifying a real valued function of a real variable. 2. Determining the domain of definition of a function. 3. Representing graphically a function point by point. 4. Recognizing if a given curve represents a function or not. 5. Recognizing and interpreting graphically the parity of a function. 6. Characterizing an increasing function, a decreasing function on an interval. 7. Recognizing, from its representative curve, if the function is odd or even, increasing or decreasing on a given interval. 	<p>The graphical representation of a function will play a fundamental role in the introduction of different notions and their acquisition by the student. On the other hand, knowing to read a graph must be an objective of the teaching of calculus in this class.</p> <p>We will precise that a function must be defined by a rule of correspondence or a curve.</p> <p>A function f can be given by writing: $f(x) = \dots$ or $x \mapsto f(x)$. We will note the difference between f and $f(x)$.</p>

CONTENTS	OBJECTIVES	COMMENTS
	<p>8. Identifying graphically a relative maximum and minimum on an interval and an absolute maximum and minimum of a function.</p> <ul style="list-style-type: none"> Knowing the definition of a function as a mapping of a part of $\mathbf{R}$ in $\mathbf{R}$; defining it on an interval, on $\mathbf{R}$ or on a part of $\mathbf{R}$. Making a function by using one of the following procedures: <ul style="list-style-type: none"> i) an explicit formula ii) an explicit relation of dependence (practical situation, table...). Knowing that the domain of definition can be: <ul style="list-style-type: none"> i) given ii) found from the explicit formula iii) deduced from the representative curve. Constructing a table of values of a function f, representing the points $(x, f(x))$ of this table in a system, and joining these different points. Recognizing that a curve represents a function if every parallel to the axis of ordinates cuts it in one point at most. Recognizing if a point $M(x, y)$ of the plane belongs to the representative curve of a function f. Recognizing a part of $\mathbf{R}$ centered at 0. Recognizing analytically an even function and linking it to the symmetry with respect to the axis of ordinates in an orthogonal system. Recognizing analytically an odd function and relating it to the symmetry with respect to the origin of axes. Recognizing analytically an increasing or decreasing function on an interval. Recognizing graphically an increasing or decreasing function on a given interval. Recognizing graphically the parity of a function. Finding according to the representative curve the intervals where the function is increasing or decreasing. Identifying graphically a relative maximum or minimum on an interval. Identifying graphically an absolute maximum or minimum on an interval. 	<p>The study of notions such as parity, increase, decrease, minimum and maximum is based especially on the shape of the graph and its reading.</p> <p>The notion of rate of variation is not in the program.</p> <p>We will mention, by examples, the existence of functions that are neither even nor odd.</p> <p>The existence of functions that do not admit a minimum or maximum is to be mentioned.</p> <p>The representative curve of the function f on an interval I is the set of points $M(x, y)$ of the plane such that: $x \in I$ and $y = f(x)$.</p> <p>We will avoid to confound: graph and representative curve of a function.</p>

CONTENTS	OBJECTIVES	COMMENTS
1.2. Solving graphically equations and inequations.	<ol style="list-style-type: none"> 1. Comparing graphically and analytically two functions on an interval. 2. Solving graphically an equation of the form $f(x) = a$ or an inequation of the form $f(x) \leq a$ (resp. $f(x) \geq a$) where a is a given constant. 3. Recognizing graphically a positive function on an interval. <ul style="list-style-type: none"> • Recognizing graphically and analytically the equality of two functions on an interval I. • Comparing graphically and analytically a function f on an interval I with: <ol style="list-style-type: none"> i) a constant function ii) an affine function iii) another function g. • Solving graphically the equation $f(x) = 0$, and the inequations $f(x) > 0$ and $f(x) < 0$. 	<p>We will divide the plane into four quadrants numbered in the direct sense, then we will recognize that a function is positive if its representative curve is in the first two quadrants.</p> <p>To compare analytically two functions f and g on an interval I, we can study the sign of the difference $f(x) - g(x)$ on I.</p> <p>The notation $f \leq g$ on I expresses that $f(x) \leq g(x)$ for every x of I.</p>
1.3. Study of functions.	<ol style="list-style-type: none"> 1. Studying and representing graphically a function. 2. Reading the representative curve of a function and reconstituting its table of variation. 3. Studying the functions defined by : $x \mapsto ax + b$; $x \mapsto x^2$; $x \mapsto \sqrt{x}$; $x \mapsto \frac{1}{x}$ and $x \mapsto x$. 4. Deducing the representative curves of functions defined by $x \mapsto f(x) + a$; $x \mapsto f(x+a)$ and $x \mapsto -f(x)$ starting by that of f. 	<p>It is good from time to time to change the variable x into t, u, or another letter so as not to be set off in certain real-life problems.</p> <p>The knowledge of the student of the equation of a straight line, justifies the given of a table leading to the study of affine functions.</p> <p>We will specify in each case the transformation that allows the deduction.</p>

Appendix I
Chapters 19, 20 & 21 From the National Mathematics Textbook Building Up
Mathematics Grade 10

INTRODUCTION

Prerequisites

- Bases and coordinate systems.
- Equations of a line in the plane.

The consumption of gasoline in a car depends on the distance traveled, the quantity of pollution fumes emitted by the chimney of a factory on the time of operation, the salary of an employee on his seniority at the Job, etc...

In Mathematics, we study more precisely and more rigorously the relation that exists among these quantities (i.e; the way a quantity depends on another).

Toward this end, we introduce the notion of a function.

In this chapter we will introduce:

- ① functions and some of their characteristics;
- ② graphical representation of a function point by point.

ACTIVITIES

The colder we

The weather services at the airport test the relation between altitude and surrounding temperature on a daily basis in order to predict tomorrow's weather. A plane equipped with an altimeter takes off and records the following results:

Altitude in hectometers	0	3	5	7.5	9	12	14	15	20
Temperature in degrees celsius	6	5	4	3	0	-2	-4	-5	-8

Consider an orthogonal system $(O; \vec{i}, \vec{j})$

1. Plot the points of the table, taking the altitude as abscissa with a unit equal to 2 hm and the temperature as ordinate with a unit equal to 1° Celsius.
2. Join the points where the abscissas are in increasing order.
3. At what altitude will the temperature be equal to 0° ? to -4° ?

4. Can we know the altitude of the place where the temperature is 1° ?
 2.5° ? -1° ?
5. Does the temperature increase with altitude ?

of a carpenter

From a rectangular plank of wood $ABCD$ with $AB = 1\text{ m}$ and $AD = 4\text{ m}$, a carpenter wants to cut a trapezoidal piece $MNDC$. He saws the wood starting with a point M on $[BC]$ in a direction parallel to (AI) , I being the midpoint of $[BC]$.

1. a) Calculate in cm^2 the area of $MNDC$ when $IM = 100\text{ cm}$.
 b) Complete the following table:

IM	0	30 cm	50 cm	75 cm	100
Area of $MNDC$	3000 cm^2				

2. Let $IM = x\text{ m}$.
- a) To what interval I of $\mathbb{R}$ does x belong ?
 b) Calculate in m^2 the area y of $MNDC$.
 c) To what interval J of $\mathbb{R}$ does y belong?
 d) Determine the position of M when $y = 3\text{ m}^2$.
3. Consider an orthonormal system $(O; \vec{i}, \vec{j})$ such that $\|\vec{i}\| = 1\text{ m}$ and $\|\vec{j}\| = 1\text{ m}^2$. We denote by (Γ) the set of points P of coordinates (x, y) when x varies in I .
- a) Verify that the point $E(1, 2)$ belongs to (Γ) .
 b) Verify that for each point P of (Γ) , the vectors $\vec{PE}$ and $\vec{a}(-1, 1)$ have the same direction. Deduce the nature of (Γ) .

The rule that assigns to each value x of I one value y is a **function** from I to $\mathbb{R}$. It is denoted by $x \mapsto y$.

I is the **domain of definition**, and J is the **range** of the function.

(Γ) is called the **graph** of the function.

equalities

Consider the function $f: x \mapsto x^2$ and denote by (Γ) its graph in an orthonormal system $(O; \vec{i}, \vec{j})$

1. Consider the points : $O(0, 0)$; $A\left(\frac{1}{2}, \frac{1}{4}\right)$; $B(1, 1)$; $C\left(\frac{3}{2}, \frac{9}{4}\right)$;

$$D(2, 4) ; A'\left(-\frac{1}{2}, \frac{1}{4}\right) ; B'(-1, 1) ; C'\left(-\frac{3}{2}, \frac{9}{4}\right) ; D'(-2, 4).$$

- a) Verify that all these points belong to (Γ) .
 b) Plot these points and join them in order to get a curve.
2. Does the curve you get have an element of symmetry ? If yes, which?
3. Is there a point where the ordinate is minimum ? If yes, which?

1. Definitions

A function will be denoted by a lower letter : f , g ...

We say also that this is a real-valued function defined on D .

x is the input, $f(x)$ is the output. The set of outputs is called **range** of f .

The notation $f(x)$ is attributed to **Euler**.

Let D be an interval or a union of intervals of $\mathbb{R}$.

Definition 1

A **function** from a set D to $\mathbb{R}$ is a rule that assigns a single element of $\mathbb{R}$ to each element of D .

The function from D to $\mathbb{R}$ is a mapping from D onto $\mathbb{R}$

Warning

Do not confuse the function f with the number $f(x)$.

2. Notations and terminology

If f is a function defined on D , then we write

$$f: D \rightarrow \mathbb{R}$$

$$x \mapsto f(x)$$

We call $f(x)$ the **image** of x and we read « f of x ».

D is called the **domain of definition** of f

($x \mapsto f(x)$ is called the arrow notation of a function).

Examples

- The rule, which associates $2x + 1$ with each real number x is a function defined on $\mathbb{R}$
- The rule, that assigns the area y of $MNDC$ in activity 2, to each element of $[0 ; 2]$ is a function defined on $[0 ; 2]$.
- The function $t \mapsto \frac{1}{t}$ is defined on $] -\infty ; 0[\cup] 0 ; +\infty[$ because zero does not have an image.

3. Domain of definition

We are often given the domain of definition but sometimes we have to determine it. This means that we have to find the set of all real numbers x such that $f(x)$ exists.

● Examples

Function	Domain of definition
$x \mapsto \frac{1}{x+1}$	$] -\infty ; -1 [\cup] -1 ; +\infty [$
$x \mapsto \sqrt{x+2}$	$[-2 ; +\infty [$
$x \mapsto \sqrt{1- x }$	$[-1 ; 1]$

4. Determining a function

A function may be determined by:

- an explicit formula** like $f: x \mapsto \sqrt{x}$. ($x \geq 0$);
- a dependency relation** like that which exists between the temperature and altitude or like that which is given by a table of values;
- a graph** which will be illustrated in the paragraph to follow.

GRAPH OF A FUNCTION

The plane is referred to an orthonormal system.

Definition 2

Let f be a function defined on a domain D . The set of points (x, y) where $x \in D$ and $y = f(x)$ is called the **graph** of f .

● Examples

- Let f be a function defined on $[-2 ; 3]$ by $f(x) = x + 1$. The graph of f is the segment of the line of equation $y = x + 1$ which corresponds to $x \in [-2 ; 3]$ (fig 1).
- The graph of the function: $x \mapsto x^2$ defined on $[-2 ; 2]$ is given by (fig 2).

(fig 1)

(fig 2)

For $x \in [-1; 1]$, there correspond two points M and N on the circle, hence two images.

☒ Remarks

- Certain curves are not graphs of function, such as the circle.
- A curve is a graph of a function if and only if every parallel to (Oy) from a point $(x; 0)$ ($x \in D$) cuts the graph at exactly one point.
- The graph of a function is not always a straight line.

Therefore, it is not always possible to draw the graph with precision. Usually we draw an approximate graph by using a certain number of points whose coordinates can be easily calculated.

A constant function such as : $x \mapsto 4$ is increasing and decreasing at the same time.

III VARIATIONS OF A FUNCTION

Definition 3

Let f be a function defined on an interval I and let $x_1 \in I$ and $x_2 \in I$ such that $x_1 < x_2$.

- f is said to be **increasing** on I if $f(x_1) \leq f(x_2)$,
- f is said to be **decreasing** on I if $f(x_1) \geq f(x_2)$.

☒ Remarks

- We say that f is **strictly increasing** on I :

if $x_1 < x_2$, then $f(x_1) < f(x_2)$. We define **strictly decreasing** on I analogously.

- A function f may be increasing on certain parts of its domain and decreasing on its remaining parts. Dividing the domain into such parts is called the discussion of the variations of f .

● Examples

x	-1	4
$f(x)$	-5	5

- The function $f: x \mapsto 2x - 3$, $x \in [-1; 4]$ is strictly increasing because whenever $x_1 \in [-1; 4]$, $x_2 \in [-1; 4]$ and $x_1 < x_2$, we have $f(x_1) - f(x_2) = 2(x_1 - x_2)$; therefore, $f(x_1) < f(x_2)$. *Not supported by a numerical app*

x	-2	0	2
$f(x)$	4	0	4

- The function : $x \mapsto x^2$, $x \in [-2; 2]$ is strictly decreasing on $[-2; 0]$ and strictly increasing on $[0; 2]$ (see Activity 3). The indicated table of variations summarizes this.

IV

Let f be a function defined on a domain D .

Definition 4

1. f has a **maximum** $f(a)$ at a point a of D if and only if for all $x \in D$, $f(x) \leq f(a)$.
2. f has a **minimum** $f(a)$ at a point a of D if and only if for all $x \in D$, $f(x) \geq f(a)$.

This maximum is sometimes called **absolute maximum** or **global maximum**.

Definition 5

f has a **local maximum** $f(a)$ for $a \in D$ if and only if there exists an open interval I containing a so that if $x \in D \cap I$, then $f(x) \leq f(a)$.

We define a **local minimum** analogously.

$f(a)$ is a **local maximum**.
 $f(b)$ is a **local minimum**.

☒ Remark

A maximum, or a minimum of a function is called an **extremum**.

● Example

The function $f: x \mapsto x^2$ defined on $[-1; 2]$ has :

- an (absolute) minimum $f(0) = 0$ because
if $x \in [-1; 2]$, then $x^2 \geq 0$ and $f(x) \geq f(0)$.
- an (absolute) maximum $f(2) = 4$ because
if $x \in [-1; 2]$, then $x^2 \leq 4$ and $f(x) \leq f(2)$.
- a (local) maximum $f(-1) = 1$ because
if $x \in [-1; 1]$, then $x^2 \leq 1$ and $f(x) \leq f(1)$.

V

Let f be a function defined on a domain D and x is any element of D .

D may be an interval or a union of intervals. 0 does not necessarily belong to D .

Definition 6

f is **even** if and only if $-x$ belongs to D and $f(-x) = f(x)$.

☒ Remark

Condition $(-x \in D)$ is essential because $f(-x)$ may not exist for certain values of x in D .

Definition 7

f is **odd** if and only if $-x$ belongs to D , and $f(-x) = -f(x)$.

Property

- The y -axis is axis or line of symmetry of the **graph** of an even function.
- The origin O is center or point of symmetry of the graph of an odd function.

● Examples

a) $f: x \mapsto x^2 + 1$ is even on $[-2; 2]$

b) $g: x \mapsto x^2 + 1$ is neither even nor odd on $[-4; 2]$

c) $h: x \mapsto \frac{1}{x}$ is odd on $[-3; 0[\cup]0; 3]$

d) $r: x \mapsto x^2 - 4x + 2$ is neither even nor odd on $[-2; 2]$.

1. The domain of definition of a function is

- either given
- or if not, it is a set of real numbers x such that $f(x)$ exists.

2. To discuss the variations of a function f , we take two real numbers x_1 and x_2 in the domain of f such that $x_1 < x_2$, then we compare $f(x_1)$ and $f(x_2)$:

if $f(x_1) \leq f(x_2)$, then f is increasing.

x	x_1	x_2
$f(x)$	$f(x_1)$	$f(x_2)$

if $f(x_1) \geq f(x_2)$, then f is decreasing.

x	x_1	x_2
$f(x)$	$f(x_1)$	$f(x_2)$

3. f is a function defined on D such that if $x \in D$, then $-x \in D$;

f is even when $f(-x) = f(x)$

f is odd when $f(-x) = -f(x)$.

⇒ **Attention** : A function, which is not even, is not necessarily odd.

4. To draw the graph of a function,

- we plot particular points $(x, f(x))$ (in sufficient number),
- we plot the extrema.
- we draw a line joining these points in according with the above rules.

EXERCISES

Consider the function f defined by $f(x) = \frac{1}{\sqrt{x^2-1}}$.

Denote by D the domain of definition of f .

Choose the right answer.

- a) $D = \mathbb{R}$
 b) $D =]-1; 1[$
 c) $D =]-\infty; -1] \cup [1; +\infty[$
 d) $D =]-\infty; -1[\cup]1; +\infty[$

State whether each of the following graphs represents a function and give the domain of definition of each of the functions.

Find the domain of definition of each of the following functions :

1. $x \mapsto \sqrt{2-x}$; 2. $t \mapsto \sqrt{t^2+4}$

3. $u \mapsto \frac{1}{\sqrt{u}} + 1$; 4. $x \mapsto \frac{x+1}{x-1}$

5. $t \mapsto \sqrt{-t+3}$; 6. $v \mapsto \sqrt{\frac{v-1}{v+1}}$

7. $x \mapsto \frac{1}{2x^2-8}$; 8. $t \mapsto \frac{\sqrt{t}}{|t-1|}$

Consider the function : *rep as obj*

$f: x \mapsto \frac{1}{2}x^2 + x, x \in [-3; 3]$

we want to plot the graph of f .

a) Complete the following table :

x	-3	-2.5	-2	-1.5	-1	-0.5	0	1	1.5	2	2.5	3
$f(x)$												

b) Plot the points $(x, f(x))$ and join these points by a smooth curve.

Answer the same question as in exercise 4

1. $x \mapsto 2x^2 - 3$; 2. $x \mapsto \sqrt{x+1}$

3. $x \mapsto \frac{x+1}{x-1}$

Let f be a function defined on an interval I . Answer true or false :

• f is either increasing or decreasing

if, for all $x \in I, f(x) \geq 0$, then f is increasing on I

The table of variations of f is

x	-1	0	2	4
$f(x)$	-2	0	-1	3

Answer True or False

1. f is decreasing on $[-1; 0]$;
 2. f is decreasing on $[3; 4]$;
 3. f is decreasing on $[1; 2]$.

- ◆ For each of the functions represented by the following graphs, set up the table of variations.

- ◆ Consider the function

$$f: x \mapsto -\frac{1}{3}x^2 + 1, x \in [-3; 3]$$

1. Complete the following table:

x	-3	-2	-1.5	0	1	2.5	3
$f(x)$							

2. Plot the graph of f .
3. Determine the maximum of the function f . For what value of x is the maximum reached?

- ◆ State if each of the following functions defined on $\mathbb{R}$, is even or odd:

1. $x \mapsto |x| + 1$; 2. $x \mapsto 3x - 1$;
3. $t \mapsto 2t^2 + 3t$; 4. $u \mapsto 3u^2 + 1$;
5. $x \mapsto x^3 + x^2$; 6. $t \mapsto -4$;
7. $x \mapsto \left(x - \frac{1}{2}\right)^2 + x$; 8. $x \mapsto \frac{x}{1+x^4}$.

- ◆ Prove that :

If $f(0)$ exists and f odd, then $f(0) = 0$.

- ◆ Answer true or false.

1. The function $x \mapsto x^2$ has at 0 an absolute and local minimum equal to zero.
2. The function defined in exercise 7 has a local minimum equal to -1 .

- ◆ f is an even function defined over $[-4; +4]$.

- a) Complete the following table :

x	-4	-3	0	1	3	4
$f(x)$			0	2	-1	-2

- b) Draw the graph given that it's made of line segments. ob)

- ◆ Repeat the preceding exercise for an odd function.

- ◆ Consider an even function f defined on $[-2; 2]$ and increasing on $[0; 2]$.

Answer true or false :

1. We can calculate $f(0)$;
2. $f(-2)$ is a minimum of f ;
3. $f(2)$ is a maximum of f .

- ◆ Answer the same question as exercise 15 for an odd function f defined on $[-2; 2]$.

- ◆ Find the minimum on $\mathbb{R}$ of :

$$1. x \mapsto |x + 2| - 1 ; 2. x \mapsto x^4 + 2x^2 - 1 .$$

- ◆ Set up the table of variations and determine the extrema of the function g defined on $[-2; 3]$ and represented graphically in the indicated figure.

◆ Consider a function defined on $\mathbb{R}$ by $f(x) = -4x^2 + 4x + 1$.

a) Solve the inequality $f(x) \leq 2$.

b) Deduce that f has a maximum and determine its abscissa.

◆ The temperature in Lebanon and neighboring countries is given in degrees Celsius and denoted by $^{\circ}\text{C}$.

In other countries, like the United States, the temperature is given in degrees Fahrenheit and

denoted by $^{\circ}\text{F}$. In Physics, we are given the following relation between $^{\circ}\text{C}$ and $^{\circ}\text{F}$:

$$\frac{^{\circ}\text{C}}{5} = \frac{^{\circ}\text{F} - 32}{9}$$

a) If the weather report gives a mean temperature of 25°C , give this temperature in $^{\circ}\text{F}$.

b) Prove that $^{\circ}\text{F}$ is an increasing function of $^{\circ}\text{C}$.

c) If the weather report gives a temperature ranging between 20 and 30°C , give the minimal and the maximal temperatures in degrees Fahrenheit.

SELF EVALUATION

◆ Which of the following curves represent a function:

◆ State if each of the following functions, is even or odd:

a) $x \mapsto 2x, x \in [-2; 3]$;

b) $t \mapsto \frac{t+1}{t}, t \in]-\infty; 0[\cup]0; +\infty[$;

c) $t \mapsto \sqrt{t^2 - 1}, t \in]-\infty; -1] \cup [1; +\infty[$;

d) $x \mapsto \frac{x}{1+x^4}, x \in [0; +\infty[$.

◆ Is the function $x \mapsto -2x + 3$ decreasing on $\mathbb{R}$?

◆ a) Find the domain of definition of the function f :

$$x \mapsto \frac{\sqrt{x}}{x-2}$$

b) Complete the following table:

x	0	1	2	2.5	3	4	4.5	5
$f(x)$								

◆ Consider the function $f: x \mapsto 2x^2$.

a) Find the domain of f .

b) Prove that f is increasing on $[0; +\infty[$ and decreasing on $]-\infty; 0]$.

c) Does f have an extremum? If yes, state precisely its nature and calculate it.

PROBLEMS

◆ a) Prove that, for all real numbers a and b in the interval $[-2; +\infty[$, $a + b + 4 \geq 0$

b) Consider the function $f: x \mapsto x^2 + 4x - 5$

1. Prove that f is increasing on $[-2; +\infty[$.

2. Prove that f is decreasing on $]-\infty; -2]$.

3. Deduce that f has a minimum to be determined.

4. Find the range of f .

◆ Consider all the rectangles of perimeter 12 cm.

a) Calculate the area $s(x)$ of such rectangle in terms of one of the dimensions x .

b) Determine the domain of definition of s .

c) Prove that $s(x) = 9 - (x - 3)^2$.

d) What is the maximum value of s ?

What is the corresponding form of the rectangle?

e) Prove that s is increasing on $[0 ; 3]$ and decreasing on $[3 ; 6]$.

f) Plot the graph of s .

◆ Consider the two functions :

$f: x \mapsto \sqrt{x}$ and $g: x \mapsto \sqrt{4x}$

a) Verify that $g(x) = f(4x)$.

b) Prove that if $0 \leq a < b$, then $f(4a) < f(4b)$.

c) Deduce that g is increasing over $[0 ; +\infty[$.

JUST FOR FUN

Complete the indicated table of variations of a function defined on $[-5 ; 5]$ such that :

x	2	5
$f(x)$	2	5

1. f is even.

2. f has three extrema which are integers, one of them is -1 and their sum is $+1$.

3. The abscissa of an extremum is an integral power of that of another

INTRODUCTION

Prerequisites

- Generalities about functions.
- Bases and coordinates system.
- Equations of a line in the plane.
- Properties of inequalities.

Some functions related to powers (integral or not) are in the program for this year. We shall consider these functions to be basic. Their discussion has the following goals:

- ① to initiate a methodical and complete discussion of these functions, by using the basic rules given in the preceding chapter;
- ② to emphasize certain specific properties of each of these functions;
- ③ to deduce from their graphs the graphs of other functions related to them by: $x \mapsto f(x) + a$, $x \mapsto f(x + a)$, $x \mapsto -f(x)$.

ACTIVITIES

another

Consider the function $f: x \mapsto -3x$.

1. Draw the graph (d) of f in an orthonormal system $(O; \vec{i}, \vec{j})$.
2. Let $M(x, y)$ be a point of (d) and $M'(x', y')$ such that $\overrightarrow{MM'} = 2\vec{j}$.

Express x' in terms of x and y' in terms of y .

3. Determine the function $g: x' \mapsto y'$ and draw its graph (d') .
4. Let $A(1, -3)$ be a point of (d) . Construct the point A' of (d') such that $\overrightarrow{AA'} = 2\vec{j}$.

M' is called the **image** of M under the **translation** with vector $2\vec{j}$.

5. $M''(x'', y'')$ is the point defined by $\overrightarrow{MM''} = \frac{3}{2} \vec{i}$. Determine the function $h: x'' \mapsto y''$ and draw its graph (d'') .

1. Complete the following table:

x	-4	-2.5	-1	-0.5	-0.2	0	0.25	0.5	1	1.5	2.5
$\frac{1}{x}$											

2. Plot the points $\left(x, \frac{1}{x}\right)$ of the table in an orthonormal system, then

draw the graph (C) of the function $g: x \mapsto \frac{1}{x}$.

3. What is the number which has no image under g ? How do you translate this fact graphically?
4. Does (C) have an element of symmetry? If yes, which?
5. Use (C) to set up the table of variations of g .

TEXT

In this chapter, the plane is referred to an orthonormal system $(O; \vec{i}, \vec{j})$, although it is not mentioned.

1. Direct variation functions

Definition 1

We call every function f defined on $\mathbb{R}$ by $x \mapsto ax$ where a is a given real number a **Direct variation function**.

For all real numbers x_1 and x_2 such that $x_1 < x_2$, we have:

$$f(x_1) - f(x_2) = a(x_1 - x_2).$$

The properties $f(x_1 + x_2) = f(x_1) + f(x_2)$.

$f(\lambda x) = \lambda f(x)$ characterizes the function $x \mapsto ax$.

The function $x \mapsto ax = y$ represents the proportionality between two quantities x and y .

- If $a > 0$, then $f(x_1) - f(x_2) < 0$ and $f(x_1) < f(x_2)$, hence, f is strictly increasing on $\mathbb{R}$;

x	$-\infty$	$+\infty$
$f(x)$		

- If $a < 0$, then $f(x_1) - f(x_2) > 0$ and hence, f is strictly decreasing on $\mathbb{R}$.

x	$-\infty$	$+\infty$
$f(x)$

• The origin $O(0,0)$ and the point $A(1, a)$ belong to the graph of f which is the line of equation $y = ax$.

☒ Remark

- The function $f: x \mapsto ax$ is odd because $f(-x) = -f(x)$. Therefore, the origin O is a center of symmetry of the graph of f .
- The graph of the function $x \mapsto x$ ($a = 1$) is the first bisector of the angle between Ox and Oy as shown in the figure below.
- The graph of the function $x \mapsto -x$ is the second bisector of $x\widehat{O}y$.

2. Linear functions

Let $f: x \mapsto kx$ be a direct variation function and (d) its graph.

The image of a point M of (d) under the translation with vector $a\vec{i}$ is a point M' of a line (d') parallel to (d) .

(d') is the graph of the function g defined by:

$$g(x) = f(x - a) = k(x - a) \text{ (See Activity 1).}$$

Therefore, $g(x) = kx + b$ ($b = -ak$). Thus, g is said to be a **linear function**.

Similarly, the translation with vector $a\vec{j}$ transforms a point M of (d) onto a point M'' . The set of points M'' is a line (d'') parallel to (d) , (d'') being the graph of the function $h: x \mapsto h(x) = f(x) + a = kx + a$.

● Example

The function $f: x \mapsto -2x + 6$ is linear. It is tied to the function $g: x \mapsto -2x$ by $f(x) = g(x - 3)$.

Thus, f is strictly decreasing on $\mathbb{R}$ and its graph is the line (d) passing through the points $O'(3, 0)$ and $A'(0, 6)$.

II

ABSOLUTE VALUE FUNCTION

Definition 2

The **absolute value function** is the function defined by $x \mapsto |x|$.

- The domain of f is $\mathbb{R}$.
- f is strictly increasing on $[0; +\infty[$, and strictly decreasing on $]-\infty; 0]$.
- f has a minimum $f(0) = 0$ because for every $x \in \mathbb{R}$, $|x| \geq 0$ and $f(x) \geq 0$.
- $O(0, 0)$, $A(1, 1)$, $A'(-1, 1)$ are points of the graph of f .

☒ Remark

f is an even function because $f(-x) = |-x| = |x| = f(x)$.

This allows us to draw the graph for $x \in [0; +\infty[$. The rest of the graph can be drawn by symmetry with respect to $y'Oy$.

● Example

To draw the graph of the function $f: x \mapsto |x + 3|$, we consider the graph (Γ) of the function $g: x \mapsto |x|$.

The translation with vector $\vec{MM}' = -3\vec{i}$ transforms a point $M(x, y)$ of (Γ) onto a point $M'(x', y')$ such that, $x' - x = -3$ and $y' - y = 0$.

Then $y' = y = |x| = |x' + 3|$.

Therefore, M' is on the graph (C) of f and we can draw (C) starting with (Γ) .

x	$-\infty$	-3	$+\infty$
$f(x)$	$\swarrow$	0	$\searrow$

☒ Remark

We could have discussed f directly as follows:

- If $x \geq -3$, then $f(x) = x + 3$ (represented by a ray).
- If $x \leq -3$, then $f(x) = -(x + 3)$ (represented by a ray).

x	0	$+\infty$
$f(x)$	0	$\nearrow$

Definition 3

The **squaring function** is the function defined by $f: x \mapsto x^2$.

(C) is called parabola.

- The domain of definition is $\mathbb{R}$
- f is even because $f(-x) = (-x)^2 = x^2 = f(x)$. Thus, we only have to consider x in the interval $[0; +\infty[$.
- f is strictly increasing on $[0; +\infty[$, because if $0 \leq x_1 < x_2$, then $f(x_1) - f(x_2) = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) < 0$, therefore, $f(x_1) < f(x_2)$.
- f has a minimum $f(0) = 0$ because for every x , $x^2 \geq 0$, thus, $f(x) \geq 0$.

Here is a table of values of f

x	0	0.5	1	1.5	2	2.5
$f(x)$	0	0.25	1	2.25	4	6.25

x	$-\infty$	0	$+\infty$
$f(x)$	$\searrow$	0	$\nearrow$

- To complete the graph of f on $]-\infty; 0]$, we use the symmetry with respect to y' .

Example

Discuss the function $g: x \mapsto x^2 + 2$.

x	$-\infty$	0	$+\infty$
$f(x)$	$\searrow$	2	$\nearrow$

Denote by (Γ) the graph of g and consider the function $f: x \mapsto x^2$ whose graph is denoted by (C)

The translation with vector $\overrightarrow{MM'} = 2\vec{j}$ transforms a point $M(x, y)$ of (C) onto a point $M'(x', y')$ of (Γ) because $x' - x = 0$ and $y' - y = 2$; therefore,
 $y' = y + 2 = x^2 + 2 = x'^2 + 2$. (Γ) is the image of (C) in the translation with vector $2\vec{j}$.

Definition 4

- The **square root function** is the function defined by $x \mapsto \sqrt{x}$

The domain of f is $\mathbb{R}_+ = [0 ; +\infty[$.

- f is strictly increasing because if $0 \leq x_1 < x_2$, then $\sqrt{x_1} < \sqrt{x_2}$.
- f is neither even nor odd.
- f has a minimum $f(0) = 0$ because for $x \geq 0$, $\sqrt{x} \geq 0$ and $f(x) \geq 0$,
- Here is a table of values of f

x	0	0.25	1	2.25	4	9
$f(x)$	0	0.5	1	1.5	2	3

- Beside is the table of variations of f .

x	0	$+\infty$
$f(x)$	0	$\nearrow$

● Example

The graph (Γ) of the function $g : x \mapsto \sqrt{x-1}$ can be deduced from the graph (C) of the function $x \mapsto \sqrt{x}$ by the translation with vector $\vec{i}$.

Here is the table of variations of g .

x	1	$+\infty$
$g(x)$	0	$\nearrow$

V INVERSE VARIATION FUNCTION

Definition 5

The **inverse variation function** is the function f defined by : $x \mapsto \frac{1}{x}$

- The domain of definition is $\mathbb{R}^* =]-\infty ; 0[\cup]0 ; +\infty[$.
- f is odd; therefore, we may discuss the variations of f and plot its graph for $x \in]0 ; +\infty[$ and complete the graph by symmetry with respect to the origin O .
- f is strictly decreasing on $]0 ; +\infty[$ because if $0 < x_1 < x_2$, then

$$\frac{1}{x_2} > \frac{1}{x_1} \text{ and } f(x_1) > f(x_2).$$

- Here is a table of values of f :

x	0.25	0.5	1	1.25	2	4
$f(x)$	4	2	1	0.8	0.5	0.25

This curve is called a **hyperbola**.

The function : $x \mapsto y = \frac{1}{x}$

represents a relation between a quantity x and a quantity y to which it is inversely proportional.

• below is the table of variations of f :

x	$-\infty$	0	$+\infty$
$f(x)$

● Example

Consider the function $g : x \mapsto \frac{x+1}{x}$

We can write $g(x) = 1 + \frac{1}{x}$

The translation with vector $\vec{j}$ transforms the graph (Γ) of the function $x \mapsto \frac{1}{x}$ onto the graph (C) of g .

VI

Consider a function $f : x \mapsto f(x)$ of graph (C) and let $g : x \mapsto -f(x)$. Denote by (Γ) the graph of g .

For every point $M(x, y)$ of (C) we have $y = f(x)$. The coordinates (x', y') of the point M' , symmetric of M with respect to $x'x$ verify $x' = x$ and $y' = -y$. Therefore, M' belongs to (Γ) and (Γ) is the symmetric of (C) with respect to $x'x$.

● Example

Consider the function $f : x \mapsto -\sqrt{x} + 2$

• The symmetry with respect to $x - axis$ leads to the graph (Γ) of the function $x \mapsto -\sqrt{x}$ starting with the graph of the function $x \mapsto \sqrt{x}$.

• The translation with vector $2\vec{j}$ transforms (Γ) onto (C) graph of f .

1. Graph of a parabola

$$y = x^2$$

Minimum 0 at $O(0, 0)$

Oy : axis of symmetry

$$y = (x - a)^2$$

Translation with $a\vec{i}$,
minimum 0 (zero) at $(a, 0)$

$x = a$ axis of symmetry

$$y = -x^2$$

Maximum 0 at $O(0, 0)$

Oy : axis of symmetry

$$y = x^2 + a$$

Translation with $a\vec{j}$

Minimum a at $(0, a)$

$x = 0$ axis of symmetry

2. Graph of a hyperbola

$$y = \frac{1}{x}$$

O : center of symmetry

located in the 1st and the 3rd quadrant

$$y = -\frac{1}{x}$$

O : center of symmetry

located in the 2nd and the 4th quadrant

$$y = \frac{1}{x-a}$$

Translation with $a\vec{i}$

$(a, 0)$: center of symmetry

$$y = \frac{1}{x} + a$$

Translation with $a\vec{j}$

$(0, a)$: center of symmetry

3. Square root function

$$y = \sqrt{x}$$

minimum 0 at O

$$y = -\sqrt{x}$$

maximum 0 at O

$$y = \sqrt{x-a}$$

Translation with $a\vec{i}$

minimum 0 at $(a; 0)$

$$y = \sqrt{x} + a$$

Translation with $a\vec{j}$

minimum a at $(0; a)$

EXERCISES

Define, if possible, a linear function in each of the following cases:

1. f verifies : $f(1) = -\frac{2}{3}$ and $f(-1) = -\frac{4}{3}$;

2. f verifies : $f(1) = \frac{1}{2}$ and $f(-1) = \frac{1}{2}$;

3. f verifies : $f(-1) = 2$ and $f(3) = f(4)$;

4. f verifies : $f(-1) = 2$ and $f(3) = f(4) + 1$.

Determine the sense of variations of the functions f , g , h defined by:

$$f(x) = -\frac{4}{5}x + 1 ;$$

$$g(x) = \left(\frac{3}{2} - \sqrt{3}\right)x - 2 ;$$

$$h(x) = x\sqrt{2} - \frac{3}{2}(x + 6).$$

The store «El-Hafi» announces a discount of 15% on the selling price of shoes.

Determine the function f which associates the new price of shoes to the old price.

Discuss the variations and draw the graph of the function defined on $\mathbb{R}$ as follows :

if $x \in]-\infty ; 1]$, $f(x) = 2x - 1$;

if $x \in]1 ; +\infty[$, $f(x) = -x + 2$.

Let $f: x \mapsto |-x + 2|$. Write the expression of f without using the symbol $| \cdot |$

Discuss the variations and draw the graph of each function:

1. $x \mapsto |x| + 1$; 2. $x \mapsto |x + 1|$

3. $x \mapsto -|x - 3|$; 4. $x \mapsto -|x| - 3$.

Find the value of a , then draw the graph f of the function $f: x \mapsto |x + a|$ in each of the following cases:

1. $a > -2$ and $f(2) = 1$ 2. $a < -2$ and $f(2) = 1$

a) Set up a table of signs of the ratio $\frac{x-2}{x}$.

b) Let $f: x \mapsto \left| \frac{x-2}{x} \right|$. Write the expression of f without using the symbol $| \cdot |$.

Find the value of a , then draw the graph of the function $f: x \mapsto x^2 + a$ in each of the following cases:

1. $f(2) = -2$; 2. $f(1) = 2$

Draw the graph of each of the following functions, then discuss its variations.

1. $x \mapsto x^2 - 3$ 2. $x \mapsto (x + 2)^2$;

3. $x \mapsto -x^2 + 2$ 4. $x \mapsto x^2 + x + \frac{1}{4}$

f is the function defined by $f(x) = ax^2 + bx + 1$

a) Determine a and b if:

$$f(-1) = \frac{1}{4} \text{ and } f(1) = \frac{9}{4} .$$

b) discuss the variations of f .

Find a number a such that :

if $x > a$, then $x^2 > 10^{16}$.

Draw the graph of each of the following functions, then discuss its variations.

1. $x \mapsto \sqrt{x} + 3$ 2. $x \mapsto \sqrt{x + 3}$

3. $x \mapsto -\sqrt{x} + 1$ 4. $x \mapsto -\sqrt{x + 1}$

Find the number a , then draw the graph of the function $f: x \mapsto -\sqrt{x} + a$ in each of the following cases :

1. $f(1) = 2$ 2. $f(1) = -\frac{1}{3}$.

◆ Find a number a such that : if $x > a$, then $\sqrt{x} > 10^{10}$.

◆ f is the function defined on $\mathbb{R}$ as follows :

if $x \geq 0$, then $f(x) = \sqrt{x}$;

if $x \leq 0$, then $f(x) = \sqrt{-x}$,

Is f even or odd? Draw the graph and set up the table of variations of f .

INVERSE VARIATION FUNCTION

◆ Draw the graph of each of the following functions, then discuss its variations.

1. $x \mapsto \frac{1}{x-1}$

2. $x \mapsto \frac{1}{x} - 1$

3. $x \mapsto \frac{-1}{x+2}$

4. $x \mapsto 2 - \frac{1}{x}$

5. $x \mapsto \frac{2x-1}{x}$

6. $x \mapsto \frac{x+2}{2x}$.

◆ Find the number a , then draw the graph of the function $f : x \mapsto \frac{1}{x+a}$ in each of the following cases:

1. $f(1) = 2$; 2. $f(1) = -2$.

◆ f is the function defined on $]0 ; +\infty[$ by :

$f(x) = -\frac{1}{x}$ if $0 < x < 1$ and $f(x) = -x^2$ if $x \geq 1$

Graph f , then discuss its variations.

20

f is the function defined by $f(x) = \frac{1}{x-1}$

a) Discuss the variations of f on $]1 ; +\infty[$.

b) Deduce that : if $x > 3$, then $\frac{1}{x-1} < \frac{1}{2}$.

◆ Find a number a such that if $x > a$, then

$\frac{1}{x+5} < 10^{-2}$.

◆ Represent in a graph the set of points $M(x, y)$ such that $x^2y^2 = 4$

SELF EVALUATION

◆ a) Find a linear function f such that : $f(0.5) = 4$ and $f(-1) = 1$

b) Determine the sense of variations of f and plot its graph.

◆ Plot the graphs of each of the following functions :

1. $x \mapsto x^2 + 6x + 9$

2. $x \mapsto -x^2 + 3$.

◆ Discuss the variations of the function f :

$x \mapsto -|x| + 0.5$ and plot its graph.

◆ Consider the function $f : x \mapsto \sqrt{x+a}$

a) Determine a so that $f(0) = 0.5$.

b) Plot the graph of f .

◆ a) Discuss the variations of f :

$x \mapsto -\frac{1}{x+3}$ on $] -\infty ; -3[$ and on $] -3 ; +\infty[$.

b) Plot the graph of f .

PROBLEMS

◆ f is the function defined by : $x \mapsto 2 - 3|x|$.

- Show that f is even.
- Discuss the variations of f on $[0 ; +\infty[$ and plot its graph.
- Deduce the graph of f on $\mathbb{R}$.

2 The phone bill consists of a fixed rate and the price of units registered during two months. A bill totals 58 000 LL for 1000 units. Another totals 83 000 LL for 1500 units.

Determine the function f which assigns to the number x of units the corresponding amount.

◆ g is the function defined by $g(x) = \frac{2}{x}$.

- Show that g is odd.
- Discuss the variations of g on $]0 , +\infty[$.
- Plot the graph of g .
- Draw the graph of each of the following functions

$$x \mapsto -\frac{2}{x} ; x \mapsto \frac{2}{x} + 1 ; x \mapsto \frac{2}{x+1}.$$

◆ Consider a semicircle with center O and radius 5 cm. Let M be a point on the diameter $[AB]$ such that $AM = x$.

We denote by $f(x)$ the area bounded by the three semicircles of diameters $[AB]$, $[AM]$, $[MB]$.

- Calculate $f(x)$ in terms of x .
- Discuss the variations of f on $[0 , 10]$.
- Find the maximum of $f(x)$ and find the corresponding point M .
- Show that $f(x) = \frac{\pi}{4} (25 - OM^2)$, and deduce the result of part c).

◆ Let $f: x \mapsto x^2 - |x|$, $x \in \mathbb{R}$.

- Show that f is even.
- Show that f is decreasing on $[0 ; \frac{1}{2}]$ and increasing on $[\frac{1}{2} ; +\infty[$.
- Complete the table.

x	0	0.25	0.5	0.75	1	2	3	4
$f(x)$								

- Draw the graph of f on $[-3 ; 3]$.

◆ k is the function defined on $\mathbb{R}$ by $k(t) = t^3$.

- Show that k is odd.
- Let $p(t) = t^2 + at + a^2$, where a is a real number.
 - Verify that $p(t) = \left(t + \frac{a}{2}\right)^2 + \frac{3a^2}{4}$.
 - Deduce that $p(t) \geq 0$.
- Show that k is increasing on $[0 , +\infty[$.
- Complete the table.

t	0	0.25	0.5	0.75	1	2
$k(t)$						

- Draw a graph of k on $[0 ; 2]$ and deduce the graph of k on $[-2 ; 2]$.

- ◆ Each of the following curves is the graph of a function, but we forgot to include the axes.

$$f(x) = (x - 1)^2 + 1$$

$$g(x) = \sqrt{x + 1} - 2$$

$$h(x) = \frac{1}{x - 1} - 1$$

Copy each graph on a (grid) and draw the coordinates axes.

- ◆ f is the function defined by $f(x) = \frac{1}{1 + x^2}$; $x \in \mathbb{R}$.

a) Verify that f is even on $\mathbb{R}$.

b) Show that f decreases on $[0; +\infty[$.

c) Complete the table.

x	0	0.5	1	2	3	5
$f(x)$						

d) Draw the graph of f on $[-5, 5]$.

- ◆ f is the function defined by $f(x) = x^2 + 2x + 2$, and (C) is the graph of f .

a) Show that $f(x) = (x + 1)^2 + 1$

b) Consider the translation of axes defined by

$$x = -1 + X, \quad y = 1 + Y \quad (\text{the new origin is } O'(-1, 1)).$$

Determine the equation of C with respect to the new axes $X'O'X, Y'O'Y$.

c) Draw (C) .

d) Deduce the table of variations of f , its elements of symmetry and its extremum.

JUST FOR FUN

Let (P) be the parabola of equation $y = x^2$ and two points A and B of (P) . I is the midpoint of $[AB]$, and the parallel through I to $(y'Oy)$ meets (P) at J . Show that

the parallel through J to (AB) cuts P at only one point
(It is the tangent to (P) at J).

INTRODUCTION

Prerequisites

- Order in $\mathbb{R}$.
- Generalities about functions.
- Basic functions.

Real-valued functions may be partially determined by their range. The order in $\mathbb{R}$, that allows the comparison of the elements of the range, also allows the comparison of functions defined on the same interval.

It is this comparison that will help to solve graphically some equations and inequalities. This solution is sometimes difficult to be made algebraically.

In this chapter, we shall learn to:

- ① compare graphically two functions on an interval;
- ② discuss graphically the sign of a function;
- ③ solve graphically an equation or an inequality.

ACTIVITY

Order between functions

Consider the two functions f and g defined on $[0 ; +\infty[$ by $f(x) = x^2$ and $g(x) = 2x + 1$.

1. Plot the graphs of f and g in the same coordinate system.
2. Let a be the abscissa of the point A , intersection of the two graphs.

Solve :

a) $f(x) = g(x)$; b) $f(x) \geq g(x)$; c) $f(x) \leq g(x)$.

3. Use the graphs to give an approximate value of a .
4. Compare $f(2.41)$ and $g(2.41)$, then $f(2.42)$ and $g(2.42)$; deduce a boundary and then an approximate value of a .

In this chapter, f and g are two functions defined on the same interval I . (C) and (Γ) are their respective graphs, and x is any element of I .

I

Definition 1

1. We say that $f \leq g$ if $f(x) \leq g(x)$.
 (C) is then below (Γ) .
2. We say that $f \geq g$ if $f(x) \geq g(x)$.
 (C) is then above (Γ) .

● Example

On the interval $\left[-\frac{3}{2}; 2\right]$, the graphs (C) and (Γ) of the functions defined by:

$f(x) = x^2$ and $g(x) = \frac{1}{2}x + 3$ are such that (C) is below (Γ) . Hence, $f \leq g$ on $\left[-\frac{3}{2}; 2\right]$.

If $x \in]-\infty; -\frac{3}{2}[\cup]2; +\infty[$, then $f > g$.

II

Definition 2

We say that f and g are equal if $f(x) = g(x)$.

We write $f = g$.

☛ Consequence

- On the interval I the graphs (C) and (Γ) coincide.
- $f \neq g$ if and only if there exists $x \in I$ such that $f(x) \neq g(x)$.

● Example

The function $f: x \mapsto |x - 1|$ and $g: x \mapsto x - 1$ are equal on $[1; +\infty[$; but they are different on $J = [0; +\infty[$ because $f(0) = 1$ and $g(0) = -1$.

Do not confuse the equality of functions $f = g$, which means that $f(x) = g(x)$ for every $x \in I$, with the equation $f(x) = g(x)$ which is only verified by the abscissa of a point of intersection of the graphs (C) and (Γ) .

Property 1

1. If (C) and (Γ) intersect at a point A , then the abscissa a of A is a solution of the equation $f(x) = g(x)$ (or $f(x) - g(x) = 0$).
2. If (C) is below (Γ) , then every $x \in I$ is a solution of the inequality $f(x) \leq g(x)$ (or $f(x) - g(x) \leq 0$).

Examples

a) Let $f(x) = x^2$ and $g(x) = \frac{1}{2}x + 3$. The solution set of the equation

$$x^2 = \frac{1}{2}x + 3 \text{ is } \{-1.5, 2\}.$$

We then have $f(2) = g(2) = 4$ and $f(-1.5) = g(-1.5) = 2.25$, but $f \neq g$ on I .

b) Consider the inequality $|x| - \frac{1}{3}x - \frac{4}{3} \geq 0$.

$$\text{We have } \frac{1}{3}x + \frac{4}{3} \leq |x|.$$

The graphs (C) and (Γ) of the functions $f: x \mapsto |x|$ and $g: x \mapsto \frac{1}{3}x + \frac{4}{3}$ intersect at $A(-1, 1)$ and $B(2, 2)$.

(Γ) is below (C) for $x \in]-\infty; -1] \cup [2; +\infty[$. This determines the solution set of the above inequality.

We verify that -1 and 2 are the roots of

$$|x| - \frac{1}{3}x - \frac{4}{3} = 0.$$

Remark

When the roots of an equation are not simple, the graphical solution is easier. In this case, we try to find graphically approximate values of the roots.

IV

Let g be a constant function : $g(x) = K$ for every $x \in I$.

The graph of g is the line (Γ) of equation $y = K$.

- The solution set of the inequality $f(x) \leq K$ is the union of intervals where (C) is below (Γ) .
- The solution set of $f(x) \geq K$ is the union of intervals where (C) is above (Γ) .
- The solution set of $f(x) = K$ contains the abscissas of the points of intersection of (C) and (Γ) .

If $K = 0$ then (Γ) becomes the x -axis.

Do not confuse the equality of functions $f = g$, which means that $f(x) = g(x)$ for every $x \in I$, with the equation $f(x) = g(x)$ which is only verified by the abscissa of a point of intersection of the graphs (C) and (Γ).

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$$\text{We have } \frac{1}{3}x + \frac{4}{3} \leq |x|.$$

The graphs (C) and (Γ) of the functions $f: x \mapsto |x|$ and $g: x \mapsto \frac{1}{3}x + \frac{4}{3}$ intersect at A (-1, 1) and B (2, 2).

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When the roots of an equation are not simple, the graphical solution is easier. In this case, we try to find graphically approximate values of the roots.

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- The solution set of $f(x) = K$ contains the abscissas of the points of intersection of (C) and (Γ).

If $K = 0$ then (Γ) becomes the x -axis.

EXERCISES

◆ f and g are the functions defined on $\mathbb{R}$ by

$$f(x) = 2x + 3 \text{ and } g(x) = \frac{1}{2}(x - 1).$$

Solve graphically :

1. $f(x) = g(x)$; 2. $f(x) \geq g(x)$.

◆ Solve graphically :

1. $|x| = \frac{5}{2}$; 2. $|x| = -1$;

3. $|x| \leq 2$; 4. $|x| > 1$.

◆ a) Draw the graph of the two functions

$$f: x \mapsto |x| \text{ and } g: x \mapsto -x + 3.$$

b) Solve graphically and algebraically the equation $|x| = -x + 3$.

c) Solve graphically and algebraically the inequality : $|x| + x - 3 < 0$.

◆ Use the graph of $f(x) = x^2$ to answer True or False :

• If $-2 \leq x \leq 0$, then $0 \leq x^2 \leq 4$;

• If $x \leq -2$, then $x^2 \leq 4$;

• If $x \leq -2$, then $x^2 \geq 4$;

• If $-1 \leq x \leq 2$, then $1 \leq x^2 \leq 4$.

◆ The following curve is the graph of the function $x \mapsto x^2 + 3x + 2$

Solve graphically the following equations and inequalities :

1. $x^2 + 3x + 2 = 0$; 2. $x^2 + 3x + 2 > 0$;

3. $x^2 + 3x \leq 0$.

◆ a) Draw the graph (P) of the function : $x \mapsto x^2 - 1$.

b) Verify graphically that every line $y = mx$, where m is a parameter, intersects (P) at two distinct points.

c) Deduce the number of solutions of the equation $x^2 - mx - 1 = 0$.

◆ Use the graph of the function $x \mapsto \sqrt{x}$ to solve :

1. $\sqrt{x} = -3$ 2. $\sqrt{x} = 1.5$

3. $\sqrt{x} \leq 2$ 4. $\sqrt{x} > 1.5$.

◆ a) Compare the functions f, g, h defined on $\mathbb{R}$ by : $f(x) = \sqrt{x^2}$; $g(x) = x$; $h(x) = |x|$.

b) Draw a conclusion concerning these functions on the interval $[0 ; +\infty[$.

◆ a) Solve graphically the inequality : $\sqrt{x-1} < 2$.

b) Find this result algebraically.

◆ Verify graphically that the equation $x^2 - \sqrt{x} = 0$ has two roots and determine them.

Use the graph of the function $x \mapsto \frac{1}{x}$, to answer True or False :

1. if $x \leq -3$, then $-\frac{1}{3} \leq \frac{1}{x} < 0$;

2. if $x_1 < 0 < x_2$, then $\frac{1}{x_1} < \frac{1}{x_2}$;

3. if $x \geq 1$, then $x > \frac{1}{x}$.

f is the function defined by $f(x) = \frac{1}{x}$. Answer True or False :

1. f decreases on $]-\infty; 0[\cup]0; +\infty[$;

2. f decreases on $]-7; 1[$;

3. f increases on $]0; 2[$.

Solve graphically :

1. $0 < \frac{1}{x} < 3$

2. $\frac{1}{x} \geq \frac{1}{2}$

3. $\frac{1}{x} < -\frac{1}{3}$

4. $\frac{1}{2} < \frac{1}{x} \leq 1$

The graph is that of the function $x \mapsto \frac{2}{x+1}$.

Solve graphically the following equations and inequalities

1. $\frac{2}{x+1} = 2$

2. $\frac{2}{x+1} = 0$

3. $\frac{2}{x+1} < 0$

4. $\frac{1}{x+1} \geq 1$

Let $f: x \mapsto x^2$ and $g: x \mapsto \frac{1}{x}$

a) Draw the graphs of f and g .

b) Verify that the graphs intersect at $A(1, 1)$.

c) Solve graphically $x^2 > \frac{1}{x}$.

SELF EVALUATION

Solve graphically :

1. $|x-1| = 2$

2. $|x+2| \leq 1$

3. $0 \leq x^2 \leq \frac{3}{4}$

4. $x^2 \geq 4$

a) Draw the graph of the functions f and g defined by :

$f(x) = \sqrt{x+2}$ and $g(x) = \frac{1}{3}x + \frac{4}{3}$

b) Solve graphically :

1. $f(x) = g(x)$

2. $f(x) \leq g(x)$

3. $f(x) \geq g(x)$

Show that the two functions f :

$x \mapsto \frac{x^2-9}{x+3}$

and $g: x \mapsto x-3$ are equal on $]-3; +\infty[$.

Can we write $f=g$ on $[-3; +\infty[$? Why?

	TRUE	FALSE
For all $x > 0$, we have $x > \frac{1}{x}$.		
On $[0, +\infty[$, the graph of the function $x \mapsto x^2$ is above the graph of the function $x \mapsto \sqrt{x}$.		
If f is a positive function on $\mathbb{R}$, then f is even.		
If f is an odd function defined at 0, then $f(0) = 0$.		
If f is even, then its graph is in the first and second quadrant.		
If f is odd, then its graph is either in the first and third quadrant, or in the second and fourth quadrant.		

PROBLEMS

- ◆ Given the function f on $[-5 ; +4]$ whose graph is below :

a) Determine the extrema of f .

b) Solve the inequality $f(x) < -3$.

c) Solve the inequalities :

$$f(x) > 2, f(x) \geq 0 \text{ and } f(x) \geq 1.$$

d) According to the values of a , find the number of solutions of the equation $f(x) = a$.

a) Draw the graphs of the functions :

$$f: x \mapsto x^2 \text{ and } g: x \mapsto -\frac{1}{x}.$$

b) Find the number of solutions of the equation

$$x^2 = -\frac{1}{x}.$$

c) What is the number of roots of the equation

$$x^3 + 1 = 0.$$

◆ a) Draw the graph of the functions f and g defined on $\mathbb{R}$ by : $f(x) = x^2$ and $g(x) = x + 1$.

b) Find graphically an approximate value for each root of the equation $x^2 - x - 1 = 0$.

c) In the following figure $ABCD$ is a rectangle of length $AB = b$ and width $AD = a$, and $AA'D'D$ is a square. The rectangle $A'BCD'$ has the same ratio of sides as the rectangle $ABCD$.

1. Let $\frac{b}{a} = K$, show that $K^2 = K + 1$.

2. Deduce an approximate value of K .

(This number is called the golden number. Its exact value is $\frac{1 + \sqrt{5}}{2}$).

◆ f is the function defined on $\mathbb{R}$ by $f(x) = x^2 + 2x - 3$.

a) Verify that $f(x) = (x + 1)^2 - 4$.

b) Deduce that f has a minimum -4 and find the value of x where the minimum is reached.

c) Show that f increases on $[-1; +\infty[$ and decreases on $] -\infty; -1]$.

d) Set up the table of variations of f .

e) Solve the equation $x^2 + 2x - 3 = 0$.

f) Sketch the graph of f and solve the inequality $x^2 + 2x - 3 < 0$.

◆ Let (C) be the graph of the function f :

$$y = -x^2 + 2x - 3.$$

a) Consider the translation of axis defined by $x = X + 1$, $y = Y - 2$.

Write the equation of (C) in the new system of axes.

b) Deduce the table of variations of the function f .

c) Solve : 1. $f(x) = 0$ 2. $f(x) < 0$.

JUST FOR FUN

Solve the inequality : $x^4 + x^2 - \frac{3}{4} < 0$.